On the Stability of a Tank and Hopfield Type Neural Network in the General Case of Complex Eigenvalues[1]

By

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Abstract

The stability of a Tank and Hopfield type neural network is investigated for the general case of practically encountered complex eigenvalues s_D of the matrix product $D_g{}^T D_f$ where D_g and D_f are approximations to the connection matrix D on the signal and constraint sides of the neural net respectively. A stability criterion in the form of an analytic expression is derived thus generalizing the results obtained by Yan [8] for the special case of purely real eigenvalues.

I. Introduction

Culhane, Peckerar and Marrian presented an electric circuit for computing the Discrete Hartley Transform (DHT) and Discrete Fourier Transform (DFT) [1]. This circuit shown in Fig. 1 is a modified Tank and Hopfield linear programming neural network [2] and has the nice feature of computing the DHT and DFT within RC time constants of the order of nanoseconds. Several variants of this circuit model have been used for solving linear and nonlinear programming problems [3-7]. With some modifications, the circuit can be used for solving linear Least Squares Error (LSE) problems [8]. The neural net of Fig. 1 has signal and constraint sides denoted by the subscripts g and f respectively. D_g and D_f are M x N interconnection conductance matrices, $\tau_g = R_g C_g$ and $\tau_f = R_f C_f$ are the relaxation time constants; u_g, v_g and α_g are respectively the input, output and gain of an operational amplifier on the signal side; u_f, v_f and α_f are their counterparts on the constraint side. Culhane, Peckerar and Marrian [1] proved the stability of the circuit under the two assumptions that : $\tau_f << \tau_g$ and $D_g{}^T D_f$ is positive definite. Yan [8] showed that the first assumption is not necessary and the second assumption can be relaxed to :

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$$s_D > \frac{-1}{\left(\alpha_g R_g\right)\left(\alpha_f R_f\right)} \tag{1}$$

where s_D is an eigenvalue of $D_g{}^T D_f$. However criterion (1) was derived under the assumption that all the eigenvalues s_D are purely real. The main objective of this paper is to generalize the stability condition (1) to the case of complex eigenvalues s_D of the matrix product $D_g{}^T D_f$, which will be done in section II. In section III the results of Yan for the computational speed of the neural net will be correspondingly generalized. In section IV an illustrative example will be presented where the matrices D_g and D_f are approximations to the matrix D of the Discrete Hartley Transform in the nonideal case and where the eigenvalues s_D of $D_g{}^T D_f$ will turn out to be complex thus demonstrating the importance of treating the general case of complex eigenvalues presented in this paper.

II. The Stability Criterion

Applying Kirchhoff's current law at the input node of the i-th amplifier on the constraint side of the neural net of Fig. 1, we get :

$$C_f \frac{du_{f_i}}{dt} + \frac{1}{R_f} u_{f_i} = -b_i + \sum_{j=0}^{N-1} D_{f_{ij}} v_{g_j} \qquad 0 \le i \le M-1 \tag{2a}$$

The counterpart equation for the signal side is :

$$C_g \frac{du_{g_i}}{dt} + \frac{1}{R_g} u_{g_i} = -a_i - \sum_{j=0}^{M-1} D_{g_{ji}} v_{f_j} \qquad 0 \le i \le N-1 \tag{2b}$$

Using the input-output relation of the operational amplifiers :

$$v_{f_i} = \alpha_f u_{f_i} \tag{3a}$$

$$v_{g_i} = \alpha_g u_{g_i} \tag{3b}$$

and writing the resulting equations in vector-matrix form, we get the following state space description of the neural net :

$$\frac{d}{dt}\begin{pmatrix} u_f \\ u_g \end{pmatrix} = \Phi \begin{pmatrix} u_f \\ u_g \end{pmatrix} + \begin{pmatrix} -\frac{1}{C_f} b \\ -\frac{1}{C_g} a \end{pmatrix} \tag{4}$$

where the square matrix Φ of order (M+N) is given by :

$$\Phi = \begin{pmatrix} -\frac{1}{\tau_f} I_M & \frac{\alpha_g}{C_f} D_f \\ -\frac{\alpha_f}{C_g} D_g{}^T & -\frac{1}{\tau_g} I_N \end{pmatrix}. \tag{5}$$

