Finite element prediction of flexural and torsional frequencies of laminated composite beams

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Abstract

A numerical model using beam element is developed for the prediction of free vibrational natural frequencies of composite beams. Both flexural and torsional natural frequencies are studied. The mass and stiffness matrices of the element are derived from kinetic and strain energies. This is done using cubic Hermit polynomials as shape functions on the basis of first-order shear deformation theory. The effects of warping stiffness, shear deformation, and rotary inertia are incorporated in the formulation. Various boundary conditions (i.e., clamped–free, clamped–clamped, simple–clamped, and simply supported) are considered. Computations and numerical results are carried out using MATLAB software. Results are presented and compared to those previously published solutions in order to demonstrate the accuracy of the present model.

Keywords: Beam Element, FEM, Natural Frequencies, Composite Beams

1. Introduction

Fiber reinforced composite materials have been widely used in many potential engineering applications including aerospace, automotive as well as in the mechanical industries. The increasing usage of composite materials, in some applications area, is preferred to conventional ones mostly because of their high specific strength and stiffness. Laminated beams find applications in a variety of structural components such as helicopter blades, robot arms, etc. The increased use of laminated composite beams requires a better understanding of vibration characteristics of such composite beams; it is quite essential in the design of composite beams subjected to dynamic loads. Therefore, this research aims to establish a numerical model using finite element method in order to predict the free vibrational frequencies of those beams. This includes out-of-plane bending, in-plane bending and torsional vibrational frequencies of composite beams.

The researches pertain to the vibration analysis of composite beams using the finite element technique have undergone rapid growth over the past few decades and are still growing. Teh and Huang [1] presented two finite element models based on a first-order theory for the free vibration analysis of fixed-free beams of general orthotropy. The discrete models include the transverse shear deformation effect and the rotary inertia effect. Chandrashekhara and Bangera [2] investigated the free vibration of angle-ply composite beams by a higher-order shear deformation theory using the shear flexible FEM. The Poisson effect, which is often neglected in one-dimensional laminated beam analysis, is incorporated in the formulation of the beam constitutive equation. Also, the effects of in-plane inertia and rotary inertia are considered in the formulation of the mass matrix. Nabi and Ganesan [3] developed a general finite element based on a first-order deformation theory to study the free vibration characteristics of laminated composite beams. The formulation accounts for bi-axial bending as well as torsion. The required elastic constants are derived from a two-dimensional elasticity matrix. Maiti and Sinha [4] developed a finite element method (FEM) to analyze the vibration behavior of laminated composite. The developed finite element methods are based on a higher-order shear deformation theory and the conventional first-order theory and are used to analyze accurately the bending and free vibration behaviour of laminated composite beams, using nine-noded isoparametric elements. Rao and Ganesan [5] investigated the harmonic response of tapered composite beams by using a finite element model. Only uniaxial bending is considered. The Poisson effect is incorporated in the formulation of the beam constitutive equations. Interlaminar stresses are evaluated by using stress equilibrium equations. The effects of in-plane inertia and rotary inertia are also considered in the formulation of the mass matrix. Bassiouni et al. [6] presented a finite element model to investigate the natural frequencies and mode shapes of the laminated composite beams. The model required all lamina had the same lateral displacement at a typical cross-section, but allowed each lamina to rotate a different amount from the other. The transverse shear deformation was included. Also, experimental investigation is carried out and the given results are used to validate the results with the finite element. Ramtekkar et al. [7] developed a six-node plane-stress mixed finite element model by using Hamilton's principle. The transverse stress components have been invoked as the nodal degrees of freedom by applying elasticity relations. Natural frequencies of cross-ply laminated beams were obtained and various mode shapes were presented. Murthy et al. [8] derived a refined 2-node beam element based on higher order shear deformation theory for axial-flexural-shear coupled deformation in asymmetrically stacked laminated composite beams. Aydogdu [9] studied the vibration of cross-ply laminated beams subjected to different sets of boundary conditions. The analysis is based on a three-degree-of-freedom shear deformable beam theory. Subramanian [10] has investigated the free vibration of LCBs by using two higher order displacement based on shear deformation theories and finite elements. Both theories assume a quintic and quartic variation of in-plane and transverse displacements in the thickness coordinates of the beam respectively. Tahani [11] developed a new layerwise beam theory for generally laminated composite beam and compared the analytical solutions for static bending and free vibration with the three-dimensional elasticity solution of cross-ply laminates in cylindrical bending and with three-dimensional finite element analysis for angle-ply laminates. Jun et al. [12] presented a dynamic finite element method for free vibration analysis of generally laminated composite beams on the basis of first order shear deformation theory. The influences of Poisson effect, couplings among extensional, bending and torsional deformations, shear deformation and rotary inertia are incorporated in the formulation. The aforementioned studies have contributed significantly to understand the vibration behavior of composite beams. Although a large number of investigators studied the problem of vibrational frequencies of composite beams using the finite element method, but most of them have been limited to out of plane bending vibrational behavior.

In the present work, a general formulation capable of predicting free vibrational frequencies of composite beams using the finite element method is presented. The formulation has the capability to determine flexural vibration (vibration in x-z plane and vibration in x-y plane) and torsional vibration of composite beams. The used beam element has two-node, developed based upon Hamilton's principle, where interdependent cubic and quadratic polynomials are used for the transverse and rotational displacements, respectively. The mass and stiffness matrices of the element are derived from kinetic and strain energies on the basis of first-order shear deformation theory using cubic Hermite polynomials as shape functions. The effects of warping stiffness, shear deformation, and rotary inertia are incorporated in the formulation.

In the following section explicit expressions for the stiffness and mass matrices of the beam element are presented.

2. Two-Node Beam Element for Free Vibration of Composite Beams

The Bernoulli-Euler theory of free motion of elastic beams has been found to be inadequate for the prediction of higher modes of vibration and also inadequate for those beams when the effect of cross sectional dimensions on frequencies cannot be neglected. Timoshenko beam theory takes into account the effects of rotary inertia and shear deformations during vibration of a beam, as it is easy to see that during vibration a typical element of a beam performs not only a translatory motion, but also rotates. With the introduction of shear deformation, the assumption of the elementary theory that plane sections remain plane is no longer valid as shown in Figure 1. Consequently, the angle of rotation, which is equal to the slope of the deflection curve, is not simply obtained by differentiating the transverse displacement owing to the shear deformation. Thus, two independent motions, $w(x, t)$ and $\theta(x, t)$ are considered in the description of flexural motion in *x-z* plane. The same is used in the description of torsional behavior, which will have two independent motions, $\psi(x, t)$ and $\varphi(x, t)$.

