Hermite-Gaussian Like Eigenvectors of the Discrete Fourier Transform Matrix Based on the Singular Value Decomposition of its Orthogonal Projection Matrices

Magdy Tawfik Hanna[1], Nabila Philip Attalla Seif[2] and Waleed Abd El Maguid Ahmed[3]

Abstract

A new technique is proposed for generating initial orthonormal eigenvectors of the Discrete Fourier Transform matrix F by the singular value decomposition of its orthogonal projection matrices on its eigenspaces and efficiently computable expressions for those matrices are derived. In order to generate Hermite-Gaussian like orthonormal eigenvectors of F given the initial ones, a new method called the Sequential Orthogonal Procrustes Algorithm (SOPA) is presented based on the sequential generation of the columns of a unitary matrix rather than the batch evaluation of that matrix as in the Orthogonal Procrustes Algorithm (OPA). It is proved that for any of the SOPA, the OPA, or the Gram-Schmidt Algorithm (GSA) the output Hermite-Gaussian like orthonormal eigenvectors are invariant under the change of the input initial orthonormal eigenvectors.

Index Terms: Discrete fractional Fourier transform, Hermite-Gaussian like eigenvectors, orthogonal procrustes algorithm, Gram-Schmidt algorithm, singular value decomposition.

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[1] Department of Engineering Mathematics and Physics, Faculty of Engineering, Cairo University / Fayoum Branch, Fayoum 63514, Egypt, Tel: 0020-2-5706033, Fax: 0020-2-5706033, Email: hanna@ieee.org

[2] Department of Engineering Mathematics and Physics, Faculty of Engineering, Cairo University, Giza, Egypt, Email: npaseif@mail.com

[3] Department of Engineering Mathematics and Physics, Faculty of Engineering, Cairo University / Fayoum Branch, Fayoum 63514, Egypt.

I. INTRODUCTION

Having developed the continuous FRactional Fourier Transform (FRFT) [1-3], current research is taking place to develop its discrete counterpart, namely the Discrete FRactional Fourier Transform (DFRFT) [4-8]. In order for the DFRFT to satisfy the requirements of unitarity and index additivity, orthonormal eigenvectors should be generated for the Discrete Fourier Transform (DFT) matrix F. In order for the DFRFT to approximate its continuous counterpart, it is logical to demand that the eigenvectors of F approximate the Hermite-Gaussian functions which are the eigenfunctions of the FRFT [9].

Candan, Kutay and Ozaktas obtained a second order difference equation by discretizing the second order differential equation satisfied by the Hermite-Gaussian functions [6,7]. Periodic solutions of this difference equation exist since its coefficients are periodic with period N. One period of each solution sequence forms the elements of an eigenvector of an almost tridiagonal real symmetric matrix S. The matrix S and the DFT matrix F have a common set of eigenvectors because they commute. Candan et. al. used those Hermite-Gaussian like orthonormal eigenvectors of F as a basis for a legitimate definition of the DFRFT. Actually the work of Candan et. al. was an extension of that of Dickinson and Steiglitz [10] who previously studied the eigenstructure of the special matrix[4] S.

Pei, Yeh and Tseng achieved markedly superior results. They regarded the orthonormal eigenvectors of matrix S only as *initial* orthonormal basis spanning the eigenspaces of matrix F. In each eigenspace they searched for other orthonormal eigenvectors that better approximate the Hermite-Gaussian functions [5]. The unitarity of matrix F implies that its eigenspaces corresponding to its distinct eigenvalues are orthogonal to each other [11] and the task reduces to finding good Hermite-Gaussian like orthonormal eigenvectors for each eigenspace individually. More specifically since matrix F has four distinct eigenvalues $(-j)^{k-1}$, $k = 1, \cdots 4$ [12,13], the corresponding *initial* eigenvectors are grouped as the columns of 4 matrices V_k, $k = 1, \cdots 4$. Pei et. al. generated a set of n vectors by sampling the Hermite-Gaussian functions and proved that they are approximate eigenvectors of matrix F corresponding to the exact eigenvalues $(-j)^{k-1}$, $k = 1, \cdots 4$ [4,5]. Those vectors are arranged as the columns of 4 matrices U_k, $k = 1, \cdots 4$. Pei et. al. proposed two techniques for getting

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[4] Strictly speaking, denoting the matrix S in the work of Dickinson et. al. [10] and Pei et. al. [5] by S_1 and the matrix S in the work of Candan et. al. [6] by S_2, the two matrices are related by $S_2 = S_1 - 4I$. Therefore S_1 and S_2 have the same eigenvectors.

orthonormal eigenvectors of F that better approximate the Hermite-Gaussian functions than the *initial* orthonormal eigenvectors forming the columns of the matrices V_k, $k = 1, \cdots 4$ [5]. The first technique is the Gram-Schmidt Algorithm (GSA) where for each value of k separately the columns of U_k are projected on the column space of V_k to get exact nonorthogonal eigenvectors of F that are next orthonormalized by applying the Gram-Schmidt method. The second technique is the Orthogonal Procrustes Algorithm (OPA) where for each value of k separately the desired superior orthonormal eigenvectors are assumed to form the columns of a matrix $\hat{U}_k$ that is expressed as $V_k Q_k$ and the unitary matrix Q_k is evaluated by minimizing the Frobenius norm of the matrix difference $\left(U_k - \hat{U}_k \right)$ [14].

One objective of the current paper is to present a direct technique for generating *initial* orthonormal eigenvectors of F without appealing to matrix S. The technique depends on the generation of the orthogonal projection matrices of matrix F on its eigenspaces and the computation of *initial* eigenvectors by applying the singular value decomposition technique. More specifically, expressions are derived for the projections matrices and simplified in order to get efficiently computable forms. A second objective is to present an alternative technique for generating good Hermite-Gaussian like eigenvectors of F given *initial* ones. In this technique – to be referred to as the Sequential Orthogonal Procrustes Algorithm (SOPA) – the columns of the unitary matrix Q_k are sequentially evaluated by solving a series of constrained minimization problems rather than batch evaluated as in the OPA. A third contribution is the proof that the superior Hermite-Gaussian like eigenvectors computed using any of the three techniques – GSA, OPA or SOPA – are invariant under the change of the *initial* eigenvectors. This implies that for any of the three refined techniques the *final* eigenvectors will be the same whether the *initial* eigenvectors are computed by finding the eigenvectors of the auxiliary matrix S or by the singular value decomposition of the projection matrices. More surprisingly, it will be proved that both the GSA and the SOPA produce identical results despite being algorithmically quite different.

