# Discrete Fractional Fourier Transform Based on the Eigenvectors of Tridiagonal and Nearly Tridiagonal Matrices

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**Abstract**

The recent emergence of the discrete fractional Fourier transform (DFRFT) has caused a revived interest in the eigenanalysis of the discrete Fourier transform (DFT) matrix **F** with the objective of generating orthonormal Hermite-Gaussian-like eigenvectors. The Grünbaum tridiagonal matrix **T** – which commutes with matrix **F** – has only one repeated eigenvalue with multiplicity two and simple remaining eigenvalues. A detailed eigendecomposition of matrix **T** is performed with the objective of deriving two orthonormal eigenvectors – common to both the **F** and **T** matrices – pertaining to the repeated eigenvalue of **T**. The nearly tridiagonal matrix **S** first introduced by Dickinson and Steiglitz and later studied by Candan et al. – which commutes with matrix **F** – is rigorously proved to reduce to a 2 x 2 block diagonal form by means of a similarity transformation defined in terms of an involutary matrix **P**. Moreover explicit expressions are derived for the elements of the two tridiagonal submatrices forming the two diagonal blocks in order to circumvent the need for performing two matrix multiplications. Although matrix **T** has the merit of being tridiagonal and does not need the tridiagonalization step as matrix **S**, the simulation results show that the eigenvectors of matrix **S** better approximate samples of the Hermite-Gaussian functions than those of matrix **T** and moreover they have a shorter computation time due to the block diagonalization result. Consequently they can serve as a better basis for developing the DFRFT.

***Keywords***: Discrete FRactional Fourier Transform (DFRFT), DFT matrix, Grünbaum tridiagonal matrix, Dickinson-Steiglitz nearly tridiagonal matrix, circular flip matrix, Hermite-Gaussian-like eigenvectors.

## 1. Introduction

The recent appearance of the discrete fractional Fourier transform (DFRFT) resulted in a revived interest in the eigenanalysis of the discrete Fourier transform (DFT) matrix **F** since having

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orthonormal eigenvectors of that matrix is a necessary condition for the DFRFT to have the desirable index additivity property. Moreover having eigenvectors that approximate samples of the Hermite-Gaussian functions is desirable in order for the DFRFT to approximate its continuous-time counterpart namely the continuous fractional Fourier transform (FRFT) since the Hermite-Gaussian functions are the eigenfunctions of both the classical Fourier transform and the FRFT [1]. In their pioneering work, McClellan and Parks arrived at the multiplicities of the eigenvalues of matrix **F** and developed an analytical method for generating a complete set of independent eigenvectors [2]. Unfortunately this set cannot be taken as a basis for defining the DFRFT due to its nonorthogonality.

Santhanam and McClellan were the first to try to develop a definition for the DFRFT through the eigendecomposition of matrix **F** [3]. Unfortunately this definition was later shown – by Pei et al. [4,5] – not to be a fully-fledged one since it is inherently the sum of four terms; namely the time-domain signal and its DFT together with their circularly reflected versions.

Since matrix **F** has only four distinct eigenvalues, the dimensions of its eigenspaces are large and a direct computation of the eigenvectors will involve a highly degenerate problem. Grünbaum discovered a real symmetric tridiagonal matrix **T** which commutes with matrix **F** [6]. This matrix has only one eigenvalue of multiplicity two and simple remaining eigenvalues. Consequently the eigenvectors of **T** corresponding to its simple eigenvalues are also eigenvectors of **F** by virtue of the commutation of both matrices. Unfortunately not every set of two linearly independent eigenvectors of **T** pertaining to its repeated eigenvalue will be eigenvectors of **F**. Clary and Mugler developed a family of Grünbaum matrices which commute with a corresponding family of shifted Fourier matrices [7]. Santhanam and Vargas-Rubio picked a particular version of Grünbaum matrix which commutes with the *centered* DFT matrix because this particular matrix is unreduced and consequently all its eigenvalues are simple [8]. Vargas and Santhanam used the orthonormal eigenvectors of this matrix as a basis for developing the *centered* discrete fractional Fourier transform (CDFRFT) [8-11].

Along a completely different line of investigation, Dickinson and Steiglitz [12] arrived at a real symmetric *nearly* tridiagonal matrix **S** which commutes with matrix **F** and proved that the maximum algebraic multiplicity of any of its eigenvalues can be two; which occurs only when the order N of the matrix is a multiple of 4. Although a common set of eigenvectors of **S** and **F** always exists, the case of a double eigenvalue of **S** requires special attention since a set of two corresponding eigenvectors – obtained by a general eigenanalysis software package – will generally neither be eigenvectors of **F** nor even be orthogonal. Candan, Kutay and Ozaktas [13,14] applied a similarity transformation defined in terms of a unitary matrix **P** to matrix[4] **S** and argued that $\mathbf{PSP}^{-1}$ is a 2 x 2 block diagonal matrix and that the two diagonal blocks are unreduced tridiagonal matrices. They started their analysis by discretizing the second order differential equation satisfied by the Hermite-Gaussian functions and obtained a second order difference equation – called Harper's equation – whose coefficients are periodic with period N implying the existence of periodic solutions with the same period. Interestingly, one period of each solution forms the elements of an eigenvector of matrix **S**. The real-valued periodic particular solutions of Harper's equation are called Harper functions [15,16]. The implication is that the eigenvectors of **S** are Hermite-Gaussian-like. Pursuing it further, Pei et al. viewed the orthonormal eigenvectors of **S** as only *initial* ones and generated *final* ones which better approximate the Hermite-Gaussian functions by using either the orthogonal procrustes algorithm (OPA) or the Gram-Schmidt algorithm (GSA) [4]. Hanna, Seif and Ahmed proved that those *final* eigenvectors are invariant under the change of the *initial* ones [17].

A digital method for computing the continuous FRFT – without using the notion of the DFRFT – was proposed in [18]; however this method does not preserve the index additivity property. The definition of the DFRFT to be adopted here is the one first proposed by Pei and Yeh [19] and later consolidated by Candan et al. [13,14] and by Pei et al. [4,5]. Other developments of the fractional Fourier transforms were made by Cariolaro et al. [20-22].

One main objective of the present paper is to perform a detailed eigenanalysis of the *original Grünbaum matrix **T** which commutes with the DFT matrix **F*** rather than the simple version of the Grünbaum matrix which commutes with the *centered* DFT matrix. More specifically two orthonormal eigenvectors common to both **T** and **F** – which pertain to the only eigenvalue of **T** of multiplicity two – will be *analytically derived*. The complete set of common orthonormal eigenvectors will be arranged according to the number of their zero crossings in ascending order in order to be compared with samples of the Hermite-Gaussian functions.

A second objective is to present a rigorous development and arrive at explicit expressions for some of the results just argued by Candan et al. [13] regarding the eigenanalysis of the nearly tridiagonal matrix **S**. More specifically the matrix $\mathbf{PSP}^{-1}$ will be rigorously proved to be block diagonal and explicit expressions will be derived for the elements of the two tridiagonal submatrices (forming the two diagonal blocks) in the general case of a matrix **S** of any order N *which was not done in [13,14]*. The main advantage of having those explicit forms is the speeding up of the generation of the

[4] Strictly speaking, denoting matrix $\mathbf{S}$ in the work of Dickinson et. al. [12] and Pei et. al. [4] by $\mathbf{S_1}$ and matrix $\mathbf{S}$ in the work of Candan et. al. [13] by $\mathbf{S_2}$, the two matrices are related by $\mathbf{S_2} = \mathbf{S_1} - 4\mathbf{I}$. Therefore $\mathbf{S_1}$ and $\mathbf{S_2}$ have the same eigenvectors. In the present paper matrix **S** in (55) is actually matrix $\mathbf{S_2}$ of [13].

desirable eigenvectors of matrix **S** by avoiding the actual performance of the two matrix multiplications appearing in the similarity transformed matrix $\mathbf{PSP}^{-1}$.

A third objective is to perform a simulation study in order to compare the behavior of the eigenvectors of the two matrices **T** and **S** and the two DFRFT defined based on those eigenvectors. Although matrix **T** has the merit of being tridiagonal rather than *nearly* tridiagonal as matrix **S**, the simulation results will show that the DFRFT based on the eigenvectors of **S** better approximates its continuous-time counterpart than that based on the eigenvectors of **T**. Moreover the computation time in the case of matrix **S** – utilizing the block diagonalization results – will be shown to be shorter than that in the case of matrix **T**.

After presenting some general properties of the DFT matrix in section 2, an eigenanalysis of Grünbaum tridiagonal matrix **T** will be performed in section 3. Some results regarding the tridiagonalization and eigendecomposition of Dickinson-Steiglitz nearly tridiagonal matrix **S** will be rigorously derived in section 4. Finally a comparative simulation study of the different techniques for generating eigenvectors of the DFT matrix – especially those based on the matrices **T** and **S** – will be carried out in section 5.

## 2. General Properties of the DFT Matrix

The discrete Fourier transform matrix $\mathbf{F} = \left(f_{m,n}\right)$ of order N is defined by:

$$f_{m,n} = \frac{1}{\sqrt{N}} W^{(m-1)(n-1)} \qquad , m,n = 1,\cdots,N \qquad \text{where} \qquad W = \exp\left(-j\frac{2\pi}{N}\right). \tag{1}$$

Matrix $\mathbf{F}$ can be expressed in partitioned form as[5]:

$$\mathbf{F} = \frac{1}{\sqrt{N}}\begin{pmatrix} 1 & \boldsymbol{\mu}^T \\ \boldsymbol{\mu} & \mathbf{G} \end{pmatrix} \tag{2}$$

where $\boldsymbol{\mu}$ is the summing vector:

$$\boldsymbol{\mu} = \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix}^T \tag{3}$$

and $\mathbf{G} = \left(g_{m,n}\right)$ is a square matrix of order $(N-1)$ whose elements are given by:

$$g_{m,n} = W^{mn} \qquad , m,n = 1,\cdots,N-1\,. \tag{4}$$

*Lemma 1*: Matrix $\mathbf{G}$ defined by (4) is centrosymmetric, i.e. $\mathbf{JGJ} = \mathbf{G}$ where $\mathbf{J}$ is the contra-identity matrix defined by:

$$\mathbf{J} = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}. \tag{5}$$

[5] The superscripts $^T$ , $^*$ , $^H$ respectively denote the transpose, the complex conjugate and the Hermitian transpose (i.e. the complex conjugate transpose).

It can be shown that [23, p. 351]:

$$\mathbf{\Gamma} \equiv \mathbf{F}^2 = \begin{pmatrix} 1 & 0 \\ 0 & \mathbf{J} \end{pmatrix}. \tag{6}$$

It follows from the above equation and the fact that $\mathbf{J}^2 = \mathbf{I}$ that:

$$\mathbf{\Gamma}^2 = \mathbf{F}^4 = \mathbf{I}. \tag{7}$$

*Definition 1*: Vector $\mathbf{y}$ is a circular flip of vector $\mathbf{x}$ if $\mathbf{y} = \mathbf{\Gamma x}$. More specifically, if vectors $\mathbf{x}$ and $\mathbf{y}$ are expressed as: $\mathbf{x} = \begin{pmatrix} x_1 \\ \tilde{\mathbf{x}} \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} y_1 \\ \tilde{\mathbf{y}} \end{pmatrix}$ - where $\tilde{\mathbf{x}}$ denotes the subvector of all elements of $\mathbf{x}$ apart from the first one – then: $y_1 = x_1$ and $\tilde{\mathbf{y}} = \mathbf{J}\tilde{\mathbf{x}}$.

Based on this definition, matrix $\mathbf{\Gamma}$ is called the *circular flip matrix*.