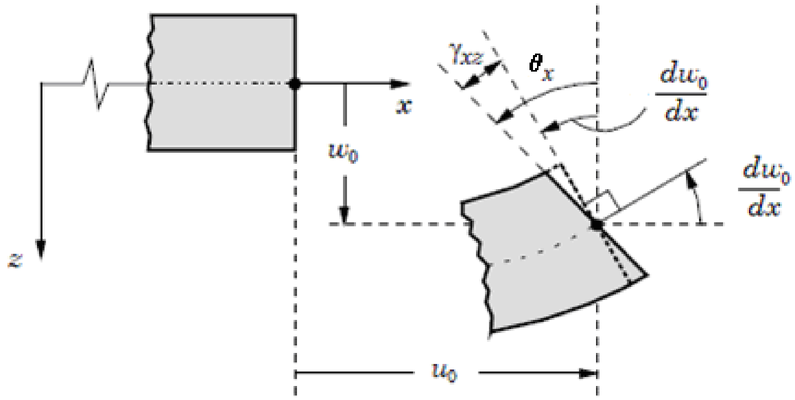

Figure 1. Undeformed and deformed geometries for a Timoshenko beam element

The following quantities can be defined as:

w_o = transverse displacement of the neutral plane of the beam
$\partial w_o/\partial x$ = slope of the neutral plane of the beam
θ_x = slope of the cross-section due to effects of the bending, or the time dependent rotation of the cross-section about y axis
γ_{xz} = shear angle
$\theta_x - \partial w_o/\partial x$ = loss of slope, equal to the shear angle γ_{xz}

2.1 Governing Equations

From the Hamilton Principle, with the use of the kinematic hypotheses of the first-order shear deformation laminate theory, it is possible to obtain the differential equations of motion for free natural uncoupled vibration of composite Timoshenko beams, which can be written in the following form:
Free flexural vibration in *x-z* plane taking into accounts both rotary inertia and transverse shear deformation effects [13]:

$$EI_{yy}\frac{\partial^2\theta^y}{\partial x^2} - S_{zz}(\theta^y - \frac{\partial w}{\partial x}) - \rho I_{yy}\frac{\partial^2\theta^y}{\partial t^2} = 0 \quad (1)$$

$$S_{zz}(\frac{\partial\theta^y}{\partial x} - \frac{\partial^2 w}{\partial x^2}) + \rho A\frac{\partial^2 w}{\partial t^2} = 0 \quad (2)$$

Free flexural vibration in *x-y* plane taking into accounts both rotary inertia and lateral shear deformation effects:

$$EI_{zz}\frac{\partial^2\theta^z}{\partial x^2} - S_{yy}(\theta^z - \frac{\partial v}{\partial x}) - \rho I_{zz}\frac{\partial^2\theta^z}{\partial t^2} = 0 \quad (3)$$

$$S_{yy}(\frac{\partial\theta^z}{\partial x} - \frac{\partial^2 v}{\partial x^2}) + \rho A\frac{\partial^2 v}{\partial t^2} = 0 \quad (4)$$

Free torsional vibration taking into accounts both warping and rotational shear deformation effects [15]:

$$EI_{\omega}\frac{\partial^2\varphi}{\partial x^2}-S_{\omega\omega}(\varphi-\frac{\partial\psi}{\partial x})-\rho I_{\omega}\frac{\partial^2\varphi}{\partial t^2}=0 \quad (5)$$

$$S_{\omega\omega}(\frac{\partial\varphi}{\partial x}-\frac{\partial^2\psi}{\partial x^2})-GI_t\frac{\partial^2\psi}{\partial x^2}+\rho I_s\frac{\partial^2\psi}{\partial t^2}=0 \quad (6)$$

Where $w(x,t)$ and $v(x,t)$ are the flexural translations in z- and y-direction respectively, while $\psi(x,t)$ represents the torsional rotation about the x-axis and $\varphi(x, t)$ represents the warping torsional angle.

(EI_{yy}) represents flexural rigidity with respect to y-axis, (EI_{zz}) represents flexural rigiditiy with respect to z-axis, (EI_{ω}) represents warping rigidity, (GI_t) represents torsional rigidity, S_{yy} is shear rigidity in x-y plane, S_{zz} is shear rigidity in x-z plane, $S_{\omega\omega}$ torsional shear rigidity, I_{yy} and I_{zz} are the moments of inertia of the cross-section about y-axis and z-axis, respectively, I_s is the polar moment of inertia of the cross-section about the shear center, I_{ω} and I_t are warping and torsion constants, A is the cross-sectional area, and ρ is the mass density.

The aforementioned rigidities for symmetric orthotropic laminated beam of solid rectangular cross section can be obtained by the following relations:

Flexural rigidity with respect to y-axis EI_{yy},

$$EI_{yy}=\frac{b}{d_{11}} \quad (7)$$

Flexural rigidity with respect to z-axis EI_{zz},

$$EI_{zz}=\frac{1}{a_{11}}\frac{b^3}{12} \quad (8)$$

Warping rigidity EI_{ω},

$$EI_{\omega}=\frac{b^3}{a_{11}}\frac{h^2}{144} \quad (9)$$

Torsional rigidity GI_t,

$$GI_t=\frac{4b}{d_{66}} \quad (10)$$

Shear rigidity in x-y plane S_{yy},

$$S_{yy}=\frac{b}{1.2a_{66}} \quad (11)$$

Shear rigidity in x-z plane S_{zz},

$$S_{ZZ}=\frac{5}{6}b\int_{-h/2}^{h/2}Q'_{55}\,dz \quad (12)$$

$$Q'_{55}=G_{13}\cos^2\beta+G_{23}\sin^2\beta \quad (13)$$

Torsional shear rigidity $S_{\omega\omega}$,

$$S_{\omega\omega}=\frac{bh^2}{1.2a_{66}} \quad (14)$$

Where b; width of the beam, h; thickness of the beam, a_{11}; element 1–1 of the laminate extensional compliance matrix, d_{11}; element 1–1 of the laminate bending compliance matrix, a_{66}; element 6-6 of the laminate extensional compliance matrix, d_{66}; element 6–6 of the laminate bending compliance matrix, Q'_{55}; transformed shear stiffness, G_{13}; lamina shear modulus in plane 1–3, G_{23}; lamina shear modulus in plane 2–3, β; angle between the fiber direction and longitudinal axis of the beam.

