Physica Scripta. Vol. 53, 714–719, 1996

On the Validity of Ideal MHD Equations for Different Collisional Regimes

J. W. Edenstrasser and M. M. M. Kassab*

Institute for Theoretical Physics, University of Innsbruck, Austria

Received October 13, 1995; accepted in revised form December 1, 1995

Abstract

This paper is devoted to clarify the discrepancy between the experimentally found, rather wide and theoretically predicted, very restricted range of validity of the ideal MHD (IMHD) equations. In the first part of the paper, the standard derivation for a collision-dominated plasma is critically reviewed, leading to even stronger limitations, than those known in the literature. In the second part of the paper, the validity range of the IMHD equations on the Alfvén timescale is, based on a multiple timescale approach, analysed for different collisional regimes. It turns out that the IMHD equations are unrestricted valid for a (strongly magnetized) weak-collisional plasma. For an intermediate- and high-collisional plasma, the adiabaticity is violated essentially due to the work done by the friction forces. However, for both collisional regimes the full set of IMHD equations is obtained for the slow-flowing case, i.e., for plasmas with a magnetic Mach number of the relative velocity $|\boldsymbol{u}_i - \boldsymbol{u}_e|/v_A \ll 1$.

1. Introduction

The development of magnetic fusion as a source of electricity requires the solution of a number of challenging physics as well as technological problems. The physics problems are traditionally separated into three basic areas: equilibrium and stability, heating, and transport. The goal is the discovery of magnetic geometries which are capable of stably confining a sufficiently high density plasma at a sufficiently high temperature for a sufficiently long time to produce net thermonuclear power.

Ideal magnetohydrodynamics (IMHD) is the most basic single-fluid model for determining the macroscopic equilibrium and stability properties of a magnetized plasma. The model describes how magnetic, inertial, and pressure forces interact within an ideal, perfectly conducting plasma in an arbitrary magnetic geometry. There is a general consensus that any configuration of a magnetically confined fusion plasma must satisfy the equilibrium and stability limits set by IMHD. If not, catastrophic termination of the plasma on a very short timescale (compared to experimental times) is the usual consequence [1]. Thus the role of IMHD in magnetic fusion is to determine the magnetic geometry which possesses attractive equilibrium and stability properties for fusion reactors.

In the literature (cf. e.g. Refs [1]–[4]), it is claimed that the basic requirement for the validity of the IMHD is that both the electrons and ions are *collision dominated*. However, this condition cannot be satisfied for plasmas of fusion interest. It is worth pointing out that other models also exist attempting to improve the reliability of IMHD. One such model is the guiding centre fluid model (the double adiabatic theory) in which the pressure is allowed to be anisotropic and its validity conditions should be questioned in the collisionless regime where the behaviour is not fluid-like along the field lines. The second model is the guiding centre plasma model. Here the perpendicular motion is fluid-like, while a one dimensional kinetic equation governs the parallel motion. These models, however, are in general far more difficult to handle than IMHD. The third model is the collisionless MHD model which represents the simplifying limit of the latter one for incompressible motions [1].

Contrary to the statements to be found in the literature of the last decades (cf e.g. Refs [1]–[4]), in Ref. [5] the IMHD equations have been obtained, on the basis of a multiple timescale approach, for the Alfvén timescale for the case of a high-temperature, *weak-collisional fusion plasma* (WCR), i.e. for the case where the particle transit time ω_α^{-1} is much shorter than the particle collision time ν_α^{-1}. This result, in fact, paved the way for further work on the validity range of the IMHD model. As a natural extension of this work, the plasma transport equations have been derived for the two other limiting regimes, namely, for the intermediate-collisional (ICR) (i.e. $\omega_\alpha^{-1} \approx \nu_\alpha^{-1}$), and the high-collisional (HCR) (i.e. $\omega_\alpha^{-1} \gg \nu_\alpha^{-1}$) ones [6]. The main object of this paper is to investigate the conditions under which the transport equations on the Alfvén timescale become for different collisional regimes identical to those of IMHD.

This paper is organized according to the following scheme. In Section 2, the standard validity conditions are critically reviewed. In Section 3, the validity conditions of IMHD are investigated for the weak- (WCR), the intermediate- (ICR), and the high-collisional (HCR) regimes. In Section 4, a brief summary and the relevant conclusions are presented.

2. Review of the standard IMHD validity conditions

In a huge amount of literature (cf. e.g. Refs [1]–[4]), it is stated that the basic requirement for the validity of the IMHD model is that both the electrons and ions are collision-dominated. This, however, would imply that the IMHD model is not applicable for the case of a high-temperature, weak-collisional fusion plasma. But on the other hand, there is an overwhelming experimental evidence that the equilibrium and gross stability behaviour of fusion plasmas is governed by the IMHD equations. This is consistent with the results of a multiple timescale expansion scheme applied to a weak-collisional fusion plasma [5], where on the Alfvén timescale the IMHD equations have

* On leave from the Dept. of Eng. Physics, Cairo University (Fayoum branch), Egypt.

been obtained. This fact has motivated us to analyse the standard IMHD validity conditions and compare them with the results obtained from the multiple timescale approach.