In section II the projection matrices on the four eigenspaces of matrix F will be derived using two completely different methods. In section III each projection matrix is decomposed by the singular value decomposition technique in order to get orthonormal basis of the corresponding eigenspace. In section IV the Gram-Schmidt Algorithm and the Orthogonal Procrustes Algorithm will be surveyed and proved to produce an output that is invariant under the change of the initial eigenvectors. The contributed Sequential Orthogonal

Procrustes Algorithm will be next presented and proved to have the same property. In section V it will be shown that the SOPA and GSA produce identical outputs despite being algorithmically distinct. Some simulation results will be presented in section VI.

II. THE ORTHOGONAL PROJECTION MATRICES ON THE EIGENSPACES OF MATRIX F

The Discrete Fourier Transform matrix $F = \left(f_{m,n} \right)$ of order N is defined by:

$$f_{m,n} = \frac{1}{\sqrt{N}} W^{(m-1)(n-1)} \qquad , m,n = 1, \cdots, N \tag{1}$$

where

$$W = \exp\left(-j\frac{2\pi}{N} \right). \tag{2}$$

Matrix F has the following 4 distinct eigenvalues [12]:

$$\lambda_k = \left(-j \right)^{k-1} \qquad , k = 1, \cdots 4 . \tag{3}$$

It is straightforward to show that matrix F is unitary and consequently is diagonalizable [15]. According to the spectral theorem [16], F has the following spectral decomposition:

$$F = \sum_{k=1}^{4} \lambda_k P_k \tag{4}$$

where P_k is the orthogonal projection matrix on the k th eigenspace of F to be denoted by E_k. Two different methods will be presented below for deriving the four projection matrices.

Method A

Since for any integer m, matrix F^m has the same eigenvectors and consequently projection matrices as matrix F, equation (4) implies that:

$$F^m = \sum_{k=1}^{4} \lambda_k^m P_k \qquad , m = 0,1, \cdots . \tag{5}$$

The special case of $m = 0$ is the resolution of the identity matrix induced by F. Since the objective is the derivation of the four projection matrices, the above equation will be written for $m = 0,1,2,3$ in order to get four matrix equations that can be expressed as:

$$A \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix} = \begin{bmatrix} I \\ F \\ F^2 \\ F^3 \end{bmatrix} \tag{6}$$

where the partitioned matrix of coefficients is given by:

$$A = \begin{bmatrix} I & I & I & I \\ \lambda_1 I & \lambda_2 I & \lambda_3 I & \lambda_4 I \\ \lambda_1^2 I & \lambda_2^2 I & \lambda_3^2 I & \lambda_4^2 I \\ \lambda_1^3 I & \lambda_2^3 I & \lambda_3^3 I & \lambda_4^3 I \end{bmatrix}. \tag{7}$$

In preparation for solving (6), one obtains the following results from (3):

$$\sum_{k=1}^{4} \lambda_k^m = \sum_{k=1}^{4} \left(e^{-j0.5\pi} \right)^{(k-1)m} = \sum_{k=1}^{4} \exp\left(-j\frac{2\pi}{4} m(k-1) \right)$$
$$= \begin{cases} 4 & \text{if } m = 4r , \text{ r integer} \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

$$\sum_{k=1}^{4} \lambda_k^m \lambda_k^{*n} = \sum_{k=1}^{4} \exp\left(-j\frac{\pi}{2}(k-1)m \right) \exp\left(j\frac{\pi}{2}(k-1)n \right) = \sum_{k=1}^{4} \exp\left(j\frac{2\pi}{4}(n-m)(k-1) \right)$$
$$= \begin{cases} 4 & \text{if } n-m = 4r , \text{ r integer} \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

Using (7)-(9), one obtains[5]:

$$AA^+ = 4I \tag{10}$$

and consequently

$$A^{-1} = 0.25 A^+. \tag{11}$$

Therefore (6) can be directly solved using (11) and (7) to get the following compact expression for the four orthogonal projection matrices:

$$P_k = 0.25 \sum_{m=0}^{3} \lambda_k^{*m} F^m \quad , \quad k = 1,\cdots,4. \tag{12}$$

In order to put the above expression in an efficiently computable form, one should utilize the fact that [17, p.351]:

$$F^2 = \Gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & J \end{pmatrix} \tag{13}$$

where J is the contra-identity matrix of order $(N-1)$ defined by:

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[5] The superscripts * and + respectively denote the complex conjugate and the complex conjugate transpose.

$$J = \begin{pmatrix} & & & 1 \\ & & \cdots & \\ & 1 & & \\ 1 & & & \end{pmatrix}.$$

(14)

Upon using the fact that [10]:

$$F^4 = I$$

(15)

and the unitarity and symmetry of F, one obtains[6]:

$$F^3 = F^{-1} = F^+ = F^*.$$

(16)

From (3), one directly gets:

$$\lambda_k^2 = (-1)^{(k-1)}.$$

(17)

Substituting (13), (16), (3) and (17) in (12), one obtains:

$$P_k = 0.25\{I + (-1)^{(k-1)}\Gamma + j^{(k-1)}[F + (-1)^{(k-1)}F^*]\}.$$

(18)

Therefore the final expressions required for computing the orthogonal projection matrices are given by:

$$P_1 = 0.25\{I + \Gamma + 2\,\mathrm{Real}(F)\},$$

(19)

$$P_2 = 0.25\{I - \Gamma - 2\,\mathrm{Imaginary}(F)\},$$

(20)

$$P_3 = 0.25\{I + \Gamma - 2\,\mathrm{Real}(F)\},$$

(21)

$$P_4 = 0.25\{I - \Gamma + 2\,\mathrm{Imaginary}(F)\}.$$

(22)

Method B

According to a corollary of the spectral theorem, each orthogonal projection matrix P_k can be expressed as [16, p. 434]:

$$P_k = g_k(F) \qquad , k = 1, \cdots, 4$$

(23)

where $g_k(F)$ is a polynomial in F which satisfies the conditions:

$$g_k(\lambda_i) = \delta_{i,k} \qquad , i = 1, \cdots, 4.$$

(24)

Since matrix F is diagonalizable and has only 4 distinct eigenvalues, the polynomial $g_k(F)$ is of the third degree. Using the Lagrange interpolation formula, $g_k(\lambda)$ can be expressed as:

$$g_k(\lambda) = \sum_{i=1}^{4} g_k(\lambda_i) f_i(\lambda)$$

(25)

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[6] Although matrix F is symmetric it is not Hermitian.

where

$$f_i(\lambda) = \prod_{\substack{r=1 \\ r \neq i}}^{4} \frac{\lambda - \lambda_r}{\lambda_i - \lambda_r} \qquad , \ i = 1, \cdots, 4 .$$

(26)

Combining (24)-(26), one gets:

$$g_k(\lambda) = f_k(\lambda) = \prod_{\substack{r=1 \\ r \neq k}}^{4} \frac{\lambda - \lambda_r}{\lambda_k - \lambda_r} .$$

(27)