2.2 Beam Element Formulation

The partial differential equations of motion of flexural vibration in x-z plane are transformed to a two-node finite element based discrete set of differential equations using “newly” developed shape functions for (w) and (θ). These functions are developed so that they exactly satisfy the homogeneous form of both of the static equations of equilibrium of an unstressed uniform Timoshenko beam.

For the static case with no external force acting on the beam, the governing equation of motion (Timoshenko beam equations) of flexural vibration in *x*-*z* plane reduces to [13]:

$$\frac{\partial}{\partial x}\left(EI_{yy}\frac{\partial\theta^{y}}{\partial x}\right)+S_{zz}\left(\frac{\partial w}{\partial x}-\theta^{y}\right)=0 \tag{15}$$

$$\frac{\partial}{\partial x}\left(S_{zz}\left(\frac{\partial w}{\partial x}-\theta^{y}\right)\right)=0 \tag{16}$$

From equation (15), it can be seen that this governing equation of the beam based on Timoshenko beam theory can only be satisfied if the polynomial order for w is selected one order higher than the polynomial order for θ. Let w be approximated by a cubic polynomial and θ be approximated by a quadratic polynomial as:

$$w(x)=a_1+a_2x+a_3x^2+a_4x^3 \tag{17}$$

$$\theta(x)=b_1+b_2x+b_3x^2 \tag{18}$$

Then the cubic shape function for (w) and a quadratic shape function for (θ) will be of the form [13]:

$$w(x)=\{1 \quad x \quad x^2 \quad x^3\}\begin{bmatrix}1 & 0 & 0 & 0\\ a_{11} & a_{21} & a_{31} & a_{41}\\ a_{12} & a_{22} & a_{32} & a_{42}\\ a_{13} & a_{23} & a_{33} & a_{43}\end{bmatrix}\begin{Bmatrix}w_1\\ \theta_1^y\\ w_2\\ \theta_2^y\end{Bmatrix} \tag{19}$$

$$\theta(x)=\{1 \quad x \quad x^2\}\begin{bmatrix}0 & 1 & 0 & 0\\ b_{11} & b_{21} & b_{31} & b_{41}\\ b_{12} & b_{22} & b_{32} & b_{42}\end{bmatrix}\begin{Bmatrix}w_1\\ \theta_1^y\\ w_2\\ \theta_2^y\end{Bmatrix} \tag{20}$$

Where (w_1), (θ_1), (w_2), (θ_2) are the nodal displacements and rotations at the beam end nodes (1) and (2), respectively, and (a_{ij}) and (b_{ij}) are unknown coefficients. Four of the (a_{ij}) and four of the (b_{ij}) coefficients can be determined in terms of the remaining 12 coefficients by enforcing that ($w(x = L) = w_2$) and ($\theta(x = L) = \theta_2$). The remaining coefficients are determined by substituting the shape functions into the equations (15) and (16) and solving. The resulting explicit form of the two shape functions are given as [13]:

$$\begin{Bmatrix}w\\ \theta\end{Bmatrix}=\begin{bmatrix}[N_w]\\ [N_\theta]\end{bmatrix}\{\delta^e\} \tag{21}$$

Where:

$$
[N_w]^T = \begin{bmatrix}
\frac{1}{(1+\phi)}\left\{2\left(\frac{x}{L}\right)^3 - 3\left(\frac{x}{L}\right)^2 - \phi\left(\frac{x}{L}\right) + (1+\phi)\right\} \\
\frac{L}{(1+\phi)}\left\{\left(\frac{x}{L}\right)^3 - \left(2+\frac{\phi}{2}\right)\left(\frac{x}{L}\right)^2 + \left(1+\frac{\phi}{2}\right)\left(\frac{x}{L}\right)\right\} \\
-\frac{1}{(1+\phi)}\left\{2\left(\frac{x}{L}\right)^3 - 3\left(\frac{x}{L}\right)^2 - \phi\left(\frac{x}{L}\right)\right\} \\
\frac{L}{(1+\phi)}\left\{\left(\frac{x}{L}\right)^3 - \left(1-\frac{\phi}{2}\right)\left(\frac{x}{L}\right)^2 - \left(\frac{\phi}{2}\right)\left(\frac{x}{L}\right)\right\}
\end{bmatrix} \tag{22}
$$

$$
[N_\theta]^T = \begin{bmatrix}
\frac{6}{(1+\phi)L}\left\{\left(\frac{x}{L}\right)^2 - \left(\frac{x}{L}\right) - \frac{\phi(1+\phi)}{6} + \frac{(1+\phi)^2 L}{6}\right\} \\
\frac{1}{(1+\phi)}\left\{3\left(\frac{x}{L}\right)^2 - (4+\phi)\left(\frac{x}{L}\right) + (1+\phi)\right\} \\
-\frac{6}{(1+\phi)L}\left\{\left(\frac{x}{L}\right)^2 - \left(\frac{x}{L}\right)\right\} \\
\frac{1}{(1+\phi)}\left\{3\left(\frac{x}{L}\right)^2 - (2-\phi)\left(\frac{x}{L}\right)\right\}
\end{bmatrix} \tag{23}
$$

(ϕ) is the ratio of the beam bending stiffness to shear stiffness and is given by:

$$
\phi = \frac{12}{L^2}\left(\frac{EI_{yy}}{S_{zz}}\right) \tag{24}
$$

And the array of nodal displacements and rotations is given as:

$$
\{\delta^e\}^T = \{w_1 \quad \theta_1 \quad w_2 \quad \theta_2\} \tag{25}
$$

It is interesting to note the dependency the shape functions have upon (ϕ), which is a ratio of the beam bending stiffness to the shear stiffness. For long slender beams ($\phi = 0$), $[N_w]$ reduces to the cubic Hermitian polynomial and $[N_\theta]$ reduces to the derivative of $[N_w]$ with respect to (x).