Starting from the standard two-fluid plasma transport equations (cf., e.g. Refs [1]–[4]), one arrives under some simplifying assumptions at the single-fluid equations in dimensional form:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0, \tag{1a}$$

$$\rho \frac{\partial \boldsymbol{u}}{\partial t} + \rho(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p - \boldsymbol{j} \times \boldsymbol{B} = -\nabla \cdot \boldsymbol{\Pi}, \tag{1b}$$

$$\frac{3}{2}\rho^{\gamma}\frac{\mathrm{d}}{\mathrm{d}t}\{\rho^{-\gamma}p_e\} = \left\{Q_e + \frac{5}{2}\frac{\boldsymbol{j}}{e}\cdot\nabla T_e - \frac{\boldsymbol{j}}{en}\cdot\nabla p_e - \boldsymbol{\Pi}_e{:}\nabla\left(\boldsymbol{u} - \frac{\boldsymbol{j}}{en}\right) - \nabla\cdot\boldsymbol{q}_e\right\}, \tag{1c}$$

$$\frac{3}{2}\rho^{\gamma}\frac{\mathrm{d}}{\mathrm{d}t}\{\rho^{-\gamma}p_i\} = \{Q_i - \boldsymbol{\Pi}_i{:}\nabla\boldsymbol{u} - \nabla\cdot\boldsymbol{q}_i\}, \tag{1d}$$

where the total pressure tensor $\mathbf{P}$ is defined by $\mathbf{P} = p\mathbf{I} + \boldsymbol{\Pi} = (p_e + p_i)\mathbf{I} + (\boldsymbol{\Pi}_e + \boldsymbol{\Pi}_i)$. Furthermore, the other higher order moments $\boldsymbol{R}_\alpha$, $\boldsymbol{q}_\alpha$, and Q_α (α refers to particle species) represent the friction force, the particle heat flux vector and the collisional energy exchange, respectively. By employing the momentum equations for both the electrons and ions, one further obtains the following generalized Ohm's law (cf., Refs [3] and [4]):

$$e^2 n\left\{\frac{m_e + m_i}{m_e m_i}\right\}\{\boldsymbol{E} + \boldsymbol{u}\times\boldsymbol{B}\} = \frac{\partial \boldsymbol{j}}{\partial t} + e\nabla\cdot\left\{n\left(\boldsymbol{u}\otimes\frac{\boldsymbol{j}}{en}\right) + n\left(\frac{\boldsymbol{j}}{en}\otimes\boldsymbol{u}\right) - n\left(\frac{\boldsymbol{j}}{en}\otimes\frac{\boldsymbol{j}}{en}\right)\right\} + \frac{e}{m_e}\left\{\left(\frac{m_e}{m_i}\right)\nabla\cdot\mathbf{P}_i - \nabla\cdot\mathbf{P}_e\right\} - \frac{e}{m_e}\boldsymbol{j}\times\boldsymbol{B} + e\left\{\frac{m_e + m_i}{m_e m_i}\right\}\boldsymbol{R}_e. \tag{2}$$

Under certain conditions the terms on the r.h.s. of eqs (1b–d) and of Ohm's law (2) may be neglected, so that one ends up with the IMHD equations. In order to find out the necessary conditions under which the r.h.s.-terms can be neglected, we first bring these equations into a dimensionless form. The dimensionless physical quantities $\bar{S}$ are obtained by normalizing the physical quantities S with respect to some characteristic values, e.g.,

$$\bar{t} = t/\tau_A,\ \bar{\boldsymbol{u}} = \boldsymbol{u}/v_A,\ \bar{T}_\alpha = 2T_\alpha/m_\alpha v_{th}^{\alpha 2},$$
$$\bar{\boldsymbol{E}} = \boldsymbol{E}/v_A B_{max},\ \bar{n} = n/n_{max}, \tag{3a}$$

$$\bar{\boldsymbol{j}} = \boldsymbol{j}/j_{max},\quad \text{with } j_{max} = e[nU^{ie}]_{max} = e[n\,|\boldsymbol{u}_i - \boldsymbol{u}_e|]_{max}, \tag{3b}$$

$$\overline{(u_i - u_e)} = (\boldsymbol{u}_i - \boldsymbol{u}_e)/U^{ie}_{max} = (\boldsymbol{u}_i - \boldsymbol{u}_e)/|\boldsymbol{u}_i - \boldsymbol{u}_e|_{max}, \tag{3c}$$

$$\bar{\nabla} = l_p\nabla = \frac{l_p}{l_t}\nabla_t + l_p\nabla_p, \tag{3d}$$

$$\overline{\rho_\alpha} = \rho_\alpha/m_\alpha n_{max},\ \overline{\Omega_\alpha} = \Omega_\alpha/\Omega_{\alpha max},\quad \text{and}\quad \overline{v_\alpha} = v_\alpha/v_{\alpha max}, \tag{3e}$$

where $U^{ie}_{max} := |\boldsymbol{u}_i - \boldsymbol{u}_e|_{max}$ refers to the maximum value of the relative velocity (that is, e.g., for the case of a fusion plasma usually the measured value at the magnetic axis), and $l_t(l_p)$ stands for the characteristic length along (perpendicular to) the magnetic field lines. Furthermore, τ_A, v_A, v_{th}^{α}, n_{max}, and B_{max} are maximum values of the Alfvén time, the Alfvén velocity, the particle thermal velocity, the particle density, and the magnetic field, respectively, all values are taken at a characteristic instant of the discharge. Furthermore, $\Omega_{\alpha max}$ and $v_{\alpha max}$ stand for the calculated value of the Larmor and the particle collision frequency at the same characteristic instant. The higher order moments $\boldsymbol{\Pi}_\alpha$, $\boldsymbol{R}_\alpha$, $\boldsymbol{q}_\alpha$ and Q_α are still undefined and have to be expressed by some approximations by the single fluid MHD variables. This was done by Braginskii in his famous work [7] under the assumption of a strongly collisional and strongly magnetized plasma. The condition for collision dominance to be valid can be written in the form [1],

$$\omega_\alpha v_\alpha^{-1} = v_{th}^{\alpha} v_\alpha^{-1}/a \ll 1,\quad \text{i.e.}\quad v_\alpha^{-1} \ll \omega_\alpha^{-1}, \tag{4}$$