Upon substituting (3) in the above equation, one obtains:

$$g_1(\lambda) = 0.25\left(1 + \lambda + \lambda^2 + \lambda^3\right) ,$$

(28)

$$g_2(\lambda) = 0.25\left(1 + j\lambda - \lambda^2 - j\lambda^3\right) ,$$

(29)

$$g_3(\lambda) = 0.25\left(1 - \lambda + \lambda^2 - \lambda^3\right) ,$$

(30)

$$g_4(\lambda) = 0.25\left(1 - j\lambda - \lambda^2 + j\lambda^3\right) .$$

(31)

Using (23) and (28)-(31), one gets the following expressions for the projection matrices:

$$P_1 = 0.25\left(I + F + F^2 + F^3\right) ,$$

(32)

$$P_2 = 0.25\left(I + jF - F^2 - jF^3\right) ,$$

(33)

$$P_3 = 0.25\left(I - F + F^2 - F^3\right) ,$$

(34)

$$P_4 = 0.25\left(I - jF - F^2 + jF^3\right) .$$

(35)

Using (3), it immediately follows that the above four expressions are identical to expression (12) obtained using the first approach.

III. INITIAL ORTHONORMAL EIGENVECTORS OF MATRIX F

The N th order square orthogonal projection matrix P_k has rank r_k which is the dimension of the k th eigenspace of matrix F given by Table 1 [12]. The objective here is to compute orthonormal basis for each eigenspace. The singular value decomposition technique will be shown to be the right technique to apply. The singular value decomposition of an arbitrary square matrix A of order N is [18]:

$$A = U\Sigma V^+$$

(36)

where U and V are unitary matrices of order N and

$$\Sigma = Diag\{\sigma_1, \cdots, \sigma_N\} .$$

(37)

In the above equation the singular values σ_i, $i = 1, \cdots N$ are real and satisfy $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_N \geq 0$.

Lemma 1:

Let A be a square Hermitian matrix of order N having a modal matrix V and eigenvalues $\lambda_1, \cdots, \lambda_N$ arranged such that $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_N|$. The singular value decomposition of A is given by (36) where

$$U = V S , \tag{38}$$

$$\Sigma = Diag\{|\lambda_1|, \cdots, |\lambda_N|\} , \tag{39}$$

$$S = Diag\{s_1, \cdots, s_N\} \tag{40}$$

and

$$s_n = \begin{cases} 1 & \text{if } \lambda_n \geq 0 \\ -1 & \text{if } \lambda_n < 0 \end{cases} , n = 1, \cdots, N . \tag{41}$$

Proof: See Appendix A.

The above lemma implies that for a Hermitian matrix the singular values are equal to the absolute values of its eigenvalues and the right singular vectors are equal to its orthonormal eigenvectors.

If the rank of A is r, the matrices V and Σ in (36) can be partitioned as:

$$V = \begin{pmatrix} V_a & V_b \end{pmatrix} , \tag{42}$$

$$\Sigma = \begin{pmatrix} \Sigma_a & O \\ O & O \end{pmatrix} \tag{43}$$

where

$$\Sigma_a = Diag\{\sigma_1, \cdots, \sigma_r\} \tag{44}$$

and $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$. In (42) the $N \times r$ submatrix V_a has orthonormal columns by the unitarity of V.

For a Hermitian matrix A of rank r, combining (36),(38),(40),(42) and (43), one obtains:

$$A = V_a S_a \Sigma_a V_a^+ \tag{45}$$

where S_a is the leading diagonal block of order r of matrix S defined by (40).

In the particular case of an orthogonal projection matrix A, all eigenvalues are either 1 or 0 [18]. Consequently the diagonal matrices Σ_a and S_a in (45) reduce to the identity matrix and (45) simplifies to:

$$A = V_a V_a^+ . \tag{46}$$

Therefore in order to find orthonormal basis for any space of dimension r given its orthogonal projection matrix A, one has only to apply the singular value decomposition technique (36) and select the first r columns of matrix V in (42). It should be emphasized that one should never try to get the same result computationally by applying an eigenvalue decomposition procedure since A might have repeated eigenvalues and the corresponding eigenvectors – as evaluated by that procedure – will not generally be orthogonal.

Applying the result (46) to the four projection matrices of last section, one gets:

$$P_k = V_k V_k^+ \qquad , \ k = 1, \cdots, 4 . \tag{47}$$

The orthonormal basis of the k th eigenspace of matrix F given by the columns of V_k will be taken as *initial* orthonormal eigenvectors of F corresponding to λ_k. They will be utilized for deriving the desired Hermite-Gaussian like orthonormal eigenvectors direly needed for defining a discrete fractional Fourier transform that approximates its continuous counterpart.

IV. HERMITE-GAUSSIAN LIKE EIGENVECTORS

By sampling the Hermite-Gaussian functions, Pei et. al. obtained approximate eigenvectors for the DFT matrix F that correspond to the exact eigenvalues λ_k of (3) as delineated in [4,5]. Those vectors are grouped to form the columns of the four $N \times r_k$ matrices U_k, $k = 1, \cdots, 4$ where the dimensions of the corresponding eigenspaces r_k are given in Table 1. Being approximate rather than exact eigenvectors, the columns of U_k do not belong to the eigenspace E_k corresponding to the eigenvalue λ_k. The objective here is to find orthonormal basis for E_k to form the columns of a matrix $\hat{U}_k$ that are as close as possible to the columns of matrix U_k. Towards achieving that goal two techniques have been proposed by Pei et. al. [5], namely the Gram-Schmidt Algorithm (GSA) and the Orthogonal Procrustes Algorithm (OPA). A third technique to be termed the Sequential Orthogonal Procrustes Algorithm (SOPA) will be proposed in this paper. In preparation for presenting the new technique, the first two techniques will be surveyed below and cast in a form that will facilitate the comparison. More importantly, they will be proved to produce an output that is invariant under the change of the *initial* orthonormal basis of E_k.

Since the eigenspaces E_k, $k = 1, \cdots, 4$ are orthogonal to each other due to the unitarity of matrix F, each eigenspace will be dealt with separately. In order to simplify the notation the

subscript k will be dropped in the remainder of this paper. The $N \times r_k$ matrices $V_k, U_k, \hat{U}_k$ will be written as the $N \times r$ matrices $V, U, \hat{U}$ respectively. The space E_k will be denoted by E.