The strain energy of the beam element undergoing flexural vibration in x-z plane depends upon the linear strain ε, the shear strain γ and is given by:

$$
U_w^e = \frac{1}{2}\int_0^L \begin{Bmatrix} \frac{\partial \theta^y}{\partial x} \\ \frac{\partial w}{\partial x} - \theta^y \end{Bmatrix}^T \begin{bmatrix} EI_{yy} & 0 \\ 0 & S_{zz} \end{bmatrix} \begin{Bmatrix} \frac{\partial \theta^y}{\partial x} \\ \frac{\partial w}{\partial x} - \theta^y \end{Bmatrix} dx \tag{26}
$$

And, the kinetic energy $[T_w^e]$ of the beam element depends on the sum of the kinetic energy due to the linear velocity dw/dt and due to angular twist θ and is given by:

$$
T_w^e = \frac{1}{2}\int_0^L \begin{Bmatrix} \frac{\partial w}{\partial t} \\ \frac{\partial \theta^y}{\partial t} \end{Bmatrix}^T \begin{bmatrix} \rho A & 0 \\ 0 & \rho I_{yy} \end{bmatrix} \begin{Bmatrix} \frac{\partial w}{\partial t} \\ \frac{\partial \theta^y}{\partial t} \end{Bmatrix} dx \tag{27}
$$

The stiffness matrix $[K_w^e]$ of the regular beam element is the sum of the bending stiffness and the shear stiffness and is written in matrix form as:

$$\left[K_w^e\right]=\int_0^L\begin{bmatrix}\frac{\partial}{\partial x}\left[N_\theta\right]\\ \left[N_\theta\right]+\frac{\partial}{\partial x}\left[N_w\right]\end{bmatrix}^T\begin{bmatrix}EI_{yy} & 0\\ 0 & S_{zz}\end{bmatrix}\begin{bmatrix}\frac{\partial}{\partial x}\left[N_\theta\right]\\ \left[N_\theta\right]+\frac{\partial}{\partial x}\left[N_w\right]\end{bmatrix}dx \tag{28}$$

Substituting the mode shape functions [N_w], [N_θ] into equation (28) and integrating, get the stiffness matrix of the regular beam element as $[K_w^e]$ which is given by [13]:

$$\left[K_w^e\right]=\frac{EI_{yy}}{(1+\phi)L^3}\begin{bmatrix}12 & 6L & -12 & 6L\\ 6L & (4+\phi)L^2 & -6L & (2-\phi)L^2\\ -12 & -6L & 12 & -6L\\ 6L & (2-\phi)L^2 & -6L & (4+\phi)L^2\end{bmatrix} \tag{29}$$

Which is the element stiffness matrix for a uniform beam element undergoing flexural vibration in *x-z* plane, with including shear deformation effect in *x-z* plane (S_{zz}).
Similarly, the element stiffness matrix for a uniform beam undergoing flexural vibration in x-y plane, with including shear deformation effect in x-y plane (S_{yy}) is expressed as $[K_v^e]$:

$$\left[K_v^e\right]=\frac{EI_{zz}}{(1+\phi)L^3}\begin{bmatrix}12 & 6L & -12 & 6L\\ 6L & (4+\phi)L^2 & -6L & (2-\phi)L^2\\ -12 & -6L & 12 & -6L\\ 6L & (2-\phi)L^2 & -6L & (4+\phi)L^2\end{bmatrix} \tag{30}$$

Where the ratio of the beam bending stiffness to shear stiffness ϕ will be:

$$\phi=\frac{12}{L^2}\left(\frac{EI_{zz}}{S_{yy}}\right) \tag{31}$$

The mass matrix of the regular beam element, under flexural vibration in *x-z* plane is the sum of the translational mass and the rotational mass, and is given in matrix form as:

$$\left[M_w^e\right]=\int_0^L\begin{bmatrix}[N_w]\\ [N_\theta]\end{bmatrix}^T\begin{bmatrix}\rho A & 0\\ 0 & \rho I_{yy}\end{bmatrix}\begin{bmatrix}[N_w]\\ [N_\theta]\end{bmatrix}dx \tag{32}$$

Substituting the mode shape functions [N_w], [N_θ] into equation (32) and integrating, give the mass matrix of the regular beam element as:

$$[M_w^e]=[M_{\rho A}]+[M_{\rho I_{yy}}] \tag{33}$$

Where [$M_{\rho A}$] and $[M_{\rho I_{yy}}]$ in equation (33) is associated with the translational inertia and rotary inertia (with the shear) as:

$$\left[M_{\rho A}\right]=\frac{\rho AL}{210(1+\phi)^2}\begin{bmatrix}\left(70\phi^2+147\phi+78\right) & \left(35\phi^2+77\phi+44\right)\frac{L}{4} & \left(35\phi^2+63\phi+27\right) & -\left(35\phi^2+63\phi+26\right)\frac{L}{4}\\ \left(35\phi^2+77\phi+44\right)\frac{L}{4} & \left(7\phi^2+14\phi+8\right)\frac{L^2}{4} & \left(35\phi^2+63\phi+26\right)\frac{L}{4} & -\left(7\phi^2+14\phi+6\right)\frac{L^2}{4}\\ \left(35\phi^2+63\phi+27\right) & \left(35\phi^2+63\phi+26\right)\frac{L}{4} & \left(70\phi^2+147\phi+78\right) & -\left(35\phi^2+77\phi+44\right)\frac{L}{4}\\ -\left(35\phi^2+63\phi+26\right)\frac{L}{4} & -\left(7\phi^2+14\phi+6\right)\frac{L^2}{4} & -\left(35\phi^2+77\phi+44\right)\frac{L}{4} & \left(7\phi^2+14\phi+8\right)\frac{L^2}{4}\end{bmatrix} \tag{34}$$

$$\left[M_{\rho I_{yy}}\right]=\frac{\rho I_{yy}}{30(1+\phi)^2L}\begin{bmatrix}36 & -(15\phi-3)L & -36 & -(15\phi-3)L\\ -(15\phi-3)L & \left(10\phi^2+5\phi+4\right)L^2 & (15\phi-3)L & \left(5\phi^2-5\phi-1\right)L^2\\ -36 & (15\phi-3)L & 36 & (15\phi-3)L\\ -(15\phi-3)L & \left(5\phi^2-5\phi-1\right)L^2 & (15\phi-3)L & \left(10\phi^2+5\phi+4\right)L^2\end{bmatrix} \tag{35}$$