where ω_α is the transit frequency of species α and a is the hydromagnetic length of the plasma (the plasma radius). Furthermore, a strongly magnetized plasma is defined by the condition

$$\delta_\alpha = :(\omega_\alpha/\Omega_\alpha) \equiv (r_{1\alpha}/a) \ll 1, \tag{5}$$

where $r_{1\alpha}$ is the Larmor radius of species α. Now we replace the higher order moments $\boldsymbol{\Pi}_\alpha$, $\boldsymbol{R}_\alpha$, $\boldsymbol{q}_\alpha$, and Q_α by the approximating expressions given by Braginskii [7] for the case of a strongly magnetized plasma (i.e., $\Omega_\alpha \omega_\alpha^{-1} \gg 1$), perform the normalization procedure, omit for convenience the overbar by the dimensionless quantities, and finally obtain:

2.1. *The dimensionless momentum equation*

$$\rho\frac{\partial \boldsymbol{u}}{\partial t} + \rho(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + a_0(A\nabla p_e + \nabla p_i) - b_0\boldsymbol{j}\times\boldsymbol{B} = -c_0\nabla\cdot\boldsymbol{\Pi}_e - d_0\nabla\cdot\boldsymbol{\Pi}_i, \tag{6}$$

where the dimensionless factors (i.e., a_0, b_0, ... etc.) in front of the dimensionless quantities are given by

$$a_0 = \left[\frac{\omega_i}{\Omega_i}\right]^2(\tau_A\Omega_i)^2 = \varepsilon_i^2\xi_i^2\Delta^2, \tag{7a}$$

$$b_0 = (\tau_A\Omega_i)\frac{j_{max}}{env_A} = \Delta\Lambda, \tag{7b}$$

$$c_0 = \frac{\omega_e^2}{\Omega_e v_e}(\tau_A\Omega_i) = \varepsilon_e^2\xi_e^2\Delta\frac{\Omega_e}{v_e} = [\kappa A^5]^{1/2}\varepsilon_i^2\xi_i^2\Delta^2(\tau_A v_i)^{-1}, \tag{7c}$$

$$d_0 = \frac{\omega_i^2}{\Omega_i v_i}(\tau_A\Omega_i) = \varepsilon_i^2\xi_i^2\Delta^2(\tau_A v_i)^{-1}. \tag{7d}$$

Furthermore, the following definitions

$$\varepsilon_\alpha := v_\alpha/\Omega_\alpha \ll 1,\ \xi_\alpha := \omega_\alpha/v_\alpha \ll 1,\ \varepsilon_\alpha\xi_\alpha = \delta_\alpha \equiv (r_{1\alpha}/a) \ll 1,$$

$$\Delta := (\tau_A\Omega_i) \gg 1,\ A := \left[\frac{T_e}{T_i}\right]_{max},\ \kappa := \frac{m_e}{m_i},\ \varepsilon_e = [\kappa A^{-3}]^{1/2}\varepsilon_i,$$

$$\xi_e = A^2\xi_i\quad \text{and}\quad \Lambda := \frac{U^{ie}_{max}}{v_A} \tag{8}$$

have been employed, where ω_α and Ω_α are the transit frequency and the gyro-frequency, respectively.

In order that the terms on the r.h.s. can be neglected, we have to require that the coefficients of the anisotropic part of the pressure tensor on the r.h.s. are much smaller than the coefficient by the isotropic part on the l.h.s., i.e. $d_0 \ll a_0$,

$$(\tau_A \nu_i)^{-1} \ll 1, \quad \text{or} \quad \nu_i^{-1} \ll \tau_A. \tag{9}$$

Thus we have arrived at the rather stringent requirement that the ion-ion collision time is much shorter than the Alfvén time. On account of the relation $\omega_e^{-1} \leqslant \tau_A \leqslant \omega_i^{-1}$ (cf., eq. (30) of Ref. [5]) this condition is consistent with the high-collisionality requirement of eq. (4) $\nu_\alpha^{-1} \ll \omega_\alpha^{-1}$ and with the result obtained in the standard derivation (cf., e.g., Ref. [1]–[3]), which reads $\nu_i^{-1} \ll \omega_i^{-1}$.

2.2. *The dimensionless electron heat-balance equation*

$$\begin{aligned}\frac{3}{2}\rho^\gamma \frac{d}{dt}\{\rho^{-\gamma}p_e\} &= a_1\rho_e\nu_e(\boldsymbol{u}_e - \boldsymbol{u}_i)^2 - b_1\rho_e\nu_e \\ &\times (T_e - A^{-1}T_i) - c_1\nabla_t \cdot n_e T_e(\boldsymbol{u}_i - \boldsymbol{u}_e)_t \\ &- d_1\nabla_p \cdot \frac{n_e T_e}{\Omega_e}\boldsymbol{b}\times(\boldsymbol{u}_i - \boldsymbol{u}_e) + e_1\nabla_t \cdot \frac{n_e^2 T_e}{\rho_e\nu_e} \\ &\times \nabla_t T_e + f_1\nabla_p \cdot \frac{n_e^2 T_e}{\rho_e\Omega_e^2}\nu_e\nabla_p T_e + g_1\nabla_p \cdot \frac{n_e T_e}{\Omega_e}\boldsymbol{b}\times\nabla_p T_e \\ &- h_1\boldsymbol{\Pi}_e:\nabla\boldsymbol{u} + i_1\boldsymbol{\Pi}_e:\nabla(\boldsymbol{j}/en) \\ &+ k_1\boldsymbol{j}\cdot\nabla T_e - l_1(\boldsymbol{j}/n)\cdot\nabla p_e,\end{aligned} \tag{10}$$