*Definition 2*: Vector $\mathbf{x}$ is *circularly even* if $\mathbf{\Gamma x} = \mathbf{x}$. More specifically, $x_1$ is unrestricted and $\mathbf{J}\tilde{\mathbf{x}} = \tilde{\mathbf{x}}$.

*Definition 3*: Vector $\mathbf{x}$ is *circularly odd* if $\mathbf{\Gamma x} = -\mathbf{x}$. More specifically, $x_1 = 0$ and $\mathbf{J}\tilde{\mathbf{x}} = -\tilde{\mathbf{x}}$. One should notice that if the length N of $\mathbf{x}$ is even, then the middle element of $\tilde{\mathbf{x}}$ in a circularly odd vector $\mathbf{x}$ will vanish.

*Lemma 2*: If vectors $\mathbf{u}$ and $\mathbf{v}$ are respectively circularly even and odd, then:

a) $\mathbf{u}$ and $\mathbf{v}$ are orthogonal

b) $\mathbf{Fu}$ and $\mathbf{Fv}$ are respectively circularly even and odd.

## 3. An Eigenanalysis of the Grünbaum Tridiagonal Matrix

The nice properties of the eigenvectors of Grünbaum matrix **T** will be first covered in order to set the stage for analytically deriving two orthonormal eigenvectors common to both **T** and **F** matrices and pertaining to the only eigenvalue of **T** of multiplicity two. Finally all N orthonormal eigenvectors of **T** will be arranged in ascending order according to the number of their zero crossings.

### 3.1. Preliminaries

The Grünbaum tridiagonal matrix of order N can be expressed as the block diagonal matrix:

$$\mathbf{T} = \begin{bmatrix} T_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{T_2} \end{bmatrix} \tag{8}$$

where $T_1$ is the scalar 0 and $\mathbf{T_2}$ is a tridiagonal matrix of order $(N-1)$ given by:

$$\mathbf{T_2} = \begin{bmatrix} \alpha_1 & \beta_1 & & & \\ \beta_1 & \alpha_2 & \beta_2 & & \\ & \beta_2 & \ddots & \ddots & \\ & & \ddots & \ddots & \beta_{N-2} \\ & & & \beta_{N-2} & \alpha_{N-1} \end{bmatrix} \tag{9}$$

with

$$\alpha_m = 2c_1 s_m^2 \quad , m = 1, \cdots, N-1 \tag{10}$$

$$\beta_m = -s_m s_{m+1} \quad , m = 1, \cdots N-2 \tag{11}$$

$$s_m = \sin(m\pi / N) \quad \forall m \tag{12}$$

$$c_1 = \cos(\pi / N). \tag{13}$$

Since $s_{N-m} = s_m \quad \forall m$, it follows that $\alpha_{N-m} = \alpha_m$ and $\beta_{N-1-m} = \beta_m$. Consequently the two vectors:

$$\boldsymbol{\alpha} = [\alpha_1 \quad \cdots \quad \alpha_{N-1}]^T \tag{14}$$

$$\boldsymbol{\beta} = [\beta_1 \quad \cdots \quad \beta_{N-2}]^T \tag{15}$$

are even symmetric, i.e.

$$\boldsymbol{\alpha} = \mathbf{J}_{\mathbf{N-1}} \boldsymbol{\alpha} \quad , \quad \boldsymbol{\beta} = \mathbf{J}_{\mathbf{N-2}} \boldsymbol{\beta} \tag{16}$$

where $\mathbf{J_m}$ is the contra-identity matrix of order m. An examination of (9) in the light of (16) shows that $\mathbf{T_2}$ can be expressed as

$$\mathbf{T_2} = [\mathbf{h_1} \quad \mathbf{h_2} \quad \mathbf{h_3} \quad \cdots \quad \mathbf{J_{N-1}h_3} \quad \mathbf{J_{N-1}h_2} \quad \mathbf{J_{N-1}h_1}] \tag{17}$$

where $\mathbf{h_m} \quad , m = 1, \cdots N-1$ are the columns of $\mathbf{T_2}$. Consequently $\mathbf{T_2}$ can be compactly expressed as

$$\mathbf{T_2} = \begin{cases} [\mathbf{H} \quad \mathbf{J_{N-1}HJ_{0.5(N-1)}}] & \text{if N is odd} \\ [\mathbf{H} \quad \mathbf{h_{0.5N}} \quad \mathbf{J_{N-1}HJ_{0.5N-1}}] & \text{if N is even} \end{cases} \tag{18}$$

where

$$\mathbf{H} = \begin{cases} [\mathbf{h_1} \quad \mathbf{h_2} \quad \cdots \quad \mathbf{h_{0.5(N-1)}}] & \text{if N is odd} \\ [\mathbf{h_1} \quad \mathbf{h_2} \quad \cdots \quad \mathbf{h_{0.5N-1}}] & \text{if N is even} \end{cases} \tag{19}$$

One should observe that $\mathbf{h_{0.5N}} = \mathbf{J_{N-1}h_{0.5N}}$ if N is even.

*Lemma 3*: Matrix $\mathbf{T_2}$ is centrosymmetric, i.e.,

$$\mathbf{J_{N-1}T_2J_{N-1}} = \mathbf{T_2}. \tag{20}$$

### 3.2. Properties of the Eigenvectors and Eigenvalues

Since matrix $\mathbf{T}$ is real and symmetric, all its eigenvalues are real and one can select its eigenvectors to be real. The same applies to matrix $\mathbf{T_2}$. Equations (9), (11) and (12) show that all elements on the first upper (or lower) diagonal of $\mathbf{T_2}$ are nonzero, i.e. $\mathbf{T_2}$ is an unreduced matrix. Consequently all eigenvalues of $\mathbf{T_2}$ are distinct [24]. It follows that any complete set of (N-1) eigenvectors of $\mathbf{T_2}$ is always orthogonal. Actually it can be taken to be orthonormal.

*Lemma 4*: All unity norm real eigenvectors of matrix $\mathbf{T_2}$ are either *even or odd symmetric*, i.e. if $\mathbf{x}$ is an eigenvector of $\mathbf{T_2}$, then

$$\mathbf{Jx} = \pm\,\mathbf{x}\,. \tag{21}$$

Proof: From $\mathbf{T_2 x} = \lambda\,\mathbf{x}$ and the fact that $\mathbf{J}^2 = \mathbf{I}$ one gets $(\mathbf{JT_2J})(\mathbf{Jx}) = \lambda(\mathbf{Jx})$ and by virtue of (20) one gets $\mathbf{T_2}(\mathbf{Jx}) = \lambda\,(\mathbf{Jx})$. The distinctness of the eigenvalues of $\mathbf{T_2}$ implies that $\mathbf{Jx} = \gamma\,\mathbf{x}$ where $\gamma$ is a real nonzero scalar. Since the eigenvectors $\mathbf{x}$ and $\mathbf{Jx}$ are unity norm, one concludes that $\gamma = \pm 1$ and (21) follows immediately.

(Q.E.D.)

The block diagonal structure of matrix $\mathbf{T}$ defined by (8) and the scalar nature of $T_1$ imply that if $\mathbf{M_2}$ is a modal matrix of $\mathbf{T_2}$, then the corresponding modal matrix $\mathbf{M}^{(1)}$ of $\mathbf{T}$ is given by

$$\mathbf{M}^{(1)} = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{M_2} \end{pmatrix}. \tag{22}$$

Obviously, the first column of $\mathbf{M}^{(1)}$ is circularly even and the remaining columns are either *circularly* even or odd depending on the corresponding columns of $\mathbf{M_2}$ being even or odd symmetric respectively. The above equation also shows that the unitarity of $\mathbf{M_2}$ implies the unitarity of $\mathbf{M}^{(1)}$.

*Lemma 5*: The eigenvalues of matrix $\mathbf{T}$ lie in the half closed interval $[0,4)$ for all finite values of the order N.

Proof: See Appendix A.

### 3.3. Eigenvectors Pertaining to the Repeated Eigenvalue

The following fact is taken from [6]:

**Fact**: $\lambda = 0$ is the only repeated eigenvalue of matrix[6] $\mathbf{T}$. It has multiplicity 2 and the two corresponding orthonormal eigenvectors of **T** can be simply taken as

$$\mathbf{x_1} = \begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix} \quad \text{and} \quad \mathbf{x_2} = \begin{pmatrix} 0 \\ \dfrac{1}{\sqrt{N-1}}\boldsymbol{\mu} \end{pmatrix} \tag{23}$$

where $\boldsymbol{\mu}$ is the summing vector (3) of dimension (N-1). (Actually $\lambda = 0$ is an eigenvalue of both $T_1$ and $\mathbf{T_2}$ in (8)).

*Unfortunately Grünbaum [6] mistakenly took* $\mathbf{x_1}$ *and* $\mathbf{x_2}$ *as eigenvectors of the DFT matrix* ***F*** of (2) which commutes with **T** although it is straightforward to show that $\mathbf{x_1}$ and $\mathbf{x_2}$ are *not* eigenvectors of **F**. The objective now is to find the two eigenvectors common to both $\mathbf{T}$ and **F** which correspond to

[6] Denoting the two **T** matrices in [6] by $\mathbf{T_a}$ and $\mathbf{T_b}$ where $\mathbf{T_b} = 2\sin^2(\pi/N)\cos(\pi/N)(\mathbf{T_a} - \mathbf{I})$, it is obvious that $\mathbf{T_a}$ and $\mathbf{T_b}$ have the same eigenvectors and that the eigenvalue $\lambda = 1$ of $\mathbf{T_a}$ corresponds to the eigenvalue $\lambda = 0$ of $\mathbf{T_b}$. In the present paper matrix **T** of (8) is actually matrix $(-\mathbf{T_b})$ in [6].

the eigenvalue $\lambda = 0$ of $\mathbf{T}$. The approach to be followed is that delineated in [25, pp. 52-54] and is summarized in the following steps:

1. Define matrix $\mathbf{X}$ as

$$\mathbf{X} = [\mathbf{x}_1 \quad \mathbf{x}_2]. \tag{24}$$

2. Find matrix $\mathbf{Y}$ such that

$$\mathbf{FX} = \mathbf{XY}. \tag{25}$$

3. Find the modal decomposition of the square matrix $\mathbf{Y}$ of order 2 as[7]

$$\mathbf{Y} = \mathbf{Z}\begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}\mathbf{Z}^{-1}. \tag{26}$$

4. Compute $\mathbf{XZ}$; the two eigenvectors common to both $\mathbf{T}$ and $\mathbf{F}$ and pertaining to the repeated eigenvalue $\lambda = 0$ of $\mathbf{T}$ are given by the columns of $\mathbf{XZ}$ and the two corresponding eigenvalues of $\mathbf{F}$ are $\mu_1$ and $\mu_2$.