From equation (33) [14]:

$$\left[M_w^e\right]=\frac{\rho AL}{(1+\phi)^2}\begin{bmatrix} m_1+\frac{r^2}{L^2}m_7 & m_2+\frac{r^2}{L^2}m_8 & m_3-\frac{r^2}{L^2}m_7 & -m_4+\frac{r^2}{L^2}m_8 \\ m_2+\frac{r^2}{L^2}m_8 & m_5+\frac{r^2}{L^2}m_9 & m_4-\frac{r^2}{L^2}m_8 & -m_6+\frac{r^2}{L^2}m_{10} \\ m_3-\frac{r^2}{L^2}m_7 & m_4-\frac{r^2}{L^2}m_8 & m_1+\frac{r^2}{L^2}m_7 & -m_2-\frac{r^2}{L^2}m_8 \\ -m_4+\frac{r^2}{L^2}m_8 & -m_6+\frac{r^2}{L^2}m_{10} & -m_2-\frac{r^2}{L^2}m_8 & m_5+\frac{r^2}{L^2}m_9 \end{bmatrix} \qquad (36)$$

Where $r^2=\frac{I_{yy}}{A}$, ϕ is given by $\phi=\frac{12}{L^2}\left(\frac{EI_{yy}}{S_{zz}}\right)$, and

$$m_1=\frac{13}{35}+\frac{7\phi}{10}+\frac{\phi^2}{3}, m_2=\frac{11L}{210}+\left(\frac{11\phi}{120}+\frac{\phi^2}{24}\right)L, m_3=\frac{9}{70}+\frac{3\phi}{10}+\frac{\phi^2}{6}, m_4=\frac{13L}{420}+\left(\frac{3\phi}{40}+\frac{\phi^2}{24}\right)L,$$

$$m_5=\frac{L^2}{105}+\left(\frac{\phi}{60}+\frac{\phi^2}{120}\right)L^2, m_6=\frac{L^2}{140}+\left(\frac{\phi}{60}+\frac{\phi^2}{120}\right)L^2, m_7=\frac{6}{5}, m_8=\left(\frac{1}{10}-\frac{\phi}{2}\right)L,$$

$$m_9=\left(\frac{2}{15}+\frac{\phi}{6}+\frac{\phi^2}{3}\right)L^2, m_{10}=\left(\frac{-1}{30}-\frac{\phi}{6}+\frac{\phi^2}{6}\right)L^2$$

Similarly, for the mass matrix of the regular beam element, under flexural vibration in *x*-*y* plane $[M_v^e]$ will be as the same as of the mass matrix, under flexural vibration in *x*-*z* plane $[M_w^e]$ but with $r^2=\frac{I_{zz}}{A}$ and $\phi=\frac{12}{L^2}\left(\frac{EI_{zz}}{S_{yy}}\right)$.

The same shape functions $[N_w]$ and $[N_\theta]$ given by equations (22) and (23) will be used for describing the torsional vibration of the beam element, including both torsional warping and shear deformation effects. Thus, the shape functions of $[N_\psi]$ and $[N_\varphi]$ will be the same as shape functions of $[N_w]$ and $[N_\theta]$, respectively.

ψ be approximated by a cubic polynomial and φ be approximated by a quadratic polynomial as:

$$\psi(x)=a_1+a_2x+a_3x^2+a_4x^3 \qquad (37)$$

$$\varphi(x)=b_1+b_2x+b_3x^2 \qquad (38)$$

Where $\psi(x)$ and $\varphi(x)$ are two independent motions.

From the strain energy of the beam element undergoing torsional vibration, including both warping and shear deformation effects, the stiffness matrix [K_ψ^e] of the beam element will be the sum of the warping stiffness, the torsional stiffness, and the torsional shear stiffness, and can be written in matrix form as:

$$\left[K_\psi^e\right]=\frac{EI_\omega}{(1+\phi)L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & (4+\phi)L^2 & -6L & (2-\phi)L^2 \\ -12 & -6L & 12 & -6L \\ 6L & (2-\phi)L^2 & -6L & (4+\phi)L^2 \end{bmatrix}+\left[K_t^e\right] \qquad (39)$$

Where the element torsional stiffness [K_t^e] can be obtained by the following relation:

$$\left[K_t^e\right]=GI_t\int_0^L\left[\frac{\partial}{\partial x}\left[N_\psi\right]\right]^T\left[\frac{\partial}{\partial x}\left[N_\psi\right]\right]dx \qquad (40)$$

After integration the above relation yields to:

$$\left[K_t^e\right]=\frac{GI_t}{30L(1+\phi)^2}\begin{bmatrix}(36+60\phi+30\phi^2) & 3L & -(36+60\phi+30\phi^2) & 3L\\ 3L & (4+5\phi+2.5\phi^2)L^2 & -3L & -(1+5\phi+2.5\phi^2)L^2\\ -(36+60\phi+30\phi^2) & -3L & (36+60\phi+30\phi^2) & -3L\\ 3L & -(1+5\phi+2.5\phi^2)L^2 & -3L & (4+5\phi+2.5\phi^2)L^2\end{bmatrix} \tag{41}$$

The mass matrix of the regular beam element under torsional vibration including warping inertia effect is:

$$\left[M_\psi^e\right]=\int_0^L\begin{bmatrix}[N_\psi]\\ [N_\varphi]\end{bmatrix}^T\begin{bmatrix}\rho I_s & 0\\ 0 & \rho I_\omega\end{bmatrix}\begin{bmatrix}[N_\psi]\\ [N_\varphi]\end{bmatrix}dx \tag{42}$$

The shape functions of $[N_\psi]$ and $[N_\varphi]$ are taken the same as shape functions of $[N_w]$ and $[N_\theta]$, respectively. Substituting these shape functions into equation (42) and integrating, give the torsional mass matrix of the regular beam element as:

$$[M_\psi^e]=[M_{\rho I_s}]+[M_{\rho I_\omega}] \tag{43}$$

$$\left[M_{\rho I_s}\right]=\frac{\rho I_s L}{210(1+\phi)^2}\begin{bmatrix}(70\phi^2+147\phi+78) & (35\phi^2+77\phi+44)\frac{L}{4} & (35\phi^2+63\phi+27) & -(35\phi^2+63\phi+26)\frac{L}{4}\\ (35\phi^2+77\phi+44)\frac{L}{4} & (7\phi^2+14\phi+8)\frac{L^2}{4} & (35\phi^2+63\phi+26)\frac{L}{4} & -(7\phi^2+14\phi+6)\frac{L^2}{4}\\ (35\phi^2+63\phi+27) & (35\phi^2+63\phi+26)\frac{L}{4} & (70\phi^2+147\phi+78) & -(35\phi^2+77\phi+44)\frac{L}{4}\\ -(35\phi^2+63\phi+26)\frac{L}{4} & -(7\phi^2+14\phi+6)\frac{L^2}{4} & -(35\phi^2+77\phi+44)\frac{L}{4} & (7\phi^2+14\phi+8)\frac{L^2}{4}\end{bmatrix} \tag{44}$$

$$\left[M_{\rho I_\omega}\right]=\frac{\rho I_\omega}{30(1+\phi)^2 L}\begin{bmatrix}36 & -(15\phi-3)L & -36 & -(15\phi-3)L\\ -(15\phi-3)L & (10\phi^2+5\phi+4)L^2 & (15\phi-3)L & (5\phi^2-5\phi-1)L^2\\ -36 & (15\phi-3)L & 36 & (15\phi-3)L\\ -(15\phi-3)L & (5\phi^2-5\phi-1)L^2 & (15\phi-3)L & (10\phi^2+5\phi+4)L^2\end{bmatrix} \tag{45}$$