where the dimensionless factors (i.e., a_1, b_1, ... etc.) in front of the dimensionless quantities, are given by

$$a_1 = \frac{\kappa}{\varepsilon_0 \xi_e^2 \Delta}\Lambda^2, \tag{11a}$$

$$b_1 = \varepsilon_e \Delta, \tag{11b}$$

$$c_1 = \frac{l_t}{l_p}\Lambda, \tag{11c}$$

$$d_1 = \varepsilon_e \Lambda, \tag{11d}$$

$$e_1 = k^{-1}\varepsilon_e \xi_e^2 \Delta\left[\frac{l_t}{l_p}\right]^2, \tag{11e}$$

$$f_1 = \kappa^{-1}\varepsilon_e^3\xi_e^2\Delta, \tag{11f}$$

$$g_1 = \kappa^{-1}\varepsilon_e^2\xi_e^2\Delta, \tag{11g}$$

$$h_1 = \kappa[\varepsilon_e\Delta]^{-1}, \tag{11h}$$

$$i_1 = \kappa[\varepsilon_e\Delta]^{-1}\Lambda, \tag{11i}$$

$$k_1 = \frac{j_{max}}{env_A} \simeq \Lambda, \tag{11j}$$

$$l_1 = \frac{j_{max}}{env_A} \simeq \Lambda. \tag{11k}$$

Furthermore, $\boldsymbol{b}$ is a unit vector along the field lines. The subscript t (p) stands for the component which is tangential (perpendicular) to the magnetic field lines.

Note, that on the r.h.s. we have replaced the Alfvén velocity normalization by the maximum relative velocity, so that all field variables are normalized to unity. The resulting ratio $\Lambda := (U_{max}^{ie}/v_A)$, however, is not taken into account in the standard derivation of the IMHD equations, where it is assumed that the fluid velocity would be of the order of the ion thermal velocity [1], i.e., $u \approx v_{th}^i$. This assumption, however, is not justified, since from an experimental point of view it follows for example $\Lambda = O(1)$ for a tokamak and $\Lambda = O(\delta)$ for a stellarator. The ratio Λ may be denoted as the magnetic Mach number of the relative velocity.

Taking into account the requirement of a strongly magnetized, collision-dominated plasma, expressed by eqs (4) and (5), it turns out that, under the assumption that $[l_t/l_p]$ and A are of order unity, the following conditions

$$\Lambda := \left[\frac{U_{max}^{ie}}{v_A}\right] \ll 1, \tag{12a}$$

$$\varepsilon_e \Delta = \kappa(\tau_A \nu_e) \ll 1, \tag{12b}$$

must be satisfied simultaneously in order to neglect all the r.h.s.-terms of eq. (11). From eq. (9), it turns out that eq. (12b) leads to the further very restricting requirement,

$$1 \ll (\tau_A \nu_i) \ll (m_i/m_e)^{1/2}. \tag{13}$$

Now let us consider the case where the conditions (12) are not fulfilled. If the fluid velocities are in the leading order parallel to the magnetic field and, furthermore, if also the electron pressure is in the leading order a flux function (cf. Ref. [8]), then the terms by the coefficients c_1 and d_1 will be negligible. The electron heat-balance equation then reads in *dimensional* form:

$$\frac{3}{2}\rho^\gamma \frac{d}{dt}\{\rho^{-\gamma}p_e\} = \rho_e\nu_e(\boldsymbol{u}_e - \boldsymbol{u}_i)^2 - \frac{3}{m_i}\rho_e\nu_e \times (T_e - T_i) + \frac{5}{2}\frac{\boldsymbol{j}}{e}\cdot\nabla T_e - \frac{\boldsymbol{j}}{en}\cdot\nabla p_e. \tag{14}$$

2.3. *The dimensionless ion heat-balance equation*

$$\begin{aligned}\frac{3}{2}\rho^\gamma \frac{d}{dt}\{\rho^{-\gamma}p_i\} &= a_2\rho_e\nu_e(T_e - A^{-1}T_i) + b_2\nabla_t \cdot \frac{n_i^2 T_i}{\rho_i\nu_i}\nabla_t T_i \\ &+ c_2\nabla_p \cdot \frac{n_i^2 T_i}{\rho_i\Omega_i^2}\nu_i\nabla_p T_i \\ &+ d_2\nabla_p \cdot \frac{n_e T_e}{\Omega_e}\boldsymbol{b}\times\nabla_p T_e - e_2\boldsymbol{\Pi}_i:\nabla\boldsymbol{u},\end{aligned} \tag{15}$$

where the dimensionless factors (i.e., a_2, b_2, ... etc.) in front of the dimensionless quantities, are given by

$$a_2 = \varepsilon_e\Delta \simeq [\kappa A^{-3}]^{1/2}\varepsilon_i\Delta, \tag{16a}$$

$$b_2 = \varepsilon_i\xi_i^2\Delta\left[\frac{l_t}{l_p}\right]^2, \tag{16b}$$

$$c_2 = \varepsilon_i^3\xi_i^2\Delta, \tag{16c}$$

$$d_2 = \varepsilon_i^2\xi_i^2\Delta, \tag{16d}$$

$$e_2 = [\varepsilon_i\Delta]^{-1}. \tag{16e}$$

Similarly, it may be easily shown that besides the conditions (5) and (9) also both of the above conditions eqs (12–13) must be satisfied, in order that the r.h.s.-terms of eq. (15) can be neglected.