From (23), one gets

$$\mathbf{X} = \begin{pmatrix} 1 & 0 \\ \mathbf{0} & \dfrac{1}{\sqrt{N-1}}\boldsymbol{\mu} \end{pmatrix}. \tag{27}$$

By exploiting the orthonormality of the columns of $\mathbf{X}$, one can directly solve (25) for $\mathbf{Y}$ to get

$$\mathbf{Y} = \mathbf{X}^H \mathbf{FX}. \tag{28}$$

By substituting (2) and (27) in the above formula, one gets

$$\mathbf{Y} = \frac{1}{\sqrt{N}}\begin{pmatrix} 1 & \sqrt{N-1} \\ \sqrt{N-1} & \dfrac{1}{(N-1)}\boldsymbol{\mu}^T\mathbf{G}\boldsymbol{\mu} \end{pmatrix}. \tag{29}$$

Using (3) and (4), one obtains

$$\boldsymbol{\mu}^T\mathbf{G}\boldsymbol{\mu} = \sum_{m=1}^{N-1}\sum_{n=1}^{N-1} W^{mn}. \tag{30}$$

Using (1), it can be shown that

$$\sum_{n=1}^{N-1} W^{mn} = -1 \quad , \quad m = 1, \cdots, N-1. \tag{31}$$

By virtue of the above two equations, (29) reduces to

$$\mathbf{Y} = \frac{1}{\sqrt{N}}\begin{pmatrix} 1 & \sqrt{N-1} \\ \sqrt{N-1} & -1 \end{pmatrix}. \tag{32}$$

The two eigenvalues of the above real symmetric matrix are $\mu_1 = 1$ and $\mu_2 = -1$ and the corresponding two orthogonal eigenvectors are

[7] The scalars $\mu_1$ and $\mu_2$ in (26) should not be confused with vector $\boldsymbol{\mu}$ in (3).

$$\mathbf{z_1} = \begin{pmatrix} \sqrt{N}+1 \\ \sqrt{N}-1 \end{pmatrix} \quad , \quad \mathbf{z}_2 = \begin{pmatrix} \sqrt{N}-1 \\ -\sqrt{N-1} \end{pmatrix}. \tag{33}$$

Forming matrix $\mathbf{Z} = (\mathbf{z_1} \quad \mathbf{z_2})$ and using (27) and (33), one gets

$$\mathbf{XZ} = \begin{pmatrix} \sqrt{N}+1 & \sqrt{N}-1 \\ \boldsymbol{\mu} & -\boldsymbol{\mu} \end{pmatrix}. \tag{34}$$

The orthonormality of the columns of **X** and the orthogonality of the columns of **Z** imply the orthogonality of the columns of $\mathbf{XZ}$. By normalizing the two columns of (34) one gets the two orthonormal eigenvectors common to both **T** and **F** and corresponding to the repeated eigenvalue $\lambda = 0$ of **T** and the two eigenvalues $\mu_1 = 1$ and $\mu_2 = -1$ of **F**. They are given respectively by:

$$\mathbf{w_1} = \frac{1}{\sqrt{2N+2\sqrt{N}}} \begin{pmatrix} \sqrt{N}+1 \\ \boldsymbol{\mu} \end{pmatrix} \quad , \quad \mathbf{w}_2 = \frac{1}{\sqrt{2N-2\sqrt{N}}} \begin{pmatrix} \sqrt{N}-1 \\ -\boldsymbol{\mu} \end{pmatrix}. \tag{35}$$

As expected, one should notice that these two eigenvectors of **F** are both circularly even since they correspond to the real eigenvalues $\pm 1$ [2]. It remains to mention that eigenvectors of **T** corresponding to its simple eigenvalues (all eigenvalues other than $\lambda = 0$) are eigenvectors of **F**.

### 3.4. Ordering the Eigenvectors According to the Number of Zero Crossings

Let $p_m(\lambda)$ be the characteristic polynomial of the m^th^ principal submatrix of $\mathbf{T_2}$ of (9), i.e.

$$p_0(\lambda) = 1 \tag{36}$$

$$p_1(\lambda) = (\alpha_1 - \lambda) \tag{37}$$

$$p_2(\lambda) = (\alpha_1 - \lambda)(\alpha_2 - \lambda) - \beta_1^2 \tag{38}$$

It can be shown that these polynomials satisfy the three-term recurrence relation [25, p. 300]:

$$p_m(\lambda) = (\alpha_m - \lambda) p_{m-1}(\lambda) - \beta_{m-1}^2 p_{m-2}(\lambda) \quad , m = 2,3,\cdots,N-1. \tag{39}$$

It can also be shown that the eigenvector of $\mathbf{T_2}$ corresponding to the eigenvalue $\lambda_i$ is given by [25, p. 316]:

$$\mathbf{v}(\lambda_i) = \begin{bmatrix} 1 & -\frac{p_1(\lambda_i)}{\beta_1} & \frac{p_2(\lambda_i)}{\beta_1\beta_2} & \cdots & (-1)^{N-2} \frac{p_{N-2}(\lambda_i)}{\beta_1\beta_2\cdots\beta_{N-2}} \end{bmatrix}^T \quad , i = 1,\cdots,N-1. \tag{40}$$

Since (11) and (12) imply that $\beta_i < 0, i = 1,\cdots,N-2$, the above equation implies that the signs of the elements of the eigenvector $\mathbf{v}(\lambda_i)$ are identical to those of the elements of the vector:

$$[p_0(\lambda_i) \quad p_1(\lambda_i) \quad \cdots \quad p_{N-2}(\lambda_i)]^T. \tag{41}$$

In order to state the Sturm sequence property of the polynomials $p_m(\lambda), m = 0,\cdots,N-1$, one starts by introducing the integer valued function $s(\lambda)$ defined as the number of agreements in sign of consecutive members of the sequence $p_m(\lambda), m = 0,\cdots,N-1$. If the value of some member $p_m(\lambda) = 0$ its sign will be chosen to be opposite to that of $p_{m-1}(\lambda)$.

**Fact**: The number of the eigenvalues of the unreduced real symmetric tridiagonal matrix $\mathbf{T_2}$ of (9) that are greater than $\lambda = a$ is given by the integer valued function $s(a)$. (See [25, p. 300] and [26, p. 534]).

Let the eigenvalues of $\mathbf{T_2}$ be arranged in descending order as $\lambda_1 > \lambda_2 > \cdots > \lambda_{N-1}$ where $\lambda_{N-1} = 0$. Since the zeros of $p_m(\lambda)$ separate those of $p_{m+1}(\lambda)$ in the strict sense [25], one can use (41) to show that

$$s(\lambda_i) = i-1 \quad , i = 1,\cdots,N-1 . \tag{42}$$

Since $p_{N-1}(\lambda_i) = 0$ and consequently the sign of $p_{N-1}(\lambda_i)$ will be opposite to that of $p_{N-2}(\lambda_i)$, one finds that all sign agreements in the sequence $p_m(\lambda_i), m = 0,\cdots,N-1$ will occur in the subsequence $p_m(\lambda_i), m = 0,\cdots,N-2$ forming the elements of vector (41).

For any vector $\mathbf{x}$ whose elements are $x_m$, a zero crossing occurs at $m$ if $x_m x_{m+1} < 0$ [14]. Consequently the number of zero crossings in vector (41) will be given by:

$$z(\lambda_i) = (N-2) - s(\lambda_i) \quad , i = 1,\cdots,N-1 . \tag{43}$$

From (42) and (43), one gets

$$z(\lambda_i) = (N-1) - i \quad , i = 1,\cdots,N-1 . \tag{44}$$

The modal matrix of $\mathbf{T_2}$ corresponding to its eigenvalues arranged in descending order is

$$\mathbf{M_2} = [\mathbf{v}(\lambda_1) \quad \mathbf{v}(\lambda_2) \quad \cdots \quad \mathbf{v}(\lambda_{N-1})]. \tag{45}$$

The corresponding modal matrix of **T** given by (22) will be

$$\mathbf{M}^{(1)} = [\mathbf{u_1} \quad \mathbf{u}_2 \quad \cdots \quad \mathbf{u}_N] \tag{46}$$

where $\mathbf{u_1} = \mathbf{e_1}$ (the first elementary column vector) and

$$\mathbf{u_m} = \begin{pmatrix} 0 \\ \mathbf{v}(\lambda_{m-1}) \end{pmatrix} \quad , m = 2,\cdots,N \tag{47}$$

with the corresponding diagonal matrix of eigenvalues given by

$$\mathbf{\Lambda}^{(1)} = Diag\{0,\lambda_1,\lambda_2,\cdots,\lambda_{N-1}\}. \tag{48}$$

Since the first and last diagonal elements of $\mathbf{\Lambda}^{(1)}$ represent the only repeated eigenvalue of **T**, the eigenvectors $\mathbf{u_1}$ and $\mathbf{u}_N$ of $\mathbf{M}^{(1)}$ should be replaced by the eigenvectors $\mathbf{w_1}$ and $\mathbf{w_2}$ of (35) in order to get the following common modal matrix of **T** and **F**:

$$\mathbf{M}^{(2)} = [\mathbf{w_1} \quad \mathbf{u_2} \quad \mathbf{u_3} \quad \cdots \quad \mathbf{u_{N-1}} \quad \mathbf{w_2}] \tag{49}$$

and the corresponding diagonal matrix of the eigenvalues of **T** will be $\mathbf{\Lambda}^{(2)} = \mathbf{\Lambda}^{(1)}$. In order to arrange all the eigenvalues of **T** in descending order, one has to define:

$$\mathbf{\Lambda}^{(3)} = Diag\{\lambda_1, \lambda_2, \cdots, \lambda_{N-1}, \lambda_N\} \tag{50}$$

where $\lambda_{N-1} = \lambda_N = 0$ and the corresponding modal matrix is

$$\mathbf{M}^{(3)} = [\mathbf{u_2} \quad \mathbf{u_3} \quad \cdots \quad \mathbf{u_{N-1}} \quad \mathbf{w_2} \quad \mathbf{w_1}]. \tag{51}$$

The number of zero crossings in the first $(N-2)$ columns of $\mathbf{M}^{(3)}$ - as evaluated by (44) - will decrease monotonically from $(N-2)$ to 1. The number of zero crossings in $\mathbf{w_2}$ and $\mathbf{w_1}$ - as can be seen from (35) - are 1 and 0 respectively. In order to arrange the eigenvectors according to the number of their zero crossings in ascending order, one flips the columns of (51) to get the final orthonormal modal matrix common to both **T** and **F** as

$$\mathbf{M} = [\mathbf{w_1} \quad \mathbf{w_2} \quad \mathbf{u_{N-1}} \quad \mathbf{u_{N-2}} \quad \cdots \quad \mathbf{u_3} \quad \mathbf{u_2}]. \tag{52}$$

The corresponding diagonal matrix of the eigenvalues of **T** is:

$$\mathbf{\Lambda_T} = Diag\{\lambda_N, \lambda_{N-1}, \cdots, \lambda_1\} \tag{53}$$

and the corresponding diagonal matrix of the eigenvalues of **F** is[8]:

$$\mathbf{\Lambda_F} = Diag\{\mu_1, \mu_2, \cdots, \mu_N\} \tag{54}$$

where $\mu_1 = 1$ and $\mu_2 = -1$.

The philosophy behind arranging the eigenvectors of **F** according to the number of their zero crossings in ascending order is to have an analogy between those eigenvectors and the Hermite-Gaussian functions.

The algorithm for generating the desired unitary modal matrix of **F** – based on the eigendecomposition of matrix **T** - can be summarized in the following steps:

1) Find the modal matrix $\mathbf{M_2}$ (given by (45)) of matrix $\mathbf{T_2}$ defined by (9).
2) Compute the eigenvectors $\mathbf{u_m}$ , $m = 2, \cdots, N-1$ using (47).
3) Compute the eigenvectors $\mathbf{w_1}$ and $\mathbf{w}_2$ using (35).
4) Form the common modal matrix $\mathbf{M}$ according to (52) where the eigenvectors correspond to the eigenvalues of **T** (expressed by (53)) arranged in ascending order.
5) Knowing the modal matrix of **F** find the corresponding diagonal matrix $\Lambda_\mathbf{F}$ of the eigenvalues.

## 4. A Modal Decomposition of Dickinson-Steiglitz Nearly Tridiagonal Matrix

The nearly tridiagonal matrix **S** introduced by Dickinson and Steiglitz [12] and explored by Candan et al. [13] will be first expressed in partitioned form. An involutary matrix **P** will be employed for defining a similarity transformation to be applied to matrix **S** in order to reduce it to a tridiagonal

[8] The elements $\mu_1, \mu_2, \cdots, \mu_N$ of the diagonal matrix $\Lambda_\mathbf{F}$ in (54) should not be confused with the elements of the summing vector $\boldsymbol{\mu}$ in (3).

form. More specifically the similarity transformed matrix will be rigorously proved to have a 2 x 2 block diagonal form where the two diagonal blocks are unreduced tridiagonal submatrices whose elements will be explicitly derived – which was not done in [13] – in order to circumvent the need for performing the two matrix multiplications involved in the definition of the similarity transformation. The emergent modal matrix of **S** will be rigorously proved to be a modal matrix of the DFT matrix **F** irrespective of the multiplicities of the eigenvalues of **S**.