(A) The Gram-Schmidt Algorithm (GSA)

First matrix U will be expressed as the partitioned matrix:

$$U = \begin{pmatrix} u_1 & \cdots & u_r \end{pmatrix}. \tag{48}$$

Each column u_n of U will be projected on E to get $\tilde{u}_n$. Since the resulting r vectors are not orthogonal, they will be orthonormalized by applying the Gram-Schmidt technique in order to get $\hat{u}_n$, $n = 1, \cdots, r$. More specifically, since the space E is spanned by the orthonormal columns of V, vector $\tilde{u}_n$ can be expressed as:

$$\tilde{u}_n = \sum_{m=1}^{r} \langle v_m, u_n \rangle v_m \qquad , \; n = 1, \cdots, r. \tag{49}$$

The above equation can be rewritten as:

$$\tilde{u}_n = \begin{pmatrix} v_1 & \cdots & v_r \end{pmatrix} \begin{pmatrix} \langle v_1, u_n \rangle \\ \vdots \\ \langle v_r, u_n \rangle \end{pmatrix}. \tag{50}$$

By virtue of the definition of matrix V, the same equation can be compactly expressed as:

$$\tilde{u}_n = VV^+ u_n \qquad , \; n = 1, \cdots, r. \tag{51}$$

Upon defining the $N \times r$ matrix $\tilde{U}$ as:

$$\tilde{U} = \begin{pmatrix} \tilde{u}_1 & \cdots & \tilde{u}_r \end{pmatrix} \tag{52}$$

the r vector equations (51) can be combinedly expressed as:

$$\tilde{U} = VV^+ U. \tag{53}$$

The Gram-Schmidt technique is next applied to orthonormalize the columns of matrix $\tilde{U}$ in order to get the columns of matrix $\hat{U}$ using the following steps:

1) $\hat{u}_1 = \dfrac{\tilde{u}_1}{\|\tilde{u}_1\|}$. $\tag{54}$

2) For $s = 2, \cdots, r$:

   a) $y_s = \tilde{u}_s - \sum_{m=1}^{s-1} \langle \hat{u}_m, \tilde{u}_s \rangle \hat{u}_m$, $\tag{55}$

b) $\hat{u}_s = \dfrac{y_s}{\|y_s\|}$. (56)

Matrix $\hat{U}$ is next defined as:

$$\hat{U} = \begin{pmatrix} \hat{u}_1 & \cdots & \hat{u}_r \end{pmatrix}.$$ (57)

Lemma 2:

The result of the Gram-Schmidt Algorithm is invariant under the change of the *initial* orthonormal basis of the space E given by the columns of matrix V.

Proof:

Consider a second set of *initial* orthonormal basis given by the columns of a matrix W defined by:

$$W = \begin{pmatrix} w_1 & \cdots & w_r \end{pmatrix}.$$ (58)

Since the columns of V form a basis of E, the columns of W can be expressed as linear combinations of those of V, i.e.,

$$w_n = \sum_{m=1}^{r} v_m \alpha_{mn} \qquad , \; n = 1, \cdots, r .$$ (59)

The above r vector equations can be compactly expressed as a matrix equation:

$$W = VG$$ (60)

where G is a square matrix of order r. It follows immediately that:

$$W^+W = G^+\left(V^+V\right)G.$$ (61)

By the orthonormality of the columns of V and W individually, the above equation implies that:

$$G^+G = I .$$ (62)

Based on this unitarity property of G, it follows from (60) that:

$$WW^+ = VGG^+V^+ = VV^+ .$$ (63)

This proves that the matrix product VV^+ is invariant under the change of the *initial* orthonormal basis of the space E. It follows from (53) that $\tilde{U}$ is invariant under the change of V. The same applies to $\hat{U}$ as can be concluded from (54)-(57).

(Q.E.D.)

(B) The Orthogonal Procrustes Algorithm (OPA)

Here the desired matrix $\hat{U}$ of the Hermite-Gaussian like eigenvectors defined by (57) will be expressed as[7]:

$$\hat{U} = VQ \tag{64}$$

where Q is a unitary matrix to be evaluated such that the square of the Frobenius norm $\left\| U - \hat{U} \right\|_F$ is minimized. The solution of this problem is given by the Orthogonal Procrustes Algorithm expounded in [14] and used in [5] and summarized in the following three steps:

1. Form matrix C:

$$C = V^+ U . \tag{65}$$

2. Find the singular value decomposition of C:

$$C = ADB^+ . \tag{66}$$

3. Compute matrix Q:

$$Q = AB^+ . \tag{67}$$

Lemma 3:

The matrix $\hat{U}$ determined by the OPA is invariant under the change of the *initial* orthonormal basis of the space E given by the columns of matrix V.

Proof:

Let V_1 and V_2 be two *initial* orthonormal bases of E. They should be related by:

$$V_2 = V_1 G \tag{68}$$

where G is a unitary matrix. (This follows along the same lines of (59)-(62)). Let Q_1 and Q_2 be the corresponding unitary matrices evaluated by the OPA and $\hat{U}_1$ and $\hat{U}_2$ be the resulting optimal matrices given by (64). It follows from (65)-(67) and (64) that:

$$C_i = V_i^+ U \qquad , \; i = 1,2 \tag{69}$$

$$C_i = A_i D_i B_i^+ \qquad , \; i = 1,2 \tag{70}$$

$$Q_i = A_i B_i^+ \qquad , \; i = 1,2 \tag{71}$$

$$\hat{U}_i = V_i Q_i \qquad , \; i = 1,2 \tag{72}$$

Our task reduces to showing that $\hat{U}_2 = \hat{U}_1$. From (68) and (69), one gets:

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[7] It should be mentioned that in [5], the OPA was erroneously applied since $\hat{U}$ was taken as $\hat{U} = QV$ rather than as $\hat{U} = VQ$.

$$C_2 = V_2^+ U = G^+\left(V_1^+ U\right) = G^+ C_1. \tag{73}$$

Substituting (70) in the above equation, one obtains:

$$C_2 = A_2 D_2 B_2^+ = G^+ A_1 D_1 B_1^+. \tag{74}$$

By the uniqueness of the singular values of C_2, the above equation implies that:

$$D_2 = D_1. \tag{75}$$

By the unitarity of the matrices A_1, B_1, A_2, B_2, G, equation (74) leads to:

$$D_2 = A_2^+ G^+ A_1 D_1 B_1^+ B_2. \tag{76}$$

Since the diagonal matrices D_2 and D_1 have real nonnegative diagonal elements arranged in decreasing order of absolute value, a direct application of Lemma 1 results in:

$$D_2 = T D_1 T^+. \tag{77}$$

By comparing (76) and (77), one gets:

$$A_2^+ G^+ A_1 = \left(B_1^+ B_2\right)^+. \tag{78}$$

It follows immediately that:

$$A_2 B_2^+ = G^+ A_1 B_1^+. \tag{79}$$

Upon utilizing (71), the above equation reduces to:

$$Q_2 = G^+ Q_1. \tag{80}$$

Substituting (68) and (80) in (72), one eventually obtains:

$$\hat{U}_2 = V_2 Q_2 = \left(V_1 G\right)\left(G^+ Q_1\right) = V_1 Q_1 = \hat{U}_1. \tag{81}$$

(Q.E.D.)