If the conditions of eqs (12) are not fulfilled, then the ion-heat balance equation is written in *dimensional* form:

$$\frac{3}{2}\rho^{\gamma}\frac{\mathrm{d}}{\mathrm{d}t}\{\rho^{-\gamma}p_{\mathrm{i}}\} = \frac{3}{m_{\mathrm{i}}}\rho_{\mathrm{e}}\nu_{\mathrm{e}}(T_{\mathrm{e}} - T_{\mathrm{i}}). \tag{17}$$

2.4. *The dimensionless Ohm's Law*

$$n(\boldsymbol{E} + \boldsymbol{u}\times\boldsymbol{B}) = a_3\frac{\partial \boldsymbol{j}}{\partial t} + b_3\nabla\cdot\left\{n\left(\boldsymbol{u}\otimes\frac{\boldsymbol{j}}{n}\right) + n\left(\frac{\boldsymbol{j}}{n}\otimes\boldsymbol{u}\right)\right\}$$
$$- c_3 n\left(\frac{\boldsymbol{j}}{n}\otimes\frac{\boldsymbol{j}}{n}\right) + d_3\nabla p_{\mathrm{i}} + e_3\nabla\cdot\boldsymbol{\Pi}_{\mathrm{i}}$$
$$- f_3\nabla_{p_{\mathrm{e}}} - g_3\nabla\cdot\boldsymbol{\Pi}_{\mathrm{e}} - h_3\boldsymbol{j}\times\boldsymbol{B} + i_3\boldsymbol{R}_{\mathrm{e}}, \tag{18}$$

where the dimensionless factors (i.e., $a_3, b_3, \ldots$ etc.) in front of the dimensionless quantities, are given by,

$$a_3 = \frac{\kappa}{\Delta}\frac{j_{\max}}{env_{\mathrm{A}}} \simeq \frac{\kappa}{\Delta}\Lambda, \tag{19a}$$

$$b_3 \simeq \frac{\kappa}{\Delta}\Lambda, \tag{19b}$$

$$c_3 = \frac{\kappa}{\Delta}\Lambda^2, \tag{19c}$$

$$d_3 = \kappa\varepsilon_{\mathrm{i}}^2\xi_{\mathrm{i}}^2\Delta, \tag{19d}$$

$$e_3 = \kappa\varepsilon_{\mathrm{i}}\xi_{\mathrm{i}}^2, \tag{19e}$$

$$f_3 = \frac{\varepsilon_{\mathrm{e}}^2\xi_{\mathrm{e}}^2\Delta}{\kappa} = A\varepsilon_{\mathrm{i}}^2\xi_{\mathrm{i}}^2\Delta, \tag{19f}$$

$$g_3 = \varepsilon_{\mathrm{e}}\xi_{\mathrm{e}}^2 = \left[\frac{\kappa\varepsilon_{\mathrm{i}}^2\xi_{\mathrm{i}}^4}{A^{-5}}\right]^{1/2}, \tag{19g}$$

$$h_3 = \left[\frac{j_{\max}}{env_{\mathrm{A}}}\right] \simeq \Lambda, \tag{19h}$$

$$i_3 = [\kappa A^{-3}]^{1/2}\varepsilon_{\mathrm{i}}\Lambda. \tag{19i}$$

It may be immediately be seen that all the terms on the r.h.s. of eq. (18) can be neglected, if – just like in the case of the heat balance equations – besides eqs (5) and (9) also eqs (12) are satisfied.

If the slow-flow condition (12a) is not fulfilled then Ohm's law reads in *dimensionless* form:

$$n(\boldsymbol{E} + \boldsymbol{u}\times\boldsymbol{B}) = \Lambda\boldsymbol{j}\times\boldsymbol{B}. \tag{20}$$

If the condition (12a) is not satisfied, thus in conclusion we arrive at the following set of single-fluid transport equations in *dimensional* form:

$$\rho\frac{\partial \boldsymbol{u}}{\partial t} + \nabla\cdot(\rho\boldsymbol{u}) = 0, \tag{21a}$$

$$\rho\frac{\partial \boldsymbol{u}}{\partial t} + \rho(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p - \boldsymbol{j}\times\boldsymbol{B} = 0, \tag{21b}$$

$$\frac{3}{2}\rho^{\gamma}\frac{\mathrm{d}}{\mathrm{d}t}\{\rho^{-\gamma}p\} = \boldsymbol{R}_{\mathrm{e}}\cdot(\boldsymbol{u}_{\mathrm{e}} - \boldsymbol{u}_{\mathrm{i}}) + \frac{3}{2}\rho^{\gamma}\frac{\boldsymbol{j}}{en}\cdot\{\rho^{-\gamma}p_{\mathrm{e}}\}, \tag{21c}$$

$$\boldsymbol{E} + \boldsymbol{u}\times\boldsymbol{B} = \frac{\boldsymbol{j}}{en}\times\boldsymbol{B}. \tag{21d}$$

Summarizing, the following assumptions and conditions have been employed in the derivation of this set of equations:

(a) high-collisionality: $\nu_\alpha^{-1} \ll \omega_\alpha^{-1}$ or almost equivalently, $\nu_{\mathrm{i}}^{-1} \ll \tau_{\mathrm{A}}$,

(b) strong magnetic field: $\delta_\alpha \equiv \varepsilon_\alpha\xi_\alpha \coloneqq \omega_\alpha/\Omega_\alpha \ll 1$,

(c) small inverse electron Hall coefficient: $\varepsilon_{\mathrm{e}} \coloneqq \nu_{\mathrm{e}}/\Omega_{\mathrm{e}} \ll 1$.

In order that the r.h.s. of eqs (21c–d) can be neglected, i.e. that one arrives at the single-fluid IMHD equations, one has further to require the slow-flow condition (12a), $\Lambda \ll 1$, where Λ is the magnetic Mach number of the relative velocity. The further restricting condition (12b), or equivalently the second inequality of eq. (13), does not influence the single-fluid IMHD equations, but would be required in order to arrive at the two-fluid IMHD equations.