### 4.1. Partitioned Form of Matrix S

A nearly tridiagonal matrix **S** is defined by:

$$\mathbf{S} = \begin{bmatrix} s_1 & 1 & & & & & 1 \\ 1 & s_2 & 1 & & & & \\ & 1 & \ddots & \ddots & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & 1 & \\ & & & & 1 & s_{N-1} & 1 \\ 1 & & & & & 1 & s_N \end{bmatrix} . \tag{55}$$

The diagonal elements of **S** are given by [13]:

$$s_n = 2\cos\left(\frac{2\pi}{N}(n-1)\right) - 4 \qquad , n = 1, \cdots, N . \tag{56}$$

It is straightforward to show that:

$$s_{N+2-n} = s_n \qquad , n = 2, \cdots, N . \tag{57}$$

Consequently the vector $\mathbf{s}$ defined as: $\mathbf{s} = \begin{bmatrix} s_1 & s_2 & \cdots & s_N \end{bmatrix}^T$ is circularly even. Let the columns of matrix $\mathbf{S}$ be denoted by $\mathbf{c_n}$ , $n = 1, \cdots, N$ , i.e.

$$\begin{aligned} \mathbf{S} &= \begin{bmatrix} \mathbf{c_1} & \mathbf{c_2} & \cdots & \mathbf{c_N} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{c_1} & \mathbf{c_2} & \mathbf{c_3} & \cdots & \mathbf{\Gamma c_3} & \mathbf{\Gamma c_2} \end{bmatrix} \end{aligned} \tag{58}$$

where the second form is obtained by examining (55) in the light of condition (57) and where the circular flip property of matrix $\mathbf{\Gamma}$ defined by (6) is utilized. It follows that $\mathbf{S}$ can be compactly expressed as:

$$\mathbf{S} = \begin{cases} \begin{bmatrix} \mathbf{c_1} & \mathbf{C} & \mathbf{\Gamma C J} \end{bmatrix} & \text{for odd N} \\ \begin{bmatrix} \mathbf{c_1} & \mathbf{C} & \mathbf{c_v} & \mathbf{\Gamma C J} \end{bmatrix} & \text{for even N} \end{cases} \tag{59}$$

where matrix $\mathbf{C}$ is defined by:

$$\mathbf{C} = \begin{cases} \begin{bmatrix} \mathbf{c_2} & \mathbf{c_3} & \cdots & \mathbf{c}_\nu \end{bmatrix} & \text{for odd N} \\ \begin{bmatrix} \mathbf{c_2} & \mathbf{c_3} & \cdots & \mathbf{c_{\nu-1}} \end{bmatrix} & \text{for even N} \end{cases} . \tag{60}$$

and[9]

$$\nu = \lfloor 0.5N \rfloor + 1 . \tag{61}$$

### 4.2. An Involutary Matrix P

One defines an elementary matrix $\mathbf{P}$ of order N such that its first $\nu$ columns are circularly even and its last $(N-\nu)$ columns are circularly odd. More specifically, matrix $\mathbf{P}$ is defined as:

$$\mathbf{P} = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \mathbf{I} & \mathbf{J} \\ 0 & \mathbf{J} & -\mathbf{I} \end{bmatrix} \text{ for odd N and } \mathbf{P} = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \mathbf{I} & 0 & \mathbf{J} \\ 0 & 0 & \sqrt{2} & 0 \\ 0 & \mathbf{J} & 0 & -\mathbf{I} \end{bmatrix} \text{ for even N} \tag{62}$$

where $\mathbf{I}$ and $\mathbf{J}$ are of order $(N-\nu)$. Matrix $\mathbf{P}$ is symmetric and it can be shown to be unitary. Therefore $\mathbf{P} = \mathbf{P}^T = \mathbf{P}^{-1}$ implying that $\mathbf{P}$ is involutary.

In preparation for finding the effect of premultiplying a vector by P, an arbitrary vector $\mathbf{x}$ of dimension N will be expressed in partitioned form as:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \mathbf{x_a} \\ \mathbf{x_b} \end{bmatrix} \text{ for odd N and } \mathbf{x} = \begin{bmatrix} x_1 \\ \mathbf{x_a} \\ x_\nu \\ \mathbf{x_b} \end{bmatrix} \text{ for even N} \tag{63}$$

where the subvectors $\mathbf{x_a}$ and $\mathbf{x_b}$ are of dimension $(N-\nu)$. Define $\mathbf{y} = \mathbf{Px}$ and use (62) and (63) to get:

$$\mathbf{y} = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2}x_1 \\ \mathbf{x_a} + \mathbf{Jx_b} \\ \mathbf{Jx_a} - \mathbf{x_b} \end{bmatrix} \text{ for odd N and } \mathbf{y} = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2}x_1 \\ \mathbf{x_a} + \mathbf{Jx_b} \\ \sqrt{2}x_\nu \\ \mathbf{Jx_a} - \mathbf{x_b} \end{bmatrix} \text{ for even N} . \tag{64}$$

*Lemma 6*: Let $\mathbf{y} = \mathbf{Px}$ where $\mathbf{x}$ is expressed by (63).

---

[9]The symbol $\lfloor b \rfloor$ denotes the largest integer less than or equal to $b$.

a) If vector $\mathbf{x}$ is circularly even, then:

$$\mathbf{y} = \frac{1}{\sqrt{2}}\begin{bmatrix}\sqrt{2}x_1 \\ 2\mathbf{x_a} \\ \mathbf{0}\end{bmatrix} \text{for odd N} \quad \text{and} \quad \mathbf{y} = \frac{1}{\sqrt{2}}\begin{bmatrix}\sqrt{2}x_1 \\ 2\mathbf{x_a} \\ \sqrt{2}x_\nu \\ \mathbf{0}\end{bmatrix} \text{for even N} . \tag{65}$$

b) If vector $\mathbf{x}$ is circularly odd, then:

$$\mathbf{y} = \frac{1}{\sqrt{2}}\begin{bmatrix}0 \\ \mathbf{0} \\ -2\mathbf{x_b}\end{bmatrix} \text{for odd N} \quad \text{and} \quad \mathbf{y} = \frac{1}{\sqrt{2}}\begin{bmatrix}0 \\ \mathbf{0} \\ 0 \\ -2\mathbf{x_b}\end{bmatrix} \text{for even N} . \tag{66}$$

c) If the last $(N-\nu)$ elements of $\mathbf{x}$ are zero, then $\mathbf{y}$ is circularly even.

d) If the first $\nu$ elements of $\mathbf{x}$ are zero, then $\mathbf{y}$ is circularly odd.

### 4.3. Tridiagonalization of S by a Similarity Transformation

*Lemma 7*: For any matrix[10] $\mathbf{S}$ of the form of (55) whose diagonal elements form a circularly even vector and for matrix $\mathbf{P}$ defined by (62), one gets:

$$\mathbf{PSP}^{-1} = \begin{bmatrix}\mathbf{EV} & 0 \\ 0 & \mathbf{OD}\end{bmatrix} \tag{67}$$

where $\mathbf{EV}$ and $\mathbf{OD}$ are square matrices of order $\nu$ and $(N-\nu)$ respectively.

Proof:

*Case a: N is odd*

Using (62) and (59) and recalling that $\mathbf{P}$ is involutary, one gets:

$$\mathbf{SP}^{-1} = \frac{1}{\sqrt{2}}\left[\sqrt{2}\mathbf{c_1} \quad (\mathbf{I}+\mathbf{\Gamma})\mathbf{C} \quad (\mathbf{I}-\mathbf{\Gamma})\mathbf{CJ}\right]. \tag{68}$$

Premultiplying by matrix $\mathbf{\Gamma}$ defined by (6) and using (7) and the fact that $\mathbf{\Gamma c_1} = \mathbf{c_1}$, one obtains:

[10] This lemma is general since it does not require that the diagonal elements of $\mathbf{S}$ be those of (56).

$$\mathbf{\Gamma SP}^{-1} = \frac{1}{\sqrt{2}}\left[\sqrt{2}\mathbf{c_1} \quad (\mathbf{I}+\mathbf{\Gamma})\mathbf{C} \quad -(\mathbf{I}-\mathbf{\Gamma})\mathbf{CJ}\right]. \tag{69}$$

By comparing (68) and (69), one concludes that the first $\nu$ columns of $\mathbf{SP}^{-1}$ are circularly even and the last $(N-\nu)$ columns are circularly odd.

*Case b: N is even*

Using (62) and (59), one gets:

$$\mathbf{SP}^{-1} = \frac{1}{\sqrt{2}}\left[\sqrt{2}\mathbf{c_1} \quad (\mathbf{I}+\mathbf{\Gamma})\mathbf{C} \quad \sqrt{2}\mathbf{c}_\nu \quad (\mathbf{I}-\mathbf{\Gamma})\mathbf{CJ}\right]. \tag{70}$$

Premultiplying by $\mathbf{\Gamma}$ and using the fact that $\mathbf{\Gamma c_1} = \mathbf{c_1}$ and $\mathbf{\Gamma c}_\nu = \mathbf{c}_\nu$ for N even , one obtains:

$$\mathbf{\Gamma SP}^{-1} = \frac{1}{\sqrt{2}}\left[\sqrt{2}\mathbf{c_1} \quad (\mathbf{I}+\mathbf{\Gamma})\mathbf{C} \quad \sqrt{2}\mathbf{c}_\nu \quad -(\mathbf{I}-\mathbf{\Gamma})\mathbf{CJ}\right]. \tag{71}$$

By comparing (70) and (71), one concludes that the first $\nu$ columns of $\mathbf{SP}^{-1}$ are circularly even and the last $(N-\nu)$ columns are circularly odd.

The same conclusion has been reached in cases a and b. Therefore premultiplying $\mathbf{SP}^{-1}$ by $\mathbf{P}$ will zero the last $(N-\nu)$ elements of the first $\nu$ columns of $\mathbf{SP}^{-1}$ by virtue of Lemma 6a and will also zero the first $\nu$ elements of the last $(N-\nu)$ columns of $\mathbf{SP}^{-1}$ by virtue of Lemma 6b. Thus the validity of (67) has been established.

(Q.E.D.)