In (4) u_f and b are M-dimensional vectors, u_g and a are N-dimensional vectors; and in (5) $\tau_f = R_f C_f$ and $\tau_g = R_g C_g$. Using the formula for the determinant of a partitioned matrix [9] :

$$\begin{vmatrix} A & D \\ C & B \end{vmatrix} = |A||B - CA^{-1}D| \tag{6a}$$

$$\begin{vmatrix} A & D \\ C & B \end{vmatrix} = |B||A - DB^{-1}C| \tag{6b}$$

we get :

$$|sI - \Phi| = \left(s + \frac{1}{\tau_f}\right)^{M-N} \left| \left(s + \frac{1}{\tau_g}\right)\left(s + \frac{1}{\tau_f}\right) I_N + \frac{\alpha_f \alpha_g}{C_f C_g} D_g{}^T D_f \right| \tag{7a}$$

$$|sI - \Phi| = \left(s + \frac{1}{\tau_g}\right)^{N-M} \left| \left(s + \frac{1}{\tau_g}\right)\left(s + \frac{1}{\tau_f}\right) I_M + \frac{\alpha_f \alpha_g}{C_f C_g} D_f D_g{}^T \right| \ . \tag{7b}$$

If M > N, Eq. (7a) implies that the net will have (M - N) poles at :

$$s = -\frac{1}{\tau_f} \tag{8}$$

and 2N poles at the roots of :

$$\left| \left(s + \frac{1}{\tau_g}\right)\left(s + \frac{1}{\tau_f}\right) I_N + \frac{\alpha_f \alpha_g}{C_f C_g} D_g{}^T D_f \right| = 0 \tag{9}$$

On the other hand if N > M, Eq. (7b) implies that the net will have (N - M) poles at :

$$s = -\frac{1}{\tau_g} \tag{10}$$

and 2M poles at the roots of :

$$\left| \left(s + \frac{1}{\tau_g}\right)\left(s + \frac{1}{\tau_f}\right) I_M + \frac{\alpha_f \alpha_g}{C_f C_g} D_f D_g{}^T \right| = 0 \ . \tag{11}$$

Defining :

s_{D_1} = an eigenvalue of $D_g{}^T D_f$, s_{D_2} = an eigenvalue of $D_f D_g{}^T$, and

$$s_D = \begin{cases} s_{D_1} & \text{if } M > N \\ s_{D_2} & \text{if } N > M \end{cases} \tag{12}$$

we find from (9) and (11) that each eigenvalue s_D will result in a pair of poles of the net determined by solving the quadratic equation :

$$\left(s + \frac{1}{\tau_g}\right)\left(s + \frac{1}{\tau_f}\right) + \frac{\alpha_f \alpha_g}{C_f C_g} s_D = 0 \ . \tag{13}$$

In the special case of square matrices D_g and D_f , i.e. when M = N, all poles of the net will be determined from (13) where $s_D = s_{D_1} = s_{D_2}$ since $D_g{}^T D_f$ and $D_f D_g{}^T$ will have the same set of eigenvalues [9].

The roots of the quadratic equation (13) are given by :

$$s = -0.5\left(\frac{1}{\tau_f} + \frac{1}{\tau_g}\right) \pm 0.5\sqrt{\left(\frac{1}{\tau_f} - \frac{1}{\tau_g}\right)^2 - 4\frac{\alpha_f \alpha_g}{C_f C_g} s_D} \quad . \tag{14}$$

Instead of making the assumption of Yan that s_D is purely real [8], we will treat the general case of complex s_D , i.e.

$$s_D = x + j\,y \tag{15}$$

Hence

$$\left(\frac{1}{\tau_f} - \frac{1}{\tau_g}\right)^2 - 4\frac{\alpha_f \alpha_g}{C_f C_g} s_D = z_r + j\,z_i \tag{16}$$

where

$$z_r = \left(\frac{1}{\tau_f} - \frac{1}{\tau_g}\right)^2 - 4\frac{\alpha_f \alpha_g}{C_f C_g} x \ , \tag{17a}$$

$$z_i = -4\frac{\alpha_f \alpha_g}{C_f C_g} y \ . \tag{17b}$$

We will use the algebraic fact that :

$$\sqrt{z_r + j\,z_i} = \alpha + j\,\beta \tag{18}$$

where

$$\alpha^2 = 0.5\left(z_r + \sqrt{z_r{}^2 + z_i{}^2}\right) \ , \tag{19a}$$

$$\beta^2 = 0.5\left(-z_r + \sqrt{z_r{}^2 + z_i{}^2}\right) \tag{19b}$$

and where the signs of α and β may be similar or not as determined by the algebraic identity :

$$z_i = 2\alpha\beta \ . \tag{20}$$

Since our objective is to investigate the stability of the neural net, we will consider only the real part of the poles. From (14),(16),(18) and (19a) we find that :