On account of the required small inverse electron Hall coefficient $\nu_{\mathrm{e}}/\Omega_{\mathrm{e}}$, the coefficient i_3 in eq. (18) can be neglected, so that no additional requirements concerning the resistivity appear.

Based on our dimensionless treatment, it is now clear that the high-collisionality condition is no longer the key condition for the validity of the IMHD equations since also the above conditions (b) and (c) are equally important. As the actual condition, however, appears the slow-flow condition $\Lambda \ll 1$.

The above derivation was based on the classical transport equations, derived for a collision-dominated plasma by Braginskii [7] and therefore no statements can be extrapolated concerning the validity range of the IMHD equations for other collisional regimes. For this reason, we believe that the old argumentation concerning the relation between the validity range of the IMHD model and fusion plasmas, which was drawn by Freidberg [1], is not an adequate one. Our belief has been supported by the results obtained within the frame of the multiple timescale approach [5].

These conclusions, together with the results obtained in [6] have motivated us to investigate the validity conditions of the IMHD, on the basis of the multiple timescale approach, for two different collisional regimes, namely for the intermediate- and the high-collisional regimes.

3. The ideal MHD equations in the frame of the multiple timescale approach

From the experimental evidence it follows that the overall equilibrium and stability behaviour of a strongly magnetized plasma is essentially governed by the stability limits set by the IMHD equations. If these stability requirements are not satisfied, a violent termination of the discharge on a very short timescale – usually in the order of the Alfvén time – will be the consequence.

Thus in the following subsections we investigate the requirements under which the plasma transport equations become for different collisional regimes on the Alfvén timescale identical to those of IMHD, where the relevant equations are taken over from the general treatment of the multiple timescale expansion schemes of Refs [5] and [6].

For an easier reading it is remarked that the lower indices, appearing in the following notations denote the order in the multiple timescale expansions of Refs [5] and [6]. This means that all physical quantities q are expanded in the form

$$q(t_1, t_2, t_3) = \sum_{n=0}\delta^n q_n(t_1, t_2, t_3);\ \delta \coloneqq \sqrt{\delta_{\mathrm{e}}\delta_{\mathrm{i}}}, \tag{22}$$

where the t_n's have – depending on the collisional regime considered – different meanings.

3.1. *The validity conditions of the IMHD equations for the case of a Weak-Collsional Plasma (WCR): $\omega_\alpha^{-1} \ll \nu_\alpha^{-1}$*

In this case, we have according to Ref. [5] the timescale ordering $\tau_A \ll \tau_c \ll \tau_{rd}$, where τ_A, τ_c, and τ_{rd} denote the Alfvén, MHD-collision, and resistive diffusion timescales, respectively. Since we are here only interested on processes occurring on the Alfvén timescale, we have to put $t_1 \equiv t_A \coloneqq t$ and disregard the dependence on the slower timescales t_2 and t_3 of eq. (22).

From eqs (22) and (23) in Ref. [5], we infer that a weakly collisional plasma is on the Alfvén timescale subject to the IMHD equations, both in the single-fluid as well as in the two-fluid description.

The corresponding first-order Ohm's law of eq. (24) in Ref. [5], which is not that of ideal MHD, is only needed in the two-fluid description to express the first-order electric field by other quantities.

In the single-fluid description, however, due to the applied multiple timescale derivative expansion scheme one ends up with the ideal equations in their *dimensional* form

$$\frac{\partial \rho_0}{\partial t} + \nabla \cdot (\rho \boldsymbol{u})_0 = 0, \tag{23a}$$

$$\rho_0 \frac{\partial \boldsymbol{u}_0}{\partial t} + \rho_0(\boldsymbol{u}_0 \cdot \nabla)\boldsymbol{u}_0 + \nabla p_0 - \frac{\delta}{c}(\boldsymbol{j}_0 \times \boldsymbol{B}_1 + \boldsymbol{j}_1 \times \boldsymbol{B}_0) = 0, \tag{23b}$$

$$\frac{3}{2}\rho_0^\gamma \frac{\mathrm{d}}{\mathrm{d}t}\{\rho_0^{-\gamma} p_0\} = 0, \tag{23c}$$

$$\operatorname{curl} \boldsymbol{E}_0 = -\frac{1}{c}\frac{\partial \boldsymbol{B}_0}{\partial t}, \tag{23d}$$

$$\boldsymbol{E}_0 + \frac{1}{c}\boldsymbol{u}_0 \times \boldsymbol{B}_0 = 0, \tag{23e}$$

$$\boldsymbol{j}_0 = \operatorname{curl} \boldsymbol{B}_0, \tag{23f}$$

$$q_0 = n_{i0} - n_{e0} = 0. \tag{23g}$$

The appearance of the expansion parameter δ by the Lorentz-force is due to the fact that the leading order magnetic field has, according to Ref. [5], to be a force-free one, i.e., $\boldsymbol{j}_0 \times \boldsymbol{B}_0 = 0$. This fact can immediately be seen, if the Lorentz-force is expanded according to eq. (22), leading to $\boldsymbol{j} \times \boldsymbol{B} = \boldsymbol{j}_0 \times \boldsymbol{B}_0 + \delta(\boldsymbol{j}_0 \times \boldsymbol{B}_1 + \boldsymbol{j}_1 \times \boldsymbol{B}_0)$.

If one disregards the force-free requirement and furthermore drops the zeroth-order subscripts then eqs (23) become identical to the standard form of the ideal MHD equations. In conclusion we thus arrive at the statement that a WCR-plasma is on the Alfvén timescale governed by the IMHD equations.

It may be remarked that on the intermediate MHD-collision timescale there one ends up with a set of non-ideal transport equations, as documented by eqs (52) in Ref. [5].