*Theorem 1*: For any matrix $\mathbf{S}$ of the form of (55) whose diagonal elements form a circularly even vector[11], the two matrices $\mathbf{EV}$ and $\mathbf{OD}$ appearing in (67) are symmetric tridiagonal and are given explicitly by:

[11] The theorem is general since it does not require that the diagonal elements of $\mathbf{S}$ be those of (56).

$$
\mathbf{EV} = \begin{bmatrix} s_1 & \sqrt{2} & & & & & & \\ \sqrt{2} & s_2 & 1 & & & & & \\ & 1 & s_3 & 1 & & & & \\ & & 1 & \ddots & \ddots & & & \\ & & & \ddots & \ddots & 1 & & \\ & & & & 1 & s_{\nu-2} & 1 & \\ & & & & & 1 & s_{\nu-1} & \gamma \\ & & & & & & \gamma & s_\nu + \delta \end{bmatrix}, \tag{72}
$$

$$
\mathbf{OD} = \begin{bmatrix} s_{\nu+1} - \delta & 1 & & & & \\ 1 & s_{\nu+2} & 1 & & & \\ & 1 & \ddots & \ddots & & \\ & & \ddots & \ddots & 1 & \\ & & & 1 & s_{N-1} & 1 \\ & & & & 1 & s_N \end{bmatrix} \tag{73}
$$

where

$$
\gamma = \begin{cases} 1 & \text{for odd N} \\ \sqrt{2} & \text{for even N} \end{cases} \quad \text{and} \quad \delta = \begin{cases} 1 & \text{for odd N} \\ 0 & \text{for even N} \end{cases}. \tag{74}
$$

Proof:

*Case a: N is odd*

Based on (62) and (55) and by virtue of (64) and Lemma 6, one gets:

$$
\mathbf{PS} = \frac{1}{\sqrt{2}} \left[ \begin{array}{ccccc|ccccc} \sqrt{2}s_1 & \sqrt{2} & & & & & & & & \sqrt{2} \\ 2 & s_2 & 1 & & & & & & 1 & s_N \\ & 1 & s_3 & 1 & & & & 1 & s_{N-1} & 1 \\ & & 1 & s_{\nu-1} & 1 & 1 & s_{\nu+2} & 1 & & \\ & & & 1 & s_\nu + 1 & s_{\nu+1} + 1 & 1 & & & \\ - & - & - & - & - & - & - & - & - & \\ & & & 1 & s_\nu - 1 & -(s_{\nu+1} - 1) & -1 & & & \\ & & 1 & s_{\nu-1} & 1 & -1 & -s_{\nu+2} & -1 & & \\ & 1 & s_3 & 1 & & & -1 & -s_{N-1} & -1 & \\ 0 & s_2 & 1 & & & & & -1 & -s_N & \end{array} \right]. \tag{75}
$$

Postmultiplying (75) by $\mathbf{P}^{-1} = \mathbf{P}$ and utilizing the key fact (57) together with the transposed form of (65) and (66), one obtains:

$$\mathbf{PSP}^{-1} = \left[\begin{array}{ccccc|cccc}
s_1 & \sqrt{2} & & & & & & & \\
\sqrt{2} & s_2 & 1 & & & & & & \\
 & 1 & s_3 & 1 & & & & & \\
 & & 1 & s_{\nu-1} & 1 & & & & \\
 & & & 1 & s_\nu + 1 & & & & \\
\hline
 & & & & & s_{\nu+1} - 1 & 1 & & \\
 & & & & & 1 & s_{\nu+2} & 1 & \\
 & & & & & & 1 & s_{N-1} & 1 \\
 & & & & & & & 1 & s_N
\end{array}\right]. \tag{76}$$

Therefore the matrices $\mathbf{EV}$ and $\mathbf{OD}$ are given respectively by (72) and (73) with $\gamma = 1$ and $\delta = 1$ in agreement with (74).

*Case b: N is even*

Based on (62) and (55) and by virtue of (64) and Lemma 6, one gets:

$$\mathbf{PS} = \frac{1}{\sqrt{2}}\left[\begin{array}{cccccc|cccc}
\sqrt{2}s_1 & \sqrt{2} & & & & & & & & \sqrt{2} \\
2 & s_2 & 1 & & & & & & 1 & s_N \\
 & 1 & s_3 & 1 & & & & 1 & s_{N-1} & 1 \\
 & & 1 & s_{\nu-2} & 1 & & 1 & s_{\nu+2} & 1 & \\
 & & & 1 & s_{\nu-1} & 2 & s_{\nu+1} & 1 & & \\
 & & & & \sqrt{2} & \sqrt{2}s_\nu & \sqrt{2} & & & \\
\hline
 & & & 1 & s_{\nu-1} & 0 & -s_{\nu+1} & -1 & & \\
 & & 1 & s_{\nu-2} & 1 & & -1 & -s_{\nu+2} & -1 & \\
 & 1 & s_3 & 1 & & & & -1 & -s_{N-1} & -1 \\
0 & s_2 & 1 & & & & & & -1 & -s_N
\end{array}\right]. \tag{77}$$

Postmultiplying (77) by $\mathbf{P}^{-1}$ and utilizing (57) together with the transposed form of (65) and (66), one obtains:

$$\mathbf{PSP}^{-1} = \left[\begin{array}{cccccc|cccc}
s_1 & \sqrt{2} & & & & & & & & \\
\sqrt{2} & s_2 & 1 & & & & & & & \\
 & 1 & s_3 & 1 & & & & & & \\
 & & 1 & s_{\nu-2} & 1 & & & & & \\
 & & & 1 & s_{\nu-1} & \sqrt{2} & & & & \\
 & & & & \sqrt{2} & s_\nu & & & & \\
\hline
 & & & & & & s_{\nu+1} & 1 & & \\
 & & & & & & 1 & s_{\nu+2} & 1 & \\
 & & & & & & & 1 & s_{N-1} & 1 \\
 & & & & & & & & 1 & s_N
\end{array}\right]. \tag{78}$$

Therefore the matrices $\mathbf{EV}$ and $\mathbf{OD}$ are given respectively by (72) and (73) with $\gamma = \sqrt{2}$ and $\delta = 0$ in agreement with (74).

(Q.E.D.)

### 4.4. The Common Modal Matrix of S and F

The modal decompositions[12]:

$$\mathbf{EV} = \mathbf{M_1 \Lambda_1 M_1^{-1}} \quad \text{and} \quad \mathbf{OD} = \mathbf{M_2 \Lambda_2 M_2^{-1}} \tag{79}$$

combined with (67) result in:

$$\mathbf{PSP}^{-1} = \mathbf{M\Lambda M}^{-1} \tag{80}$$

where

$$\mathbf{\Lambda} = \begin{pmatrix} \mathbf{\Lambda_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Lambda_2} \end{pmatrix} \quad \text{and} \quad \mathbf{M} = \begin{pmatrix} \mathbf{M_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{M_2} \end{pmatrix}. \tag{81}$$

Consequently the modal decomposition of **S** is:

$$\mathbf{S} = \mathbf{Q\Lambda Q}^{-1} \quad \text{where} \quad \mathbf{Q} = \mathbf{PM}\,. \tag{82}$$

*Lemma 8*: The modal matrix **Q** of matrix **S** given by (82) is always a modal matrix of **F** irrespective of the multiplicities of the eigenvalues of **S**.

Proof: It was shown in [12] that the maximum multiplicity of any eigenvalue $\lambda$ of $\mathbf{S}$ can be 2. If $\lambda$ is a simple eigenvalue of $\mathbf{S}$, then the corresponding eigenvector will also be an eigenvector of $\mathbf{F}$ since $\mathbf{S}$ and $\mathbf{F}$ commute [12]. It remains to consider the case when $\lambda$ is an eigenvalue of $\mathbf{S}$ with multiplicity 2. Since the matrices **EV** and **OD** given by (72) and (73) are unreduced tridiagonal matrices, the eigenvalues of each of them are distinct [24]. Based on (81) and (82) matrix **S** can have an eigenvalue of multiplicity 2 only if one element of $\Lambda_1$ happens to equal one element of $\Lambda_2$. In that case one of the two corresponding eigenvectors of **S** - to be denoted by **u** - will be circularly even and the other one - to be denoted by **v** - will be circularly odd based on Lemma 6(c,d). Since $\mathbf{Su} = \lambda\mathbf{u}$, $\mathbf{Sv} = \lambda\mathbf{v}$ and exploiting the commutativity of $\mathbf{S}$ and $\mathbf{F}$, one gets: $\mathbf{S}(\mathbf{Fu}) = \lambda(\mathbf{Fu})$ and $\mathbf{S}(\mathbf{Fv}) = \lambda(\mathbf{Fv})$. This implies that $\mathbf{Fu}$ and $\mathbf{Fv}$ are eigenvectors of $\mathbf{S}$ corresponding to the same $\lambda$. Consequently they can be expressed as linear combinations of $\mathbf{u}$ and $\mathbf{v}$ as follows:

[12] The matrix $\mathbf{M_2}$ in (79) should not be confused with the matrix $\mathbf{M_2}$ appearing in (22) and (45).

$$\mathbf{Fu} = \alpha_1 \mathbf{u} + \alpha_2 \mathbf{v}\,, \tag{83}$$

$$\mathbf{Fv} = \beta_1 \mathbf{u} + \beta_2 \mathbf{v}\,. \tag{84}$$

Premultiplying the above two equations by $\mathbf{v}^H$ and $\mathbf{u}^H$ respectively and applying Lemma 2, one gets $\alpha_2 = 0$ and $\beta_1 = 0$; and consequently $\mathbf{Fu} = \alpha_1 \mathbf{u}$ and $\mathbf{Fv} = \beta_2 \mathbf{v}$. Therefore $\mathbf{u}$ and $\mathbf{v}$ are also eigenvectors of $\mathbf{F}$.

(Q.E.D.)

The algorithm for generating the desired modal matrix common to both **S** and **F** can be summarized in the following steps:

1) Given matrix **S** of (55), generate the matrices **EV** and **OD** using (72) and (73).
2) Find the modal decomposition of **EV** and **OD** according to (79) where the eigenvalues of each matrix are arranged in descending order. (See [13] for the reason of this arrangement).
3) Form matrix **M** according to (81).
4) Compute the target modal matrix **Q** using (82).

## 5. A Comparative Simulation Study

### 5.1. The T and S Matrix Methods for the Evaluation of the Eigenvectors

The modal matrix of the Grünbaum matrix **T** of order[13] $N = 11$ is computed according to (52) and is given in Table 1 where the bottom two rows list the eigenvalues of **T** and **F** corresponding to the common set of eigenvectors. Since a vector of length N of samples of the Hermite-Gaussian functions of order $m$ is an *approximate* eigenvector of matrix **F** of order N corresponding to the exact eigenvalue $(-j)^m$ as delineated in [4], the eigenvalues of **F** are to be next arranged as $(-j)^{n_m}$, $m = 1, \cdots, N$ where the set of indices $\{n_m\}$ is given by $\{0,1,\cdots N-2, N-1\}$ for odd N and by $\{0,1,\cdots N-2, N\}$ for even N. Rearranging the columns of matrix **M** of (52) to consecutively correspond to repeated cycles of the ordered set $\{1,-j,-1,j\}$ of distinct eigenvalues of matrix **F**, one obtains matrix **V** shown in Table 2 starting from matrix **M** shown in Table 1. Rearranging the diagonal

---

[13] The order $N = 11$ was selected because it is large enough to allow the illustration of the eigenstructure of the matrices and small enough to allow the inclusion of the corresponding modal matrices within the page limits.

elements of matrix $\mathbf{\Lambda_F}$ of (54) accordingly, one gets the diagonal matrix **D**. The final modal decomposition of **F** is given by

$$\mathbf{F} = \mathbf{V}\mathbf{D}\mathbf{V}^H \tag{85}$$

and the kernel matrix of the discrete fractional Fourier transform (DFRFT) of order $a$ (corresponding to an angle of rotation $\alpha = 0.5\pi\, a$ ) – according to the definition proposed in [19,4,5,13,14] – is given by:

$$\mathbf{F}^a \equiv \mathbf{V}\mathbf{D}^a\mathbf{V}^H . \tag{86}$$

The modal matrix **Q** of the nearly tridiagonal matrix **S** of order $N = 11$ is computed according to (82) and is given in Table 3 where the bottom two rows list the eigenvalues of **S** and **F** corresponding to the common set of eigenvectors. Rearranging the columns of **Q** to get matrix **V** in the manner explained above, one gets Table 4.