(C) The Sequential Orthogonal Procrustes Algorithm (SOPA)

Although the desired matrix $\hat{U}$ will be expressed as in (64), the unitary matrix Q will not be evaluated by minimizing the square of the Frobenius norm $\left\|U - \hat{U}\right\|_F$ as in the OPA but rather its columns $q_s, s = 1, \cdots, r$ will be evaluated sequentially in the manner to be next delineated. By virtue of the definition of the Frobenius norm of a matrix and the Euclidean norm of a vector [18] and using (48), (57) and (64), one obtains:

$$J = \sum_{s=1}^{r} J_s \tag{82}$$

where

$$J = \left\|U - \hat{U}\right\|_F^2 = \left\|U - VQ\right\|_F^2, \tag{83}$$

$$J_s = \left\| u_s - \hat{u}_s \right\|_2^2 = \left\| u_s - V q_s \right\|_2^2 , \tag{84}$$

$$Q = \begin{pmatrix} q_1 & \cdots & q_r \end{pmatrix} . \tag{85}$$

In the OPA, matrix Q has been evaluated by minimizing the total performance index J of (83). In the SOPA there will be r stages of sequential minimizations. In stage s, the column q_s of Q will be evaluated by minimizing the partial performance index J_s of (84) subject to the constraints that q_s will be orthogonal to the previously evaluated columns $q_k, k = 1, \cdots, s-1$ and be of unit norm in order to satisfy the unitarity of Q. These s constraints can be expressed as:

$$q_k^+ q_s = 0 \qquad , k = 1, \cdots, s-1 \tag{86}$$

$$q_s^+ q_s = 1 . \tag{87}$$

The first set of $(s-1)$ orthogonality constraints can be compactly expressed as:

$$C_{s-1} q_s = 0 \tag{88}$$

where

$$C_{s-1} = \begin{pmatrix} q_1^+ \\ \vdots \\ q_{s-1}^+ \end{pmatrix} . \tag{89}$$

It should be mentioned that the rows of the $(s-1) \times r$ matrix C_{s-1} are linearly independent because of being orthogonal due to the way they have been sequentially generated. For mathematical tractability, one will set aside the quadratic constraint (87) for a while and minimize J_s subject to the linear constraints (88) and call the resulting vector x_s. Next by normalizing x_s in order to satisfy the normalization condition (87), one obtains q_s. More specifically, the constrained minimization problem defined by the quadratic criterion (84) and the linear constraints (88) is solved in Appendix B and its solution is given by:

$$x_s = \left(V^+ V \right)^{-1} \left\{ I - C_{s-1}^+ \left[C_{s-1} \left(V^+ V \right)^{-1} C_{s-1}^+ \right]^{-1} C_{s-1} \left(V^+ V \right)^{-1} \right\} V^+ u_s . \tag{90}$$

By the orthonormality of the columns of matrix V, the above equation reduces to:

$$x_s = \left\{ I - C_{s-1}^+ \left[C_{s-1} C_{s-1}^+ \right]^{-1} C_{s-1} \right\} V^+ u_s . \tag{91}$$

By utilizing the orthonormality of the rows of matrix C_{s-1} of (89), the above result simplifies to:

$$x_s = \left(I - C_{s-1}^+ C_{s-1} \right) V^+ u_s \qquad , s = 2, \cdots, r . \tag{92}$$

In the special case of generating x_1, one has the unconstrained minimization problem defined by criterion J_1 of (84) whose solution is given by:

$$x_1 = V^+ u_1. \tag{93}$$

The quadratic constraint (87) accounting for the normalization condition will be satisfied by computing:

$$q_s = \frac{1}{\|x_s\|} x_s \qquad , \ s = 1, \cdots, r. \tag{94}$$

Therefore the sequential orthogonal procrustes algorithm can be summarized in the following steps:

1) For $s = 1$:

    a) $x_1 = V^+ u_1$

    b) $q_1 = \dfrac{1}{\|x_1\|} x_1$

    c) $Q = q_1$

    d) $C = [\ ]$ (the null matrix)

2) For $s = 2, \cdots, r$:

    a) Augment matrix C by the row vector q_{s-1}^+

    b) $x_s = \left(I - C^+ C\right) V^+ u_s$

    c) $q_s = \dfrac{1}{\|x_s\|} x_s$

    d) Augment matrix Q by the column vector q_s

3) Generate $\hat{U}$ according to (64).

Lemma 4:

The matrix $\hat{U}$ determined by the SOPA is invariant under the change of the *initial* orthonormal basis of the space E given by the columns of matrix V.

Proof:

Let V_1 and V_2 be two *initial* orthonormal bases of E which should be related by (68). Let Q_i be the Q matrix determined by the SOPA corresponding to V_i and let $\hat{U}_i$ be the resulting matrix $\hat{U}$. An extra subscript i will be introduced in the vectors x_s and q_s and matrix C_{s-1}

to become $x_{i,s}$, $q_{i,s}$ and $C_{i,s-1}$ respectively when they are computed based on matrix V_i. From (93) and (68), one obtains:

$$x_{2,1} = V_2^+ u_1 = G^+ V_1^+ u_1 = G^+ x_{1,1} \ . \tag{95}$$

By virtue of (94) and the unitarity of G, one gets:

$$q_{2,1} = G^+ q_{1,1} \ . \tag{96}$$

From (89) and (96), it follows that:

$$C_{2,1} = q_{2,1}^+ = q_{1,1}^+ G = C_{1,1} G \ . \tag{97}$$

From (92), (97) and (68), one obtains:

$$x_{2,2} = \left(I - C_{2,1}^+ C_{2,1}\right) V_2^+ u_2 = \left(I - G^+ C_{1,1}^+ C_{1,1} G\right) G^+ V_1^+ u_2 = G^+ \left(I - C_{1,1}^+ C_{1,1}\right) V_1^+ u_2$$
$$= G^+ x_{1,2} \tag{98}$$

It follows from the above equation and (94) that:

$$q_{2,2} = G^+ q_{1,2} \ . \tag{99}$$

By the same token, it can be shown that:

$$q_{2,s} = G^+ q_{1,s} \qquad , \ s = 1, \cdots, r \ . \tag{100}$$

From (85) and (100), one obtains:

$$Q_2 = G^+ Q_1 \ . \tag{101}$$

The above equation together with (64) and (68) lead to:

$$\hat{U}_2 = V_2 Q_2 = \left(V_1 G\right)\left(G^+ Q_1\right) = V_1 Q_1 = \hat{U}_1 \ . \tag{102}$$

(Q.E.D.)