3.2. *The validity conditions of the IMHD equations for the case of an Intermediate-Collisional (ICR) Plasma: $\omega_\alpha^{-1} \approx \nu_\alpha^{-1}$*

According to Ref. [6], we have in this case to consider the timescale ordering $\tau_A \approx \tau_c \ll \tau_{rd}$. Thus, in eq. (22) we have to put $t_1 = t_A = t_c \coloneqq t$. The dependence on slower timescales is not considered.

Following the derivation scheme outlined in Ref. [6], it turns out that the plasma transport equations on the Alfvén timescale are written in the following *dimensional* form:

$$\frac{\partial \rho_0}{\partial t} + \nabla \cdot (\rho_0 \boldsymbol{u}_0) = 0, \tag{24a}$$

$$\rho_0 \frac{\partial}{\partial t}\boldsymbol{u}_0 + \rho_0(\boldsymbol{u}_0 \cdot \nabla)\boldsymbol{u}_0 + \nabla p_0 - \delta(1/c)(\boldsymbol{j}_1 \times \boldsymbol{B}_0 + \boldsymbol{j}_0 \times \boldsymbol{B}_1) = 0, \tag{24b}$$

$$\frac{\partial}{\partial t}\{p_0 \rho_0^{-\gamma}\} + \boldsymbol{u}_0 \cdot \nabla\{p_0 \rho_0^{-\gamma}\} = -\tfrac{2}{3}\rho_0^{-\gamma}\boldsymbol{R}_{e0} \cdot (\boldsymbol{u}_{e0} - \boldsymbol{u}_{i0}) + (1/en_0)\boldsymbol{j}_0 \cdot \nabla\{p_{e0}\rho_0^{-\gamma}\}. \tag{24c}$$

In the light of eqs (24), it turns out that the introduction of moderate collisions leads on the Alfvén timescale to the violation of the plasma adiabaticity, where this violation is essentially due to the work done by the friction forces. Note that, if we consider the expansion for the Lorentz force from above, then this set of equations become *identical* to eqs (21), derived by the standard theory for a *high-collisional plasma.*

Similarly, in Ref. [6] it is shown that the first-order Maxwell's equations (on the Alfvén timescale) are written in the following *dimensionless* form:

$$\operatorname{curl} \boldsymbol{B}_1 = \boldsymbol{j}_1, \tag{25a}$$

$$\operatorname{div} \boldsymbol{B}_1 = 0, \tag{25b}$$

$$\operatorname{curl} \boldsymbol{E}_0 = \operatorname{curl}(\boldsymbol{B}_0 \times \boldsymbol{u}_0) = -(\Omega_i \tau_A \delta)\frac{\partial \boldsymbol{B}_0}{\partial t_1}, \tag{25c}$$

$$\boldsymbol{j}_1 = (\Omega_i \tau_A)\left\{\left[\frac{\delta_i}{\delta_e}\right]^{1/2}(n\boldsymbol{u})_{i1}^A - \left[\frac{\delta_e}{\delta_i}\right]^{1/2}(n\boldsymbol{u})_{e1}^A\right\},$$

$$(n\boldsymbol{u})_{\alpha 1}^A \coloneqq n_{\alpha 0}\boldsymbol{u}_{\alpha 1}^A + n_{\alpha 1}\boldsymbol{u}_{\alpha 0}^A, \tag{25d}$$

$$q_1 = \left[\frac{\delta_i}{\delta_e}\right]^{1/2} n_{i1} - \left[\frac{\delta_e}{\delta_i}\right]^{1/2} n_{e1} = 0, \tag{25e}$$

where t_1 in eqs (25) is related to Alfvén time. The upper index $(\ )^A$ in eq. (25d) denotes normalization with respect to the Alfvén velocity. From the above equations we infer that the introduction of moderate collisions leaves the ideal behaviour of the magnetic field lines on the Alfvén timescale unchanged. The behaviour of the plasma itself, however, is nonideal, due to the violation of the adiabatic nature of the energy-balance eq. (24c). Thus the validity of the IMHD equations requires that the r.h.s. of eq. (24c) can be neglected.

From eq. (24c), we infer that this is the case if the condition (12b) is satisfied, i.e., $j/env_A \approx O(\delta)$ or equivalently $(\boldsymbol{u}_i - \boldsymbol{u}_e)/v_A \approx O(\delta)$. This further implies for the validity of the IMHD equations in the case of an intermediate-collisional plasma the requirement:

$$\boldsymbol{j}_0 = 0 \quad \text{and} \quad \boldsymbol{u}_{i0} - \boldsymbol{u}_{e0} = 0. \tag{26}$$

3.3. *The validity conditions of the IMHD equations for the case of a High-collisional (HCR) Plasma: $\nu_\alpha^{-1} \ll \omega_\alpha^{-1}$*

In this case, the MHD-collision and Alfvén timescales are interchanged, so that we have to consider the timescale ordering $\tau_c \ll \tau_A \ll \tau_{rd}$. This implies that in eq. (22) we have to put $t_1 = \tau_c$ and $t_2 = \tau_A$. In order to get the transport equations on the Alfvén timescale, one first has to perform the time-average over the shorter timescale t_1 and can then put $t_2 = \tau_A \coloneqq t$. The dependence on t_3 again is not considered.