From equation (43):

$$\left[M_\psi^e\right]=\frac{\rho I_s L}{(1+\phi)^2}\begin{bmatrix}m_1+\frac{r^2}{L^2}m_7 & m_2+\frac{r^2}{L^2}m_8 & m_3-\frac{r^2}{L^2}m_7 & -m_4+\frac{r^2}{L^2}m_8\\ m_2+\frac{r^2}{L^2}m_8 & m_5+\frac{r^2}{L^2}m_9 & m_4-\frac{r^2}{L^2}m_8 & -m_6+\frac{r^2}{L^2}m_{10}\\ m_3-\frac{r^2}{L^2}m_7 & m_4-\frac{r^2}{L^2}m_8 & m_1+\frac{r^2}{L^2}m_7 & -m_2-\frac{r^2}{L^2}m_8\\ -m_4+\frac{r^2}{L^2}m_8 & -m_6+\frac{r^2}{L^2}m_{10} & -m_2-\frac{r^2}{L^2}m_8 & m_5+\frac{r^2}{L^2}m_9\end{bmatrix} \tag{46}$$

Where $r^2=\frac{I_\omega}{I_s}$, $\phi=\frac{12}{L^2}\left(\frac{EI_\omega}{S_{\omega\omega}}\right)$, and

$$m_1=\frac{13}{35}+\frac{7\phi}{10}+\frac{\phi^2}{3}, m_2=\frac{11L}{210}+\left(\frac{11\phi}{120}+\frac{\phi^2}{24}\right)L, m_3=\frac{9}{70}+\frac{3\phi}{10}+\frac{\phi^2}{6}, m_4=\frac{13L}{420}+\left(\frac{3\phi}{40}+\frac{\phi^2}{24}\right)L,$$

$$m_5=\frac{L^2}{105}+\left(\frac{\phi}{60}+\frac{\phi^2}{120}\right)L^2, m_6=\frac{L^2}{140}+\left(\frac{\phi}{60}+\frac{\phi^2}{120}\right)L^2, m_7=\frac{6}{5}, m_8=\left(\frac{1}{10}-\frac{\phi}{2}\right)L,$$

$$m_9=\left(\frac{2}{15}+\frac{\phi}{6}+\frac{\phi^2}{3}\right)L^2, m_{10}=\left(\frac{-1}{30}-\frac{\phi}{6}+\frac{\phi^2}{6}\right)L^2$$

It is clear that if the shear deformations and rotary inertia effects are neglected that is (ϕ = 0 and r = 0), all the stiffness and mass matrices will be the stiffness and mass matrices of a Bernoulli-Euler model.

2.3 Solution procedure for eigenvalue Problem

Applying Lagrange's equation yields the element equation of motion for free vibration of laminated composite beams as:

$$[M^e]\{\ddot{\delta}^e\}+[K^e]\{\delta^e\}=0 \tag{47}$$

Upon assembly of the element equations, the equations of motion can be determined for the entire structure in the form:

$$[M]\{\ddot{\delta}\}+[K]\{\delta\}=0 \tag{48}$$

Where *K* is the global stiffness matrix, obtained, by proper assembly of the local stiffness matrices (K^e) and *M* is the global inertia matrix, obtained, by proper assembly of the local mass matrices (M^e).

The 12×12 stiffness matrix *K* takes the following form:

$$K=\begin{bmatrix} K_w & 0 & 0 \\ 0 & K_v & 0 \\ 0 & 0 & K_\psi \end{bmatrix} \tag{49}$$

Similarly, the mass matrix *M* can be written as:

$$M=\begin{bmatrix} M_w & 0 & 0 \\ 0 & M_v & 0 \\ 0 & 0 & M_\psi \end{bmatrix} \tag{50}$$

And the vector containing 12 nodal displacements and slopes, (δ^T), can be written as:

$$\delta^T=[w_1 \quad \theta_1^y \quad w_2 \quad \theta_2^y \quad v_1 \quad \theta_1^z \quad v_2 \quad \theta_2^z \quad \psi_1 \quad \varphi_1 \quad \psi_2 \quad \varphi_2] \tag{51}$$

The finite element approximations to the natural frequencies and mode shapes are obtained using eigenvalue - eigenvector problem method; that is, the natural frequency approximations are the square roots of the eigenvalues of [$M^{-1}K$], and the mode shapes are developed from the eigenvectors.

Imposing harmonic motion to equation (48) in the form:

$$\{\delta\}=\{\Delta\}\sin\omega t \tag{42}$$

Get:

$$[K]\{\Delta\}=\{M\}\omega^2\{\Delta\} \tag{53}$$

Hence:

$$[M]^{-1}[K]\{\Delta\}=\omega^2\{\Delta\} \tag{54}$$

This is the eigenvalue problem that was solved to obtain the natural frequencies and mode shapes of beams under investigation.

Where ω is the angular natural frequency and Δ is the mode shape of the structure for the corresponding natural frequency.

3. Validation and numerical results

3.1 Validation

In order to validate the accuracy and applicability of the developed model, various examples are presented and the results are compared with the available in the literature.

Example 1. Out-of-plane bending frequencies of a simply supported (0^0) graphite epoxy beam:

Two simply supported orthotropic (0^0) graphite epoxy beams with different aspect ratios are considered. The shape of the cross-section of the beams is assumed to be a square (*h/b*=1). The material properties are given in Table 1, which are taken from the literature [5]. The first five out-of-plane bending natural frequencies of the thin beam (*L/h*=120) and thick beam (*L/h*=15) using the developed FEM model are presented in Table 6.2, where *L* is the beam length. Those results are compared with the results of [2], [3], and [5]. As seen from the table, the present model yielded results in good agreements with the results given in the literature.