V. THE EQUALITY OF THE OUTPUTS OF THE GSA AND SOPA

In the GSA, one starts by projecting the vectors u_s, $s = 1, \cdots, r$ on the column space of matrix V to get the nonorthogonal vectors $\tilde{u}_s$, $s = 1, \cdots, r$. Next by applying the Gram-Schmidt orthonormalization technique, one sequentially obtains the orthonormal vectors $\hat{u}_s$, $s = 1, \cdots, r$. In the SOPA, one sequentially obtains the columns q_s, $s = 1, \cdots, r$ of the unitary matrix Q. This corresponds to the sequential evaluation of the vectors $\hat{u}_s$, $s = 1, \cdots, r$ since (64), (57) and (85) imply that:

$$\hat{u}_s = V q_s \qquad , \ s = 1, \cdots, r \ . \tag{103}$$

It will be shown below that the GSA and SOPA produce identical results. Towards that goal one starts by a further manipulation of the equations pertaining to both techniques only for the sake of proving the equality of their outputs.

(A) The GSA

Manipulating (55), one obtains:

$$
\begin{aligned}
y_s &= \tilde{u}_s - \sum_{m=1}^{s-1} \hat{u}_m \left(\hat{u}_m^+ \tilde{u}_s \right) \\
&= \tilde{u}_s - \left(\sum_{m=1}^{s-1} \hat{u}_m \hat{u}_m^+ \right) \tilde{u}_s \\
&= \tilde{u}_s - \hat{U}_{s-1} \hat{U}_{s-1}^+ \tilde{u}_s
\end{aligned}
\tag{104}
$$

where

$$
\hat{U}_{s-1} = \left(\hat{u}_1 \quad \cdots \quad \hat{u}_{s-1} \right).
\tag{105}
$$

Substituting (51) in (104), one gets:

$$
\begin{aligned}
y_s &= VV^+ u_s - \hat{U}_{s-1} \hat{U}_{s-1}^+ VV^+ u_s \\
&= (A - B_{s-1}A) u_s
\end{aligned}
\quad , \quad s = 2, \cdots, r
\tag{106}
$$

where

$$
A = VV^+,
\tag{107}
$$

$$
B_{s-1} = \hat{U}_{s-1} \hat{U}_{s-1}^+ .
\tag{108}
$$

Since the vectors $\hat{u}_s$, $s = 1, \cdots, r$ lie in the column space of matrix V, it follows from (105) that

$$
\hat{U}_{s-1} = VW
\tag{109}
$$

where W is an $r \times (s-1)$ matrix. Consequently

$$
\hat{U}_{s-1}^+ \hat{U}_{s-1} = W^+ \left(V^+ V \right) W .
\tag{110}
$$

By the orthonormality of the columns of $\hat{U}_{s-1}$ and those of V, the above equation results in:

$$
W^+ W = I .
\tag{111}
$$

From (107), (108) and (109), one obtains:

$$
B_{s-1}A = VWW^+ \left(V^+ V \right) V^+ = (VW)(W^+ V^+) = \hat{U}_{s-1} \hat{U}_{s-1}^+ = B_{s-1} .
\tag{112}
$$

Therefore (106) simplifies to:

$$
y_s = (A - B_{s-1}) u_s \quad , \quad s = 2, \cdots, r .
\tag{113}
$$

In the special case of $s = 1$, (51) and (107) result in:

$$y_1 = \tilde{u}_1 = VV^+ u_1 = Au_1 .$$ (114)

(B) The SOPA

One starts by defining:

$$z_s = Vx_s \qquad , \ s = 1, \cdots, r$$ (115)

where the vectors x_s, $s = 1, \cdots, r$ are given by (92) and (93). The orthonormality of the columns of V implies that:

$$\|z_s\| = \|x_s\| \qquad , \ s = 1, \cdots, r.$$ (116)

Equations (115), (116), (94) and (103) result in:

$$\frac{z_s}{\|z_s\|} = Vq_s = \hat{u}_s \qquad , \ s = 1, \cdots, r.$$ (117)

Since (94) implies that x_s is an unnormalized version of q_s, the above equation implies that z_s defined by (115) can be interpreted as an unnormalized version of $\hat{u}_s$. Combining (115) and (92), one obtains:

$$z_s = \left[VV^+ - VC_{s-1}^+ \left(VC_{s-1}^+ \right)^+ \right] u_s \qquad , \ s = 2, \cdots, r.$$ (118)

Using (89), (103) and (105), it follows that:

$$VC_{s-1}^+ = V \begin{pmatrix} q_1 & \cdots & q_{s-1} \end{pmatrix} = \begin{pmatrix} \hat{u}_1 & \cdots & \hat{u}_{s-1} \end{pmatrix} = \hat{U}_{s-1} .$$ (119)

Substituting the above equation in (118) and utilizing (107) and (108), one gets:

$$z_s = \left(VV^+ - \hat{U}_{s-1} \hat{U}_{s-1}^+ \right) u_s = \left(A - B_{s-1} \right) u_s \qquad , \ s = 2, \cdots, r.$$ (120)

In the special case of $s = 1$, it follows from (115), (93) and (107) that:

$$z_1 = Vx_1 = VV^+ u_1 = Au_1 .$$ (121)

(C) The identity

From (107) and (108), it is obvious that matrix A is the same for both the GSA and the SOPA since it is an input matrix while the matrices B_{s-1}, $s = 2, \cdots, r$ are computed for each algorithm separately. From (114) and (121) it is clear that $y_1 = z_1$; and from (56) and (117) it follows that $\hat{u}_1$ is the same for both algorithms. By virtue of (105) and (108), one concludes that $\hat{U}_1$ and B_1 are identical for both algorithms. Equations (113) and (120) imply that $y_2 = z_2$ and consequently $\hat{u}_2$ are the same for both algorithms. Proceeding in the same

manner, one concludes that both the GSA and SOPA result in exactly the same set of vectors $\hat{u}_s$, $s = 1, \cdots, r$.

VI. SIMULATION RESULTS

Orthonormal eigenvectors of the DFT matrix F have been computed using the following three techniques:

1) The P method where one only obtains *initial* orthonormal vectors by the singular value decomposition of the projection matrices P_k, $k = 1, \cdots, 4$ of F according to (47) as was explained in section III.

2) The Orthogonal Procrustes Algorithm (OPA) explained in section IV (B).

3) The Sequential Orthogonal Procrustes Algorithm (SOPA) explained in section IV (C).