In Ref. [6] it is shown that the application of the multiple timescale approach to a suitable high-collisional plasma leads to a set of transport equations, which are formally identical to those of eqs (21), arising from the standard derivation. Here, however, one has to be aware of the following definitations and expansions:

$$\frac{\mathrm{d}}{\mathrm{d}t} \coloneqq \frac{\partial}{\partial t} + \boldsymbol{u}_0 \cdot \nabla \quad (t \text{ is related to Alfvén time}), \tag{27a}$$

$$j(B) = \sum_{n=0}^{2} \delta^n \boldsymbol{j}_n(\boldsymbol{B}_n), \quad \rho, p, \boldsymbol{u} = \rho_0, p_0, \boldsymbol{u}_0, \tag{27b}$$

$$\boldsymbol{R}_e \cdot (\boldsymbol{u}_e - \boldsymbol{u}_i) = \boldsymbol{R}_{e0} \cdot (\boldsymbol{u}_{e1} - \boldsymbol{u}_{i1}) + \boldsymbol{R}_{i1} \cdot (\boldsymbol{u}_{e0} - \boldsymbol{u}_{i0}). \tag{27c}$$

In this case, just as in the case of the standard derivation of eqs (21), it durns out that the violation of the plasma adiabaticity is mainly due to the first-order work done by the friction forces.

For completeness, it may be further shown (cf. Ref. [6]) that the *dimensionless* Maxwell's equations read

$$\operatorname{curl} \boldsymbol{B}_2 = \boldsymbol{j}_2, \tag{28a}$$

$$\operatorname{div} \boldsymbol{B}_2 = 0, \tag{28b}$$

$$\operatorname{curl} \boldsymbol{E}_0 = \operatorname{curl} (\boldsymbol{B}_0 \times \boldsymbol{u}_0) = -(\Omega_i \tau_A \delta^2)\left\{\frac{\partial \boldsymbol{B}_0}{\partial t_2} + \frac{\partial \boldsymbol{B}_1}{\partial t_1}\right\}, \tag{28c}$$

$$j_2 = (\Omega_i \tau_A)\left\{\left[\frac{\delta_i}{\delta_e}\right](un)^A_{i2} - \left[\frac{\delta_e}{\delta_i}\right](un)^A_{e2}\right\}, \tag{28d}$$

$$q_2 = \left[\frac{\delta_i}{\delta_e}\right] n_{i2} - \left[\frac{\delta_e}{\delta_i}\right] n_{e2} = 0, \tag{28e}$$

where t_1 and t_2 are related to both the MHD-collision and Alfvén timescale respectively. To get rid of the t_1-dependence of the MHD-collision timescale, we may assume a harmonic dependence and perform the time averaging.

Equations (28) show that despite of the high-collisionality the magnetic field lines preserve their ideal behaviour on the Alfvén timescale. From eqs (27) together with the energy-balance equation (21c) we infer that the slow-flow condition (12a) must be satisfied in order that the IMHD equations are valid for the case of the high-collisional regime (HCR).

4. Summary and conclusions

In Section 2, the standard derivation of the IMHD equations for a collision-dominated plasma (with $\nu_i^{-1} \ll \tau_A$) has been critically reviewed, where the slow-flow condition $\Lambda \coloneqq |\boldsymbol{u}_i - \boldsymbol{u}_e|/v_A \ll 1$ turns out to be essential in order that the adiabaticity is conserved.

Based on a multiple timescale approach of [5] and Ref. [6], in Section 3 the validity of the IMHD equations on the Alfvén timescale is investigated for different collisional regimes. It turns out, that the continuity and momentum equations of IMHD are valid independently from the collisional regime considered.

Furthermore, it turns out that also the ideal behaviour of the magnetic field lines remains unrestricted valid. In the ICR- and HCR-regime, however, the adiabaticity is violated, essentially due to the work done by the friction forces, arising from the leading-order relative-flow along the field lines. In order that the adiabaticity is conserved, the slow-flow condition $|\boldsymbol{u}_i - \boldsymbol{u}_e|/v_A \simeq O(\delta)$ has to be fulfilled. In terms of the applied expansion, this implies $(\boldsymbol{u}_{i0} - \boldsymbol{u}_{e0}) = 0$ and $\boldsymbol{j}_0 = 0$.

In the HCR-regime the only difference to the standard derivation arises in the disappearance of the Hall-term in Ohm's law, when the slow-flow condition is not satisfied. This is due to the fact that the magnetic field is in leading-order force-free, i.e., $\boldsymbol{j}_0 \times \boldsymbol{B}_0 = 0$.

The outcome of this paper, namely that the momentum equation and Ohm's law of the IMHD are valid independently from the collisional regime considered, is consistent with the experimental results, where it is found that the equilibrium and stability behaviour of a strongly magnetized plasma is over a wide range of parameters governed by the IMHD equations.

Acknowledgements

This work was supported in part by the Austrian Science Foundation (FWF) under Contract No. P9641-TEC.

References

1. Freidberg, J. P., "Ideal Magnetohydrodynamics" (Plenum Press, New York 1987), Chapter 2.
2. Krall, N. A. and Trivelpiece, A. W., "Principles of Plasma Physics" (McGraw Hill, New York 1973), p. 93–97.
3. Mayamoto, K., "Plasma Physics for Nuclear Fusion" (MIT Press, Cambridge 1980), p. 110.
4. Spatschek, K. H., "Theoretische Plasmaphysik" (Teubner, Stuttgart, 1990), p. 155–159.
5. Edenstrasser, J. W., Phys. Plasmas **2**, 1192 (1995).
6. Edenstrasser, J. W. and Kassab, M. M., "Transport Equations on Different Timescales For Intermediate- and High-Collisional Regimes", Accepted for publication in J. Plasma Phys., Oct. 1995.
7. Braginskii, S. I., in: "Reviews of Plasma Physics" (Edited by M. A. Lentovitch) (Consultant Bureau, New York 1995), Vol. I, p. 216–218.
8. Hinton, F. L. and Hazeltine, R. D., Rev. Mod. Phys. **2**, 240 (1976).