A desired feature to be sought in developing the DFRFT is to have it resemble its continuous counterpart, namely the FRFT and the way for satisfying this requirement is to make the columns of the unitary modal matrix **V** in (86) as close as possible to samples of the Hermite-Gaussian functions. In order to compare the eigenvectors of the matrices **T** and **S** from that perspective, the Euclidian norms of the error vectors between the columns of matrix **V** and the approximate eigenvectors (i.e. the samples of the Hermite-Gaussian functions) are computed and plotted in Fig. 1 for $N = 256$. It should be mentioned that the $k^{th}$ column of matrix **V** is compared with the Hermite-Gaussian function of order $n_k$ for $k = 1, \cdots, N$ where the set of indices $\{n_k\}$ has been introduced above. Figure 1 clearly shows *the relative merit of the eigenvectors of matrix* ***S*** *compared to those of matrix* ***T***. The reason behind this result is that matrix **S** was generated in [13] in the context of discretizing the second order differential equation satisfied by the Hermite-Gaussian functions while matrix **T** was only weakly shown in [6] to be related to the Hermite operator in the limit as $N \to \infty$. Since samples of the Hermite-Gaussian functions become a poor approximation of the eigenvectors of matrix **F** for large values of N [4,5], the approximation error tends generally to increase with the order k.

The first two columns of Table 5 list the computation time in seconds of the eigenvectors of the two matrices for different values of the order N of the matrix where the computation was carried out on a Pentiuum 4 PC. It is obvious that the **S** matrix method has the *extra merit of being more computationally efficient* than the **T** matrix method. The reason behind this finding is that the **S** matrix method is implemented by decomposing the problem into two subproblems of almost half the size of the original problem as expressed by (67). Since the estimated computation time of an algorithm running on a PC is affected by the unavoidable system related tasks running concurrently in the background, the times listed in Table 5 are reliable mainly for large values of N. The conclusion is that the **S** matrix method takes approximately half the time required by the **T** matrix method.

### 5.2. Other Methods for the Evaluation of the Eigenvectors

In addition to the **T** and **S** matrix methods for the computation of Hermite-Gaussian-like eigenvectors of the DFT matrix **F**, three other methods have been included in the comparative simulation study; namely:

1. The orthogonal procrustes algorithm (OPA) where the eigenvectors of matrix **S** are taken as *initial* ones and *final* superior ones are computed collectively for each eigenspace separately [4].
2. The algorithm based on an alternative implementation of the OPA by the direct utilization of the orthogonal projection matrices on the eigenspaces of matrix **F** *without* having to first find initial orthonormal bases for those spaces [27].
3. The Direct Batch Evaluation by constrained Optimization Algorithm (DBEOA) which is quite distinct algorithmically from – despite being theoretically equivalent to – the OPA [28].

The computation time in seconds of the above three algorithms are given in the last three columns of Table 5 where the above two implementations of the OPA – i.e. the first two methods listed above - are denoted by OPA (1) and OPA (2) respectively. The above three methods were essentially derived using the same minimization criterion and set of constraints and consequently are supposed to produce identical outputs in the absence of roundoff error which is unavoidable. It has been practically found that for values of the order N of matrix **F** as large as 256 there is no noticeable difference in the outputs of the three algorithms. However for larger values of N, the discrepancy between the outputs becomes noticeable. Figure 1 has a plot of the Euclidian norms of the approximation error vectors for the OPA (1) in addition to the **T** and **S** methods for $N = 256$. As expected the OPA (1) has a better degree of approximation than both the **T** and **S** methods.

In order to assess the accuracy of the numerical computation, the orthonormality error matrix defined as $\left(\mathbf{V}^{\mathbf{H}}\mathbf{V} - \mathbf{I}\right)$ - which is theoretically zero – is computed for each of the five methods; namely the **T** and **S** methods together with the above three algorithms. The maximum absolute value of the elements of the orthonormality error matrix as well as its Frobenius norm are calculated and given in Tables 6 and 7 respectively. An examination of these two tables shows that the orthonormality error is negligible for the **T** method, **S** method and OPA (1) for all values of N while for the OPA (2) and DBEOA this error is negligible only for $N \leq 256$. Therefore although the OPA (2) and DBEOA have the merit of being faster than the first three methods listed in Table 5 as evidenced by that table, they do not have the numerical robustness for large values of N enjoyed by the first three methods as testified by Tables 6 and 7. Since section 5.1 has pointed out the superiority of the **S** method to the **T** method, the comparison among the 5 methods can be narrowed down to that between the **S** method and the OPA (1). One should say that the **S** method has the merit of being faster than the OPA (1) as evidenced by Table 5 and the OPA (1) has the advantage of having a lower approximation error[14] than the **S** method as testified by Fig. 1.

### 5.3. The DFRFT

The DFRFT – with its kernel defined by (86) and modal matrix **V** computed using each of the **T** method, **S** method and OPA (1) – of the discrete time unit sample signal $\delta[n]$ is calculated for the set of angles $\alpha = 0.3\,\pi, 0.45\,\pi, 0.48\,\pi$ and $0.5\,\pi$ radians for $N = 65$ and plotted in Figs. 2-4 where the solid and dashed lines represent the real and imaginary parts respectively. The continuous FRFT of the continuous-time unit impulse – selected as a test signal only because of its simplicity - is given by:

$$FRFT[\delta(t)] = K_{\alpha}(0,u) = \sqrt{\frac{1 - j\cot\alpha}{2\pi}}\exp\left[j\left(\frac{u^2}{2}\cot\alpha\right)\right] \quad \text{if } \alpha \text{ is not a multiple of } \pi \tag{87}$$

where $K_{\alpha}(t,u)$ is the kernel of the transform. Figure 5 shows the plot of the above equation - using an increment of $\Delta u = \sqrt{2\pi / N}$ where $N = 65$ - for the same set of angles used for the DFRFT. Since the angle $\alpha = 0.5\pi$ corresponds to the classical transform, namely the Fourier transform in the continuous-time case and the DFT in the discrete-time case, (87) was scaled by the factor $\sqrt{2\pi / N}$ before plotting so that it will coincide with the DFRFT of $\delta[n]$ when $\alpha = 0.5\pi$. By comparing Figs 2-4 with Fig 5 it is obvious that Figs 3 and 4 better resemble Fig 5 than does Fig 2. This observation furnishes more evidence that the **S** method is superior to the **T** method.

It should be emphasized that the DFRFT computed using a modal matrix **V** – evaluated by any of the five methods included in this comparative simulation study – is a chirp fractional transform (CFRT) rather than a weighted fractional transform (WFRT) according to the terminology of [21] because in all cases the kernel of the transform is given by (86) following the fully-fledged definition adopted in [19,4,5,13,14] which is regarded as a CFRT definition.

## 6. Conclusion

An investigation of the eigenstructure of both Grünbaum tridiagonal matrix **T** and Dickinson-Steiglitz nearly tridiagonal matrix **S** – which commute with the DFT matrix **F** – is made with the objective of obtaining Hermite Gaussian like eigenvectors of matrix **F**. For matrix **T**, two orthonormal eigenvectors pertaining to its only repeated eigenvalue of multiplicity two – and common to both **T** and **F** matrices – are *algebraically derived*. For matrix **S** which can be tridiagonalized by a similarity transformation defined in terms of an involutary matrix **P**, *explicit expressions are derived* for the elements of the two tridiagonal submatrices resulting from the transformation in order to circumvent

[14] The approximation error plotted in Fig. 1 should not be confused with the orthonormality error given by Tables 6 and 7.

the need for performing the two matrix multiplications involved in the transformation. Although *matrix **T** seems to have the merit of being tridiagonal* and does not need the tridiagonalization step, the simulation results have shown that the computation time of the eigenvectors of matrix **S** is shorter than that of matrix **T**. More importantly the eigenvectors of matrix **S** have been found to be more Hermite-Gaussian-like than those of matrix **T**. Consequently the discrete fractional Fourier transform developed using the eigenvectors of matrix **S** has been found to better approximate its continuous-time counterpart than that developed using the eigenvectors of matrix **T**.

**Appendix A**

**(Proof of Lemma 5)**

Applying Gerschgorin circle theorem, one concludes that all eigenvalues of $\mathbf{T_2}$ of (9) are contained in the interval $[a,b]$ with

$$a = \underset{1\le i\le N-1}{Min}\left\{\alpha_i - |\beta_i| - |\beta_{i-1}|\right\} \qquad \text{(A-1)}$$

$$b = \underset{1\le i\le N-1}{Max}\left\{\alpha_i + |\beta_i| + |\beta_{i-1}|\right\} \qquad \text{(A-2)}$$

where $\beta_0 = \beta_{N-1} = 0$. (See [26, pp. 500-502 and p. 538] ).

For $2 \le i \le N-2$, (10)-(13) result in:

$$\begin{aligned}\alpha_i - |\beta_i| - |\beta_{i-1}| &= 2\cos(\pi/N)\sin^2(i\pi/N) - \sin(i\pi/N)\sin((i+1)\pi/N) - \sin((i-1)\pi/N)\sin(i\pi/N)\\ &= 2\cos(\pi/N)\sin^2(i\pi/N) - \sin(i\pi/N)[2\sin(i\pi/N)\cos(\pi/N)]\\ &= 0\end{aligned} \qquad \text{(A-3)}$$

For $i = 1$:

$$\begin{aligned}\alpha_1 - |\beta_1| - |\beta_0| &= 2\cos(\pi/N)\sin^2(\pi/N) - \sin(\pi/N)\sin(2\pi/N)\\ &= 0\end{aligned} \qquad \text{(A-4)}$$

For $i = N-1$:

$$\alpha_{N-1} - |\beta_{N-1}| - |\beta_{N-2}| = \alpha_1 - |\beta_0| - |\beta_1| = 0\,. \qquad \text{(A-5)}$$

Therefore the above three equations together with (A-1) imply that $a = 0$.

Similarly one gets:

$$\alpha_i + |\beta_i| + |\beta_{i-1}| = 4\cos(\pi/N)\sin^2(i\pi/N) \quad 2 \le i \le N-2\,. \qquad \text{(A-6)}$$

$$\alpha_1 + |\beta_1| + |\beta_0| = 4\cos(\pi/N)\sin^2(\pi/N)\,. \qquad \text{(A-7)}$$

$$\alpha_{N-1} + |\beta_{N-1}| + |\beta_{N-2}| = \alpha_1 + |\beta_0| + |\beta_1|\,. \qquad \text{(A-8)}$$

The above three equations together with (A-2) imply that:

$$b = 4\cos(\pi/N)\underset{1\le i\le N-1}{\mathrm{Max}}\left\{\sin^2(i\pi/N)\right\} \qquad \text{(A-9)}$$

More specifically

$$b = \begin{cases} 4\cos(\pi / N) & \text{if N is even} \\ 4\cos(\pi / N)\cos^2(\pi /(2N)) & \text{if N is odd and N} \geq 3 \end{cases} \tag{A-10}$$

Therefore for all finite values of N the eigenvalues of $\mathbf{T_2}$ are contained in the half closed interval $[0,4)$. The same applies to matrix **T** of (8) since $T_1$ has only one eigenvalue equal to zero.