It should be mentioned that even if one intends to apply a refined technique such as the second or third one, he should start by applying the first technique in order to generate *initial* orthonormal eigenvectors of F to be used as input to the advanced technique. Since the main goal is to generate Hermite-Gaussian like orthonormal eigenvectors of F, the error vectors between the eigenvectors of F and samples of the Hermite-Gaussian functions of the same order have been computed as it was done in [5]. The norms of these error vectors have been plotted versus k for $k = 1, \cdots, N$ where k denotes the columns of the modal matrix of F. Figures 1 and 2 show the results for $N = 64$ and 128 respectively. Since in the P method the act of approximating the Hermite-Gaussian functions was not taken into account, it is quite expected that the P method has the largest error among the three techniques being compared. For the OPA or SOPA the error tends to increase on the average with k. The interpretation is that the samples of the Hermite-Gaussian functions are approximate eigenvectors of F with an approximation error that grows with the order of those functions [4,5]. Consequently the error between the exact Hermite-Gaussian like eigenvectors determined by the OPA or SOPA and the approximate eigenvectors tends to increase with k. Upon comparing the OPA and SOPA, one clearly notices that the OPA has a lower rate of growth of the error. The interpretation is that for the SOPA the number of linear constraints expressed by (88) and (89) increases with the order thus restricting the freedom left in the solution space for minimizing the criterion J_s given by (84). This is to be contrasted to the OPA where matrix $\hat{U}$ rather than its individual columns is batch evaluated. On the other hand, the SOPA has the merit that the error begins to be noticeable at a value of k larger than that for the OPA.

VII. CONCLUSION

A new technique has been developed for generating *initial* orthonormal eigenvectors of the Discrete Fourier Transform matrix F based on the singular value decomposition of the projection matrices of F on its eigenspaces after deriving efficiently computable expressions for these projection matrices. In order to generate Hermite-Gaussian like eigenvectors of F given the *initial* ones, a new method called the Sequential Orthogonal Procrustes Algorithm (SOPA) has been proposed based on the sequential evaluation of the columns of a unitary matrix rather than the batch evaluation of that matrix as in the Orthogonal Procrustes Algorithm (OPA). Surprisingly the output of the SOPA has been proved to be equal to that of the Gram-Schmidt Algorithm (GSA). Furthermore It has been proved that for any of the GSA, OPA or SOPA, the output is invariant under the change of the input *initial* orthonormal eigenvectors of F .

APPENDIX A

Proof of Lemma 1

The modal decomposition of a Hermitian matrix A is:

$$A = V\Lambda V^+ \tag{A1}$$

where

$$\Lambda = Diag\{\lambda_1, \cdots, \lambda_N\} \tag{A2}$$

and all eigenvalues λ_n, $n = 1, \cdots, N$ are real [15]. The diagonal matrix Λ can be expressed as:

$$\Lambda = S\Sigma \tag{A3}$$

where the real diagonal matrices Σ and S are defined by (39)-(41). Substituting (A3) in (A1), one gets:

$$A = (V\,S)\Sigma V^+ \ . \tag{A4}$$

By comparing (36) and (A4), one obtains (38).

(Q.E.D.)

APPENDIX B

Statement of the problem:

Find the r-dimensional vector x that minimizes:

$$J = \|u - Vx\|_2^2 \tag{B1}$$

subject to the constraints:

$$Cx = 0 \tag{B2}$$

where u is an N-dimensional vector, V is an $N \times r$ matrix with linearly independent columns, C is an $(s-1) \times r$ matrix with linearly independent rows and $(s-1) < r < N$.

Solution:

Augmenting the constraints (B2) to the criterion (B1) by means of the complex vector λ of Lagrange multipliers, one gets the following real augmented criterion:

$$J_a = J - \lambda^+ Cx - (Cx)^+ \lambda \ . \tag{B3}$$

By virtue of the definition of the Euclidean norm, one obtains from (B1) and (B3):

$$J_a = u^+ u + x^+ V^+ Vx - u^+ Vx - x^+ V^+ u - \lambda^+ Cx - x^+ C^+ \lambda \ . \tag{B4}$$

Since J_a is a real-valued scalar function of the complex vector x and its complex conjugate x^*, a necessary and sufficient condition for minimization is:

$$\nabla_{x^*} J_a = 0 \tag{B5}$$

where in finding the gradient vector, one should view x and x^* as two different vectors, i.e. one should treat x as a constant vector when evaluating ∇_{x^*} [19]. Consequently it follows from (B4) that:

$$\nabla_{x^*} J_a = V^+ Vx - V^+ u - C^+ \lambda \ . \tag{B6}$$

Upon applying condition (B5), one gets:

$$x = (V^+ V)^{-1} (V^+ u + C^+ \lambda) \ . \tag{B7}$$

By applying condition (B2) in order to evaluate vector λ, one obtains:

$$\lambda = -\left[C(V^+ V)^{-1} C^+ \right]^{-1} C(V^+ V)^{-1} V^+ u \ . \tag{B8}$$

Substituting (B8) in (B7), one obtains:

$$x = (V^+ V)^{-1} \left\{ I - C^+ \left[C(V^+ V)^{-1} C^+ \right]^{-1} C(V^+ V)^{-1} \right\} V^+ u \ . \tag{B9}$$

REFERENCES

[1] V. Namias, "The fractional order Fourier transform and its application to quantum mechanics," *J. Inst. Math. Appl.*, vol. 25, pp. 241-265, 1980.

[2] A.C. McBride and F.H. Kerr, "On Namias's fractional Fourier transforms," *IMA J. Appl. Math.*, vol. 39, pp. 159-175, 1987.

[3] L.B. Almeida, "The fractional Fourier transform and time-frequency representations," *IEEE transactions on Signal Processing*, vol. SP-42, pp. 3084-3091, Nov 1994.

[4] S.-C. Pei, C.-C. Tseng and M.-H. Yeh, "A new discrete fractional Fourier transform based on constrained eigendecomposition of DFT matrix by Lagrange multiplier method," *IEEE Transactions on Circuits and Systems, Part II*, vol. 46, pp. 1240-1245, September 1999.

[5] S.-C. Pei, M.-H. Yeh and C.-C. Tseng, "Discrete fractional Fourier transform based on orthogonal projections," *IEEE Transactions on Signal Processing*, vol. SP-47, pp. 1335-1348, May 1999.

[6] Ç. Candan, M.A. Kutay and H.M. Ozaktas, "The discrete fractional Fourier transform," *IEEE Transactions on Signal Processing*, vol. SP-48, pp. 1329-1337, May 2000.

[7] Ç. Candan, "The discrete fractional Fourier transform," M.S. Thesis, Bilkent University, Ankara, Turkey, 1998.

[8] Haldun M. Ozaktas, Zeev Zalevsky and M. Alper Kutay, The Fractional Fourier Transform with Applications in Optics and Signal Processing, Chichester, England: John Wiley, 2001.

[9] H. Dym and H.P. McKean, *Fourier Series and Integrals*, San Diego, California: Academic Press, 1972.

[10] B.W. Dickinson and K. Steiglitz, "Eigenvectors and functions of the discrete Fourier transform," *IEEE Transactions on Acoustics, Speech and Signal Processing*, vol. ASSP-30, pp. 25-31, February 1982.

[11] F.R. Gantmacher, *The Theory of Matrices*, vol. 1, 2nd Edition, New York: Chelsea, 1990.