$$\mathrm{Re}(s) = -\frac{1}{2}\left(\frac{1}{\tau_f} + \frac{1}{\tau_g}\right) + \frac{1}{2\sqrt{2}}\sqrt{z_r + \sqrt{z_r{}^2 + z_i{}^2}} \tag{21}$$

where we only considered positive α since negative α is guaranteed to result in nagative Re(s). The stability condition : Re(s) < 0 results in :

$$z_r + \sqrt{z_r^{\ 2} + z_i^{\ 2}} < 2\left(\frac{1}{\tau_f} + \frac{1}{\tau_g}\right)^2 \ . \qquad (22)$$

Using (17a), the above inequality reduces to :

$$z_r^{\ 2} + z_i^{\ 2} < \left(\frac{1}{\tau_f^{\ 2}} + \frac{1}{\tau_g^{\ 2}} + \frac{6}{\tau_f \tau_g}\right)^2 + 8\frac{\alpha_f \alpha_g}{C_f C_g}\left(\frac{1}{\tau_f^{\ 2}} + \frac{1}{\tau_g^{\ 2}} + \frac{6}{\tau_f \tau_g}\right)x + \left(4\frac{\alpha_f \alpha_g}{C_f C_g}\right)^2 x^2 . \qquad (23)$$

Substituting (17a) and (17b) into the above inequality we get after some algebraic manipulation :

$$x > \left(\alpha_f R_f\right)\left(\alpha_g R_g\right)\frac{\tau_f \tau_g}{\left(\tau_f + \tau_g\right)^2} y^2 - \frac{1}{\left(\alpha_f R_f\right)\left(\alpha_g R_g\right)} \ . \qquad (24)$$

This inequality which should be satisfied by the real and imaginary parts of the eigenvalues s_D of (15) of the matrix product $D_g^{\ T} D_f$ or $D_f D_g^{\ T}$ is the sole stability condition for the neural network of Fig. 1. In the special case of purely real s_D , the above analytic result reduces to the inequality (1), which is the special result obtained earlier by Yan [8]. We should notice that in the general case of complex s_D , the stability condition (24) depends on the input capacitances C_f and C_g through the time constants τ_f and τ_g .

Yan [8] showed that the neural net of Fig. 1 can be used for approximating the solution of the Least Squares Error (LSE) problem, and that the accuracy of the computation is determined by the condition :

$$\left|s_D\right|_{\min} >> \frac{1}{\left(\alpha_f R_f\right)\left(\alpha_g R_g\right)} \qquad (25)$$

where no assumption of a real s_D was required. Fig. 2 visually depicts the relation between the stability condition (24) and the accuracy condition (25). The stable region in the complex plane is enclosed by a parabola while the high accuracy region lies outside a circle centered at the origin. The parabola encloses the circle and both are tangential at the point :

$$s_D = -\frac{1}{\left(\alpha_f R_f\right)\left(\alpha_g R_g\right)} \ . \qquad (26)$$

In order to attain a high accuracy without violating the stability condition, the eigenvalues s_D must be inside the parabola and as far to the right as possible from the circle.

III. Computational Speed

The computational speed of the neural network is determined by its response time which is determined by the pole having the smallest magnitude of the real part. For a stable neural net, i.e. one with Re(s) < 0 , we find from (21) and (17) that :

$$\left|\mathrm{Re}(s)\right| = \frac{1}{2}\left(\frac{1}{\tau_f} + \frac{1}{\tau_g}\right) - \frac{1}{2\sqrt{2}}\sqrt{f(x,y)} \qquad (27)$$

where

$$f(x,y) = a - bx + \sqrt{(a-bx)^2 + (by)^2} \;, \tag{28}$$

$$a = \left(\frac{1}{\tau_f} - \frac{1}{\tau_g}\right)^2 \;, \tag{29}$$

$$b = 4\frac{\alpha_f \alpha_g}{C_f C_g} \;. \tag{30}$$

From (27), we find that :

$$\left|\mathrm{Re}(s)\right|_{\min} = \frac{1}{2}\left(\frac{1}{\tau_f} + \frac{1}{\tau_g}\right) - \frac{1}{2\sqrt{2}}\sqrt{f(x,y)_{\max}} \tag{31}$$