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Table 1: The common modal matrix of the matrices **T** and **F** of order $N = 11$.

| Eigenvectors of matrix T | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.8067 | 0.591 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0.1869 | -0.2551 | 0.5636 | 0.5827 | -0.4102 | -0.241 | -0.1177 | -0.0481 | -0.0161 | 0.0042 | -0.0007 |
| 0.1869 | -0.2551 | 0.3464 | 0.075 | 0.3277 | 0.5288 | 0.4936 | 0.3328 | 0.1693 | -0.0637 | 0.0156 |
| 0.1869 | -0.2551 | 0.2157 | -0.1387 | 0.383 | 0.1861 | -0.234 | -0.5084 | -0.4903 | 0.2961 | -0.1075 |
| 0.1869 | -0.2551 | 0.1196 | -0.2389 | 0.2632 | -0.1492 | -0.3973 | -0.1156 | 0.3718 | -0.5543 | 0.3469 |
| 0.1869 | -0.2551 | 0.0385 | -0.2802 | 0.0916 | -0.3247 | -0.1729 | 0.3393 | 0.3039 | 0.3178 | -0.6065 |
| 0.1869 | -0.2551 | -0.0385 | -0.2802 | -0.0916 | -0.3247 | 0.1729 | 0.3393 | -0.3039 | 0.3178 | 0.6065 |
| 0.1869 | -0.2551 | -0.1196 | -0.2389 | -0.2632 | -0.1492 | 0.3973 | -0.1156 | -0.3718 | -0.5543 | -0.3469 |
| 0.1869 | -0.2551 | -0.2157 | -0.1387 | -0.383 | 0.1861 | 0.234 | -0.5084 | 0.4903 | 0.2961 | 0.1075 |
| 0.1869 | -0.2551 | -0.3464 | 0.075 | -0.3277 | 0.5288 | -0.4936 | 0.3328 | -0.1693 | -0.0637 | -0.0156 |
| 0.1869 | -0.2551 | -0.5636 | 0.5827 | 0.4102 | -0.241 | 0.1177 | -0.0481 | 0.0161 | 0.0042 | 0.0007 |
| Eigenvalues of matrix T | | | | | | | | | | |
| 0 | 0 | 0.0587 | 0.1327 | 0.274 | 0.4865 | 0.791 | 1.2072 | 1.7584 | 2.4711 | 3.3749 |
| Eigenvalues of matrix F | | | | | | | | | | |
| 1 | -1 | -j | 1 | j | -1 | -j | 1 | j | -1 | -j |

Table 2: Matrix **V** corresponding to the modal matrix of **T** of order $N = 11$.

| Matrix V corresponding to the eigenvectors of matrix T | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.8067 | 0 | 0.591 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0.1869 | 0.5636 | -0.2551 | -0.4102 | 0.5827 | -0.1177 | -0.241 | -0.0161 | -0.0481 | -0.0007 | 0.0042 |
| 0.1869 | 0.3464 | -0.2551 | 0.3277 | 0.075 | 0.4936 | 0.5288 | 0.1693 | 0.3328 | 0.0156 | -0.0637 |
| 0.1869 | 0.2157 | -0.2551 | 0.383 | -0.1387 | -0.234 | 0.1861 | -0.4903 | -0.5084 | -0.1075 | 0.2961 |
| 0.1869 | 0.1196 | -0.2551 | 0.2632 | -0.2389 | -0.3973 | -0.1492 | 0.3718 | -0.1156 | 0.3469 | -0.5543 |
| 0.1869 | 0.0385 | -0.2551 | 0.0916 | -0.2802 | -0.1729 | -0.3247 | 0.3039 | 0.3393 | -0.6065 | 0.3178 |
| 0.1869 | -0.0385 | -0.2551 | -0.0916 | -0.2802 | 0.1729 | -0.3247 | -0.3039 | 0.3393 | 0.6065 | 0.3178 |
| 0.1869 | -0.1196 | -0.2551 | -0.2632 | -0.2389 | 0.3973 | -0.1492 | -0.3718 | -0.1156 | -0.3469 | -0.5543 |
| 0.1869 | -0.2157 | -0.2551 | -0.383 | -0.1387 | 0.234 | 0.1861 | 0.4903 | -0.5084 | 0.1075 | 0.2961 |
| 0.1869 | -0.3464 | -0.2551 | -0.3277 | 0.075 | -0.4936 | 0.5288 | -0.1693 | 0.3328 | -0.0156 | -0.0637 |
| 0.1869 | -0.5636 | -0.2551 | 0.4102 | 0.5827 | 0.1177 | -0.241 | 0.0161 | -0.0481 | 0.0007 | 0.0042 |
| Eigenvalues of matrix F | | | | | | | | | | |
| 1 | -j | -1 | j | 1 | -j | -1 | j | 1 | -j | -1 |

Table 3: The common modal matrix of the matrices **S** and **F** of order $N = 11$.

| Eigenvectors of matrix S | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.6609 | -0.4994 | 0.4494 | -0.3157 | -0.1097 | -0.0113 | 0 | 0 | 0 | 0 | 0 |
| 0.4854 | 0.0869 | -0.3384 | 0.3391 | 0.1636 | 0.0254 | 0.5343 | -0.4058 | 0.2145 | 0.0616 | -0.0054 |
| 0.2061 | 0.4968 | -0.0473 | -0.305 | -0.3264 | -0.0944 | 0.4274 | 0.3068 | -0.4248 | -0.2048 | 0.0278 |
| 0.0583 | 0.321 | 0.3543 | -0.0405 | 0.4286 | 0.2876 | 0.1717 | 0.4352 | 0.2652 | 0.4447 | -0.1142 |
| 0.0128 | 0.1248 | 0.3232 | 0.2994 | 0.027 | -0.538 | 0.0476 | 0.2202 | 0.4212 | -0.3988 | 0.3359 |
| 0.0029 | 0.0485 | 0.2287 | 0.3884 | -0.4198 | 0.3439 | 0.0088 | 0.0573 | 0.1607 | -0.3122 | -0.611 |
| 0.0029 | 0.0485 | 0.2287 | 0.3884 | -0.4198 | 0.3439 | -0.0088 | -0.0573 | -0.1607 | 0.3122 | 0.611 |
| 0.0128 | 0.1248 | 0.3232 | 0.2994 | 0.027 | -0.538 | -0.0476 | -0.2202 | -0.4212 | 0.3988 | -0.3359 |
| 0.0583 | 0.321 | 0.3543 | -0.0405 | 0.4286 | 0.2876 | -0.1717 | -0.4352 | -0.2652 | -0.4447 | 0.1142 |
| 0.2061 | 0.4968 | -0.0473 | -0.305 | -0.3264 | -0.0944 | -0.4274 | -0.3068 | 0.4248 | 0.2048 | -0.0278 |
| 0.4854 | 0.0869 | -0.3384 | 0.3391 | 0.1636 | 0.0254 | -0.5343 | 0.4058 | -0.2145 | -0.0616 | 0.0054 |
| Eigenvalues of matrix S | | | | | | | | | | |
| -0.5312 | -2.3481 | -3.5059 | -4.148 | -4.9833 | -6.4835 | -1.5175 | -3.0735 | -4.2984 | -5.6418 | -7.4688 |
| Eigenvalues of matrix F | | | | | | | | | | |
| 1 | -1 | 1 | -1 | 1 | -1 | -j | j | -j | j | -j |

Table 4: Matrix **V** corresponding to the modal matrix of **S** of order $N = 11$.

| Matrix V corresponding to the eigenvectors of matrix S | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.6609 | 0 | -0.4994 | 0 | 0.4494 | 0 | -0.3157 | 0 | -0.1097 | 0 | -0.0113 |
| 0.4854 | 0.5343 | 0.0869 | -0.4058 | -0.3384 | 0.2145 | 0.3391 | 0.0616 | 0.1636 | -0.0054 | 0.0254 |
| 0.2061 | 0.4274 | 0.4968 | 0.3068 | -0.0473 | -0.4248 | -0.305 | -0.2048 | -0.3264 | 0.0278 | -0.0944 |
| 0.0583 | 0.1717 | 0.321 | 0.4352 | 0.3543 | 0.2652 | -0.0405 | 0.4447 | 0.4286 | -0.1142 | 0.2876 |
| 0.0128 | 0.0476 | 0.1248 | 0.2202 | 0.3232 | 0.4212 | 0.2994 | -0.3988 | 0.027 | 0.3359 | -0.538 |
| 0.0029 | 0.0088 | 0.0485 | 0.0573 | 0.2287 | 0.1607 | 0.3884 | -0.3122 | -0.4198 | -0.611 | 0.3439 |
| 0.0029 | -0.0088 | 0.0485 | -0.0573 | 0.2287 | -0.1607 | 0.3884 | 0.3122 | -0.4198 | 0.611 | 0.3439 |
| 0.0128 | -0.0476 | 0.1248 | -0.2202 | 0.3232 | -0.4212 | 0.2994 | 0.3988 | 0.027 | -0.3359 | -0.538 |
| 0.0583 | -0.1717 | 0.321 | -0.4352 | 0.3543 | -0.2652 | -0.0405 | -0.4447 | 0.4286 | 0.1142 | 0.2876 |
| 0.2061 | -0.4274 | 0.4968 | -0.3068 | -0.0473 | 0.4248 | -0.305 | 0.2048 | -0.3264 | -0.0278 | -0.0944 |
| 0.4854 | -0.5343 | 0.0869 | 0.4058 | -0.3384 | -0.2145 | 0.3391 | -0.0616 | 0.1636 | 0.0054 | 0.0254 |
| Eigenvalues of matrix F | | | | | | | | | | |
| 1 | -j | -1 | j | 1 | -j | -1 | j | 1 | -j | -1 |

Table 5: The computation time (in seconds) of the eigenvectors.

| N | T Method | S Method | OPA (1) | OPA (2) | DBEOA |
|---|---|---|---|---|---|
| 37 | 0.046875 | 0.046875 | 0.015625 | 0 | 0.03125 |
| 128 | 0.109375 | 0.0625 | 0.0625 | 0.03125 | 0.03125 |
| 256 | 0.484375 | 0.296875 | 0.375 | 0.1875 | 0.1875 |
| 512 | 4.78125 | 2.40625 | 2.921875 | 1.265625 | 1.1875 |
| 1024 | 36.29688 | 18.46875 | 22.84375 | 8.25 | 8.25 |
| 2048 | 282.1875 | 149.3594 | 198 | 76.42188 | 73.67188 |

Table 6: Maximum orthonormality error.

| N | T Method | S Method | OPA (1) | OPA (2) | DBEOA |
|---|---|---|---|---|---|
| 37 | 1.14631E-14 | 2.10942E-15 | 2.5535E-15 | 2E-15 | 3.54E-15 |
| 128 | 8.34721E-13 | 5.44009E-15 | 3.9968E-15 | 8.09E-13 | 4.85E-10 |
| 256 | 1.28672E-11 | 4.66294E-15 | 3.3307E-15 | 1.03E-08 | 0.00778 |
| 512 | 2.86842E-11 | 6.21725E-15 | 6.6613E-15 | 0.067112 | 0.269404 |
| 1024 | 3.61431E-10 | 1.06581E-14 | 5.9952E-15 | 0.095463 | 0.733461 |
| 2048 | 3.94696E-09 | 1.34337E-14 | 1.5099E-14 | 0.096422 | 0.379373 |

Table 7: Frobenius norm of the orthonormality error matrix.

| N | T Method | S Method | OPA (1) | OPA (2) | DBEOA |
|---|---|---|---|---|---|
| 37 | 2.94927E-14 | 6.96041E-15 | 9.13158E-15 | 1.6E-14 | 1.87E-14 |
| 128 | 1.90672E-12 | 2.31327E-14 | 2.8328E-14 | 1.21E-11 | 2E-09 |
| 256 | 3.50961E-11 | 4.33601E-14 | 5.19285E-14 | 1.63E-07 | 0.043024 |
| 512 | 5.83399E-11 | 8.3416E-14 | 9.77425E-14 | 2.843433 | 4.185355 |
| 1024 | 8.07624E-10 | 1.65572E-13 | 1.99409E-13 | 6.324341 | 16.83342 |
| 2048 | 1.02527E-08 | 3.24143E-13 | 4.13468E-13 | 19.08835 | 14.13938 |