[12] J.H. McClellan and T.W. Parks, "Eigenvalue and eigenvector decomposition of the discrete Fourier transform," *IEEE Transactions on Audio and Electroacoustics*, vol. AU-20, pp. 66-74, March 1972.

[13] L. Auslander and R. Tolimieri, "Is computing with the finite Fourier transform pure or applied mathematics," *Bulletin of the American Mathematical Society*, vol. 1, no. 6, pp. 847-897, November 1979.

[14] G.H. Golub and C.F. Van Loan, *Matrix Computations*, Baltimore, M.D.: Johns Hopkins University Press, 1989.

[15] G. Strang, *Linear Algebra and its Applications*, 3rd Edition, San Diego: Harcourt Brace Jovanovich Publishers, 1988.

[16] S.H. Friedberg, A.J. Insel and L.E. Spence, *Linear Algebra*, Englewood Cliffs, N.J.: Prentice-Hall, 1979.

[17] Z.I. Borevich and I.R. Shafarevich, *Number Theory*, English translation by N. Greenleaf, New York, N.Y.: Academic Press, 1966.

[18] G.W. Stewart, *Introduction of Matrix Computations*, New York: Academic Press, 1973.

[19] D.H. Brandwood, "A complex gradient operator and its application in adaptive array theory," *IEE Proceedings, vol. 130, Parts F and H*, no. 1, pp. 11-16, February 1983.

Table 1: The dimensions r_k of the four eigenspaces

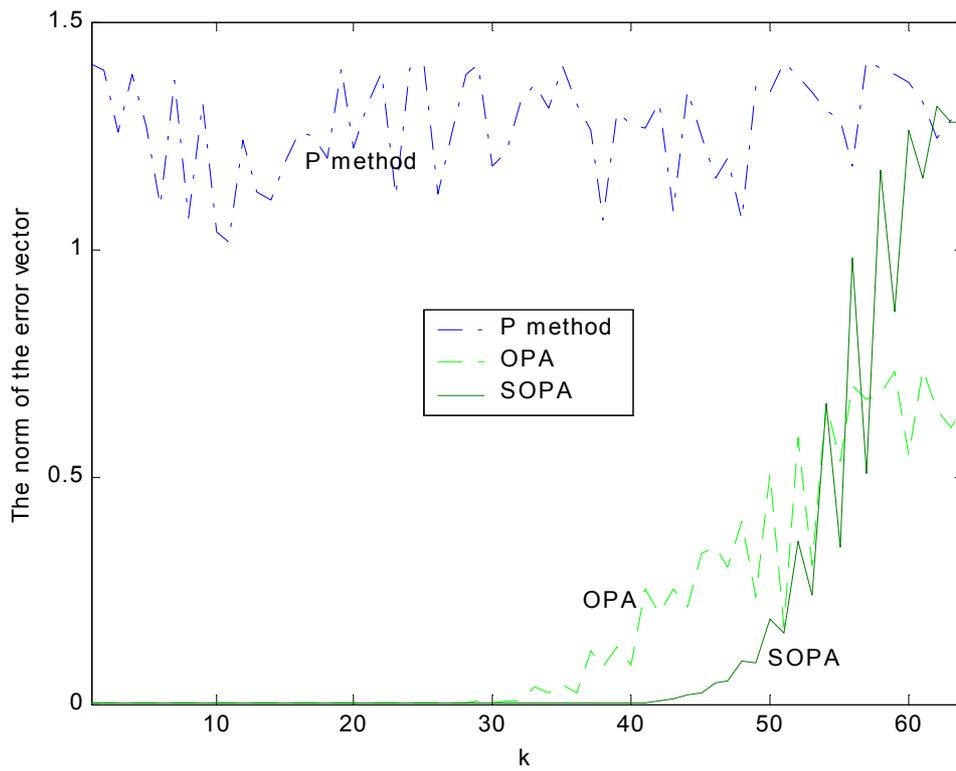| k		1	2	3	4
λ_k		1	-j	-1	j
N	4m	m+1	m	m	m-1
	4m+1	m+1	m	m	m
	4m+2	m+1	m	m+1	m
	4m+3	m+1	m+1	m+1	m

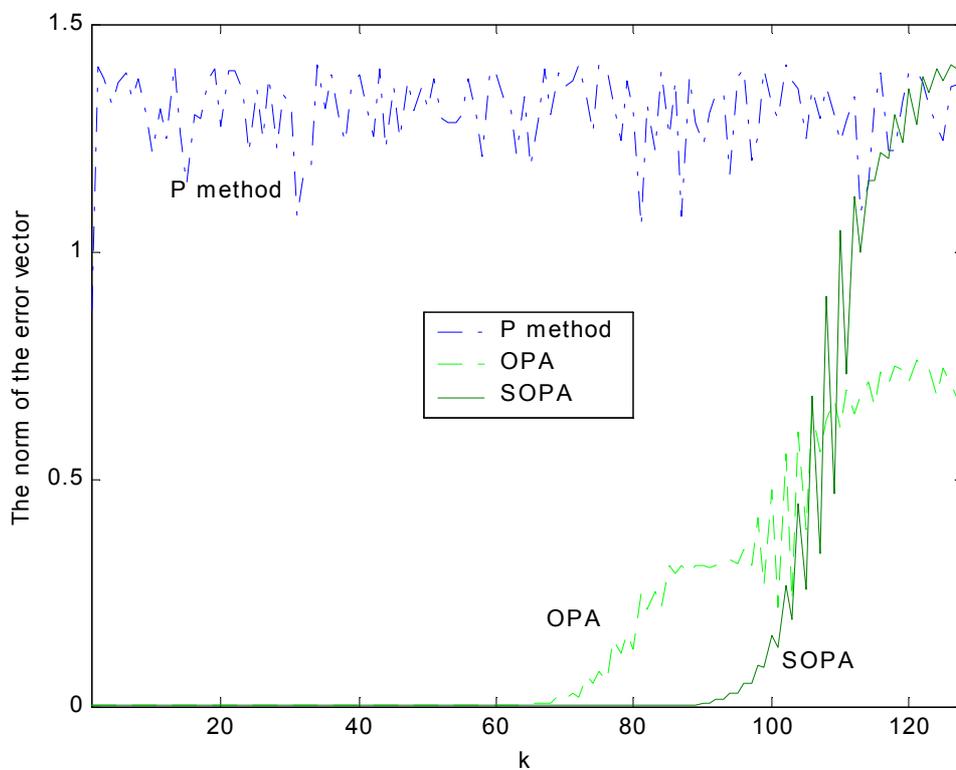Fig. 1: Norm of the error vectors between the exact and approximate eigenvectors for $N = 64$.

Fig. 2: Norm of the error vectors between the exact and approximate eigenvectors for $N = 128$.

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