Since some of the poles of the neural net can be at $s = -\frac{1}{\tau_f}$ or $s = -\frac{1}{\tau_g}$ as was found earlier in (8) and (10), we conclude that the minimum decay rate is given by :

$$d_{\min} = \begin{cases} \min\left\{\frac{1}{\tau_f} \;,\; \left|\mathrm{Re}(s)\right|_{\min}\right\} & \text{if M } > \text{N} \\ \left|\mathrm{Re}(s)\right|_{\min} & \text{if M } = \text{ N} \\ \min\left\{\frac{1}{\tau_g} \;,\; \left|\mathrm{Re}(s)\right|_{\min}\right\} & \text{if M} < \text{ N} \end{cases} \tag{32}$$

Since x and y are the real and imaginary parts of the eigenvalues s_D of the matrix product $D_g{}^T D_f$ or $D_f D_g{}^T$, the only way for finding $f(x,y)_{\max}$ and hence $\left|\mathrm{Re}(s)\right|_{\min}$ and $d_{\min}$ is by enumeration.

IV. An Illustrative Example

We consider the neural network for computing the Discrete Hartley Transform [1], where in the ideal case the elements of the symmetric square connection matrix D are given by :

$$D_{ij} = cas\left(\frac{2\pi ij}{N}\right) \qquad \text{i, j } = 0, \ldots, \text{N - 1} \tag{33}$$

where

$$cas(\phi) = \cos(\phi) + \sin(\phi) \tag{34}$$

The matrices D_g and D_f are approximations to the ideal matrix D; departure from idealization occurs when a neural net is fabricated on an integrated circuit. The elements of D_g and D_f are obtained from those of D through multiplication by the factor $[1 + rand(\varepsilon)]$ where $rand(\varepsilon)$ is a uniformly distributed random number whose value lies in the range : $0 < rand(\varepsilon) < \varepsilon$. We have taken N = 8 , $R_f = R_g = 10^3\ \Omega$, $C_f = 10^{-15}$ F , $C_g = 10^{-12}$ F , $\alpha_f = \alpha_g = 1$. For these values of the parameters the general criterion (24) and the special criterion (1) reduce to :

$$x \;>\; 998.003y^2 - 10^{-6} \tag{35}$$

and

$$s_D > -10^{-6} \qquad (36)$$

respectively.

In the ideal case of no perturbation ($\varepsilon = 0$) we have $D_f = D_g = D$ and ${D_g}^T D_f = D^2$ is a symmetric matrix whose eigenvalues are real and equal to N [1]. The corresponding minimum decay rate as determined by (32) and (31) is : $d_{min} = 5.005 \times 10^{11}\ sec^{-1}$.

In the nonideal case ($\varepsilon \neq 0$) , the matrices D_f , D_g and ${D_g}^T D_f$ are nonsymmetric. For 1 % perturbation ($\varepsilon = 0.01$) the computed eigenvalues of ${D_g}^T D_f$ are :

8.0377 , 8.0756 , 8.0928 ± j0.0205 , 8.0700 ± j0.0246 , 8.0794 ±j0.0110

and the corresponding minimum decay rate is : $d_{min} = 3.6365 \times 10^{11}\ sec^{-1}$.

Stability criterion (35) shows that the neural network is still stable although its computational speed (as determined by d_{min}) is lower than that for the ideal case.

For 4 % perturbation in the values of the elements of matrices D_g and D_f , the computed eigenvalues of the matrix product ${D_g}^T D_f$ are :

8.1512 , 8.3040 , 8.3200 ± j0.0448 , 8.3744 ± j0.0832 , 8.2814 ± j0.0997 .

Applying stability criterion (35), we find that the neural network becomes unstable because of the last pair of complex conjugate eigenvalues. It is true that the eigenvalues are dominated by the real part but the stability has been largely affected by the small imaginary part.

V. Conclusion

The stability of a Tank and Hopfield type neural network has been investigated and found to depend substantially on the location of the eigenvalues s_D of the matrix product ${D_g}^T D_f$ (or $D_f {D_g}^T$) where D_g and D_f are respectively the interconnection conductance matrices on the signal and constraint sides of the neural net. The general case of complex eigenvalues s_D was treated and an analytic expression for checking the stability of the neural net was derived thus extending the results of Yan [8] who treated the case of purely real eigenvalues s_D . The issues of the computational speed of the neural net and its accuracy of computation were also considered. An illustrative example of a neural net for computing the Discrete Hartley Transform was presented where the eigenvalues of the matrix product ${D_g}^T D_f$ were computed to be complex thus demonstrating the significance of treating the general case presented in this paper.

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Figure Captions

Fig. 1 : A Tank and Hopfield type neural network.

Fig. 2 : The stability and high accuracy regions.