Table 1. The transversely isotropic material properties

Properties	E_1 (GPa)	$E_2 = E_3$ (GPa)	$G_{12}=G_{13}$ (GPa)	G_{23} (GPa)	$\upsilon_{12}= \upsilon_{13}$	ρ (kg/m^3)
graphite epoxy (AS4/3501-6)	144.8	9.65	4.14	3.45	0.3	1389.23

Table 2. Out-of-plane bending frequencies (kHz) for a simply supported (0^0) graphite epoxy composite beam

L/h	Mode	Present	Reference		
		FEM	[2]	[3]	[5]
120 (*L*=762 mm)	1	0.052	0.051	0.054	0.051
	2	0.209	0.203	0.213	0.202
	3	0.468	0.454	0.472	0.454
	4	0.824	0.804	0.801	0.804
	5	1.275	1.262	-	1.252
١٥ (*L*=381 mm)	1	0.751	0.755	0.789	0.754
	2	2.537	2.548	2.656	2.555
	3	4.707	4.716	4.895	4.753
	4	6.960	6.960	7.165	7.052
	5	9.212	9.194	-	9.383

Example 2. Out-of-plane bending frequencies of symmetric angle ply [β/-β/-β/β] graphite epoxy beam with different boundary conditions:

The material and geometrical properties used in this example are same as the properties of the thick beam (L/h=15, h/b=1) given in the previous example. Different boundary conditions are considered to study the performance of the present model.

The fundamental out-of-plane bending frequencies of symmetric angle-ply [β/-β/-β/β] composite beam for various boundary conditions are presented in Table 3. It can be seen that the present results using FEM demonstrate good agreement with the results of reference [12] for all values of β.

Table 3. Fundamental out of plane bending frequencies (Hz) of symmetric angle ply [β/-β/-β/β] composite beam

Boundary conditions	Method	β						
		0^0	15^0	30^0	45^0	60^0	75^0	90^0
C-F	**FEM**	277.0	208.0	138.0	89.0	75.0	74.0	74.0
	[12]	278.4	207.2	137.9	89.3	74.7	73.7	74.2
C-C	**FEM**	1373.0	1130.0	812.0	549.0	463.0	456.0	459.0
	[12]	1376.4	1125.7	811.2	548.4	462.9	456.4	459.1
C-S	**FEM**	1055.0	836.0	580.0	385.0	323.0	319.0	321.0
	[12]	1058.5	856.1	592.4	386.3	323.2	318.6	320.6
S-S	**FEM**	752.0	572.0	384.1	250.1	209.4	206.5	207.8
	[12]	753.2	656.6	428.6	256.4	209.2	206.1	207.4
C-F Clamped-Free C-C Clamped -Clamped C-S Clamped-simply supported S-S simply supported-simply supported								

Example 3. Torsional natural frequencies of simply supported I-beam:

In this example, using FEM, the torsional natural frequencies for the simply supported (S–S) I-beam with symmetric lamination of [45/-45/-45/45] are evaluated. The I-beam has flange width b=600 mm and the height d =600 mm, as shown in Figure 2. The beam length L is 12 m and the thicknesses t_b of flanges and web are 30 mm. All constituent flanges and web are assumed to be symmetrically laminated with respect to its mid-plane. The graphite-epoxy is used for the beam with its material as given in Table 1.

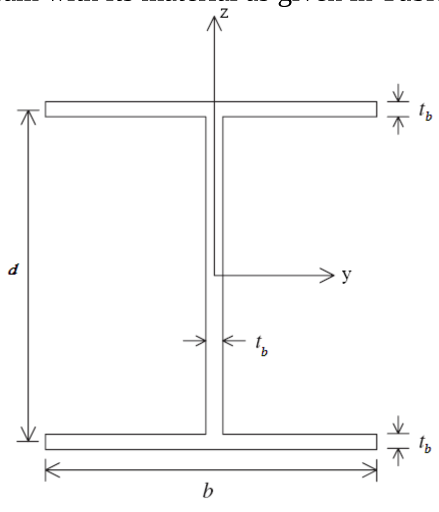

Figure 2. Cross section of the I-beam

The lowest four torsional natural frequencies for the S-S I beam obtained using FEM are presented in Table 4. It can be shown that the present results are in good agreement with those from [16] and [17]

Table 4. Torsional natural frequencies (Hz) of S–S I beam with [45/-45/-45/45]

Method		Mode			
		1	**2**	**3**	**4**
Present	**FEM**	10.69	27.71	53.61	88.88
Reference	**[16]**	10.96	28.03	53.85	89.05
	[17]	10.96	28.10	54.18	90.02

From the validation process, it can be concluded that results from the present model demonstrate a good agreement with the results of references, for both flexural and torsional frequencies.

3.2 Effects of Shear deformation and Rotary Inertia

The investigation is performed to a graphite-epoxy composite beam of square cross-section of 4 layers of equal thickness with lay-up of symmetric cross-ply $[0/90]_s$. Clamped-free and clamped-clamped boundary conditions are considered. Material properties are given in Table 1. The investigation includes the thin beam (L/h=120, L=762 mm) and thick beam (L/h=15, L=381 mm). The natural frequencies are calculated with and without including the effects of shear deformation and rotary inertia to study their influence on the natural frequencies. The effects of shear deformation and rotary inertia on the first six out-of-plane and in-plane bending frequencies of the thin beam are tabulated in Tables 5-8 for C-F and C-C boundary conditions. The percentage difference between Timoshenko's and Bernoulli's frequencies is determined as:

$$\text{Percentage difference} = \frac{\left|f_{\text{Timoshenko}} - f_{\text{Bernoulli}}\right|}{f_{\text{Timoshenko}}} \times 100$$

Table 5. The shear and rotary inertia effects on out-of-plane bending frequencies (Hz) of C-F graphite-epoxy beam (L/h=120, L=762mm)

Mode	Neglecting shear deformation		Including shear deformation		Percentage difference (%)
	Neglecting rotary inertia	Including rotary inertia	Neglecting rotary inertia	Including rotary inertia	
1	16.2	16.2	16.2	16.2	0.0
2	101.6	101.6	101.3	101.3	0.3
3	284.5	284.5	282.2	282.2	0.8
4	557.6	557.3	549.3	549.1	1.5
5	921.7	921.1	900.0	899.5	2.5
6	1377	1375.5	1330.4	1329.3	3.6