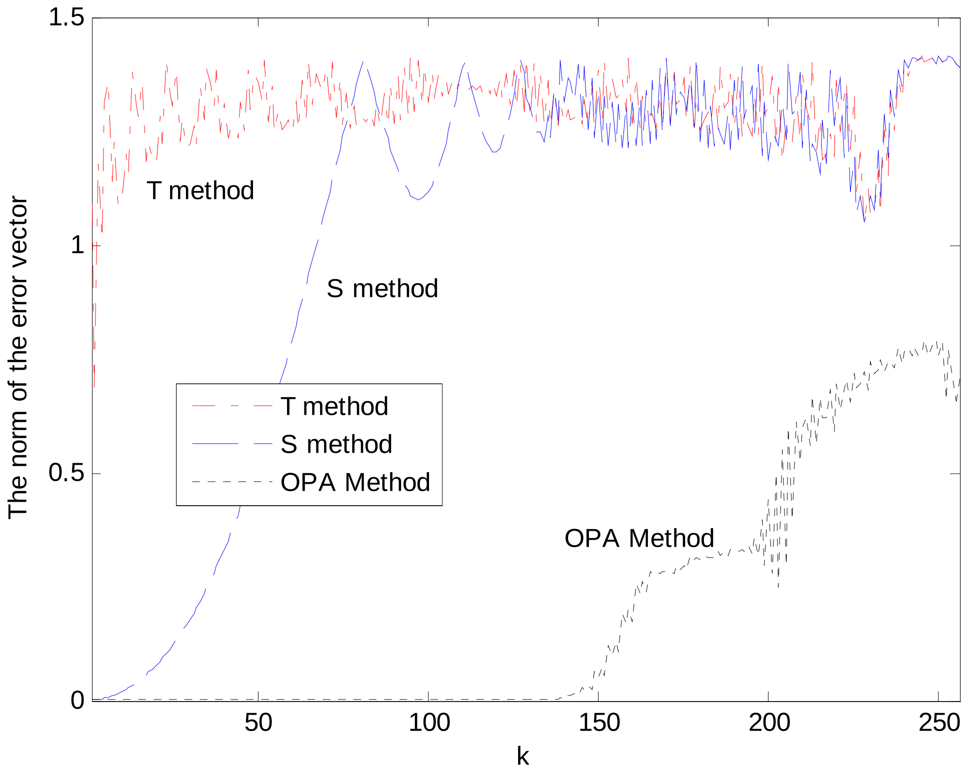


Fig. 1: Norm of the error vectors between the exact and approximate eigenvectors for N =256.

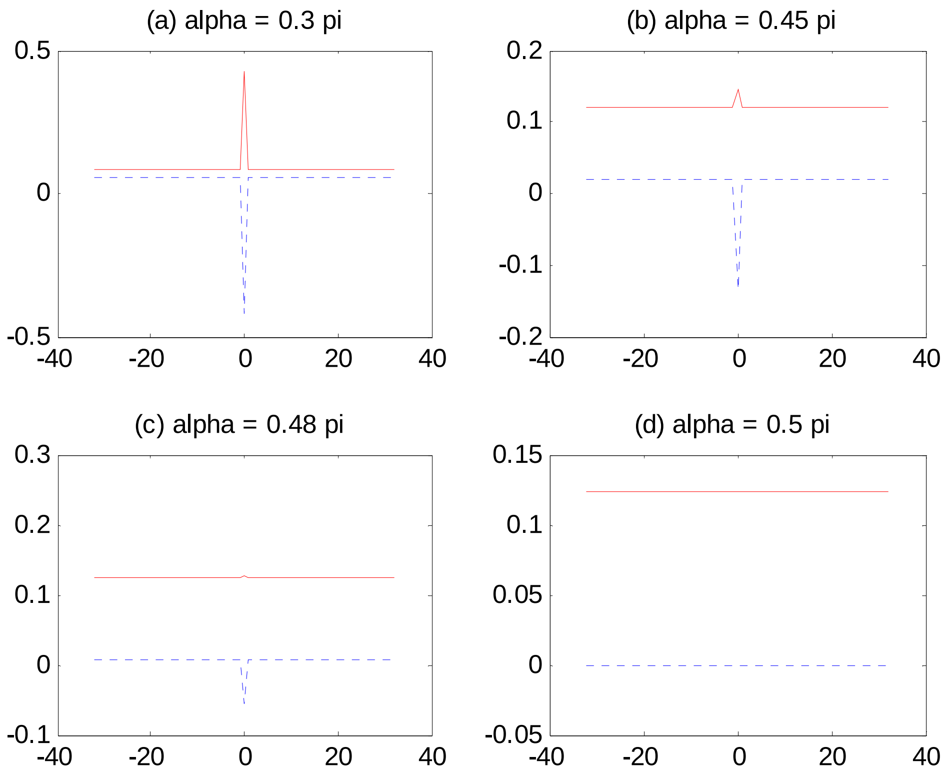


Fig. 2: The DFRFT (based on the eigenvectors of matrix T) of a discrete-time impulse. (N = 65)

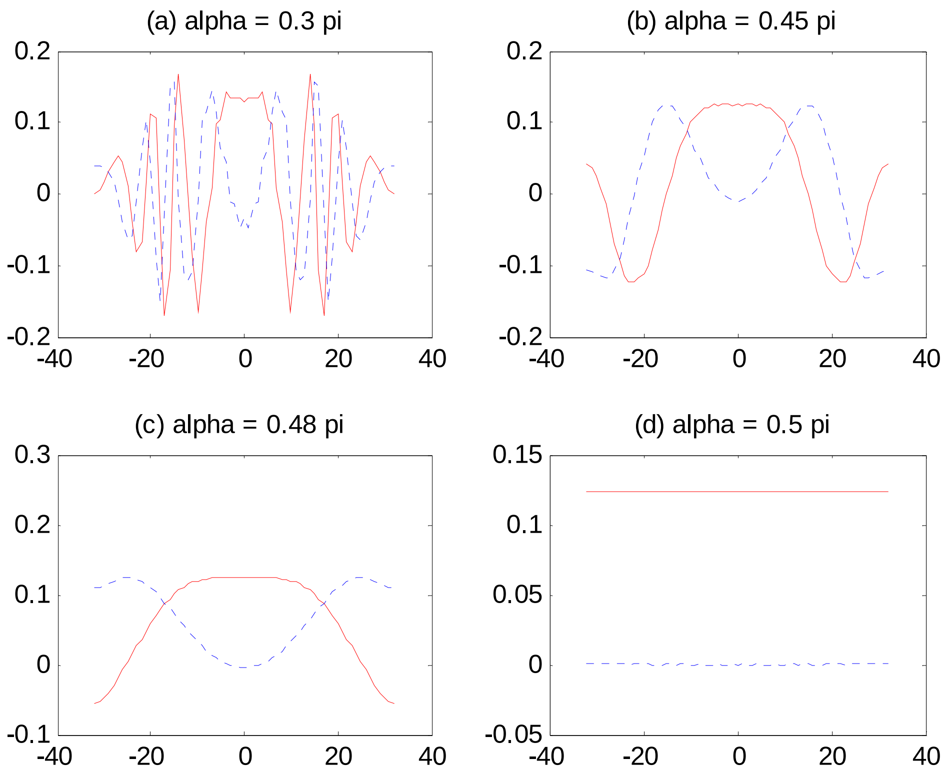


Fig. 3: The DFRFT (based on the eigenvectors of matrix S) of a discrete-time impulse. (N = 65)

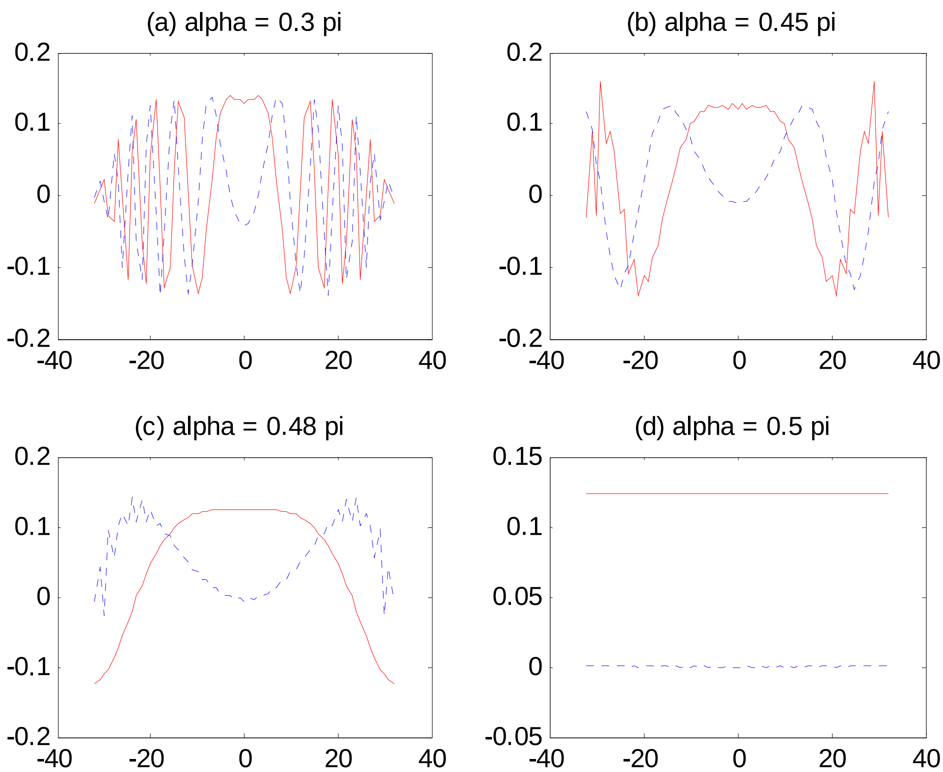


Fig. 4: The DFRFT (based on the OPA (1)) of a discrete-time impulse. (N = 65)

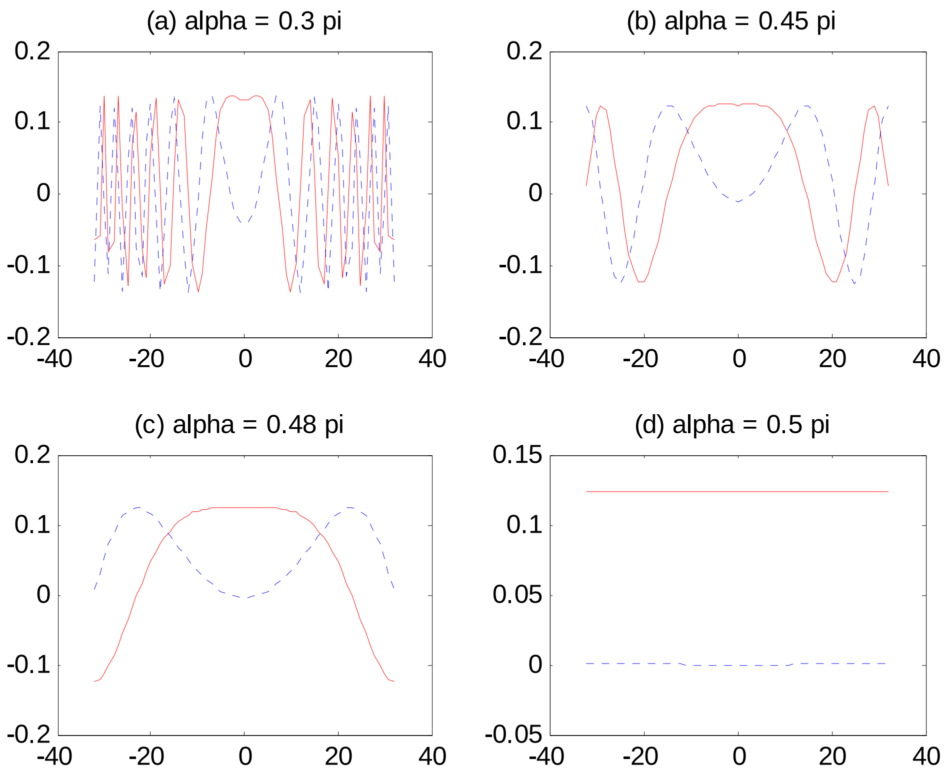


Fig. 5: The (scaled) FRFT of a continuous-time impulse. (N = 65)

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