Table 6. The shear and rotary inertia effects on out-of-plane bending frequencies (Hz) of C-C graphite-epoxy beam (L/h=120, L=762mm)

Mode	Neglecting shear deformation		Including shear deformation		Percentage difference (%)
	Neglecting rotary inertia	Including rotary inertia	Neglecting rotary inertia	Including rotary inertia	
1	103.0	103.0	102.7	102.7	0.3
2	285.0	285.0	281.3	281.3	1.3
3	558.0	558.0	547.0	546.9	2.0
4	922.0	922.0	895.3	894.9	3.0
5	1378.0	1377.0	1321.8	1320.9	4.3
6	1924.0	1922.0	1821.7	1820.1	5.7

Table 7. The shear and rotary inertia effects on in-plane bending frequencies (Hz) of C-F graphite-epoxy beam (L/h=120, L=762mm)

Mode	Without shear deformation		With shear deformation		Percentage difference (%)
	Without rotary inertia	With rotary inertia	Without rotary inertia	With rotary inertia	
1	13.3	13.3	13.3	13.3	0.0
2	83.3	83.3	83.1	83.1	0.2
3	233.3	233.2	232.1	232.1	0.5
4	457.1	456.9	452.9	452.7	0.9
5	755.6	755.1	744.6	744.1	1.5
6	1128.7	1127.6	1104.9	1103.9	2.2

Table 8. The shear and rotary inertia effects on in-plane bending frequencies (Hz) of C-C graphite-epoxy beam (L/h=120, L=762mm)

Mode	Neglecting shear deformation		Including shear deformation		Percentage difference (%)
	Neglecting rotary inertia	Including rotary inertia	Neglecting rotary inertia	Including rotary inertia	
1	84.6	84.6	84.4	84.4	0.2
2	233.3	233.3	231.6	231.6	0.7
3	457.3	457.2	451.8	451.7	1.2
4	756.0	755.6	742.2	741.9	1.9
5	1129.3	1128.5	1100.5	1099.8	2.7
6	1577.4	1575.6	1524.0	1522.6	3.6

As presented in Tables 5-8, it can be seen that, the classical theory for the thin beam with shear deformation and rotary inertia neglected yields accurate results for the lower modes and when the number of modes increases percentage differences increase.

The effects of shear deformation and rotary inertia on out-of-plane and in-plane bending frequencies of the thick beam are presented in Tables 9-12 for C-F and C-C boundary conditions. As shown, ignoring the shear deformation and rotary inertia leads to an over prediction of the natural frequencies of the thick beam. The classical theory yields strikingly inaccurate results for the thick beam (high percentage differences). The percentage differences increase with increasing number of modes and increasing number of constrains for boundary conditions. The percentage difference is considerably higher for the higher frequencies, for example, the percentage difference for the sixth natural frequency of out-of-plane vibration is 140.6% for clamped-free ends. This value reaches 193.9% for clamped-clamped ends.

Table 9. The shear and rotary inertia effects on out-of-plane bending frequencies (Hz) of C-F graphite-epoxy beam (L/h=15, L=381mm)

Mode	Neglecting shear deformation		Including shear deformation		Percentage difference (%)
	Neglecting rotary inertia	Including rotary inertia	Neglecting rotary inertia	Including rotary inertia	
1	270	270	263	262	3.1
2	1700	1690	1395	1390	22.3
3	4770	4700	3267	3252	46.7
4	9340	9110	5342	5316	75.7
5	15450	14830	7488	7453	107.3
6	23070	21770	9629	9588	140.6

Table 10. The shear and rotary inertia effects on out-of-plane bending frequencies (Hz) of C-C graphite-epoxy beam (L/h=15, L=381mm)

Mode	Neglecting shear deformation		Including shear deformation		Percentage difference (%)
	Neglecting rotary inertia	Including rotary inertia	Neglecting rotary inertia	Including rotary inertia	
1	1730	1730	1307	1306	32.5
2	4770	4730	2931	2926	63.0
3	9350	9180	4835	4824	93.8
4	15450	14980	6853	6837	126.0
5	23080	22030	8918	8899	159.4
6	32240	30200	10990	10970	193.9

Table 11. The shear and rotary inertia effects on in-plane bending frequencies (Hz) of C-F graphite-epoxy beam (L/h=15, L=381mm)

Mode	Neglecting shear deformation		Including shear deformation		Percentage difference (%)
	Neglecting rotary inertia	Including rotary inertia	Neglecting rotary inertia	Including rotary inertia	
1	212	210	209	208	1.9
2	1330	1324	1181	1176	13.1
3	3730	3678	2911	2892	29.0
4	7310	7123	4956	4917	48.7
5	12090	11603	7149	7091	70.5
6	18060	17031	9389	9317	93.8

Table 12. The shear and rotary inertia effects on in-plane bending frequencies (Hz) of C-C graphite-epoxy beam (L/h=15, L=381mm)

Mode	Neglecting shear deformation		Including shear deformation		Percentage difference (%)
	Neglecting rotary inertia	Including rotary inertia	Neglecting rotary inertia	Including rotary inertia	
1	1350	1350	1136	1135	18.9
2	3730	3700	2689	2682	39.1
3	7320	7190	4573	4555	60.7
4	12100	11730	6622	6592	83.6
5	18070	17250	8756	8717	107.3
6	25240	23640	10927	10882	131.9

From the results, it can be seen that the shear deformation and the rotary inertia have great effects on out-of-plane and in-plane bending frequencies of the thick beam L/h=15. It can be shown that the classical theory, which neglects the effects of rotary inertia and shear deformation, over-predicts the natural frequencies for the specified boundary conditions. In other words, the effects of shear deformation and rotary inertia are to decrease the out-of plane and in-plane bending natural frequencies.

4. Conclusion

A finite element model applicable to the dynamic behavior of laminated composite beams has been developed on the basis of first-order shear deformation theory. This model has the capability for determining flexural and torsional frequencies of laminated beams. The shear deformation and rotary inertia effects has been investigated. It has been found that neglecting shear deformation and rotary inertia effects yields strikingly inaccurate results for thick beams. It can be drawn from this research that dynamic problems of laminated composites must be solved employing the rotary inertia and the shear deformation effects in the mathematical model to obtain solutions that are more realistic. Finally, it is shown that the developed model using finite element method provides an appropriate and efficient means in predicting accurate natural frequencies of flexural and torsional vibrations for various configurations and boundary conditions of the composite beams.

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