Fayoum University
Faculty of Engineering
Arch. Eng. Department
First Year

Mathematics & Statistics
Jan., 16, 2010
Final Exam - Final Mark = 75
Time Allowed : 3 hours

Model Answer

1-A) $P(A \cup B) = P(A)+(B) - P(A \cap B)$

$\Rightarrow P(A) + (B) - P(A \cup B) = P(A \cap B)$

$\Rightarrow P(A) + (B) - P(A \cup B) \le 1$

1-B) We choose three-letters from four letters (without order , without replacement)

$\Rightarrow$ Using combination, we may compute the number (N) of all possible three-letter words as $N = {}^4C_3 = 4$

1-C) We may use the Baye's theorem and combination, where,

The event *A* represents : drawing 4 black balls and 2 white balls

The event *I* represents : drawing the box *I*

The event *II* represents : drawing the box *II*

$$P(II/A) = \frac{P(II)P(A/II)}{P(I)P(A/I)+P(II)P(A/II)} = \frac{1/2 * {}^5C_4\,{}^5C_2 \,/\, {}^{10}C_6}{1/2 * {}^5C_4\,{}^5C_2 \,/\, {}^{10}C_6 + 1/2 * {}^7C_4\,{}^3C_2 \,/\, {}^{10}C_6}$$

$\Rightarrow P(II/A) = 0.677$

2-A) The pdf function of the negative binomial distribution is

$$f(x) = {}_{x+r-1}C_x \; p^r \; q^x \quad , \quad x = 0,\ 1,\ 2,\ 3,\ \ldots\ldots$$

$$\Rightarrow \sum_{x=0}^{\infty} f(x) = \sum_{x=0}^{\infty} {}^{x+r-1}C_x \; p^r \; q^x = p^r \sum_{x=0}^{\infty} {}^{x+r-1}C_x \; q^x$$

$$= p^r \left(1 + \; rq + \frac{r(r+1)}{2!} q^2 + \frac{r(r+1)(r+2)}{3!} q^3 + \ldots\ldots\ldots\ldots\ldots\ldots \right)$$

$$= p^r \left(1 + \; (-r)(-q) + \frac{-r(-r-1)}{2!} (-q)^2 + \frac{-r(-r-1)(-r-2)}{3!} (-q)^3 + \ldots\ldots\ldots\ldots\ldots\ldots \right)$$

$\Rightarrow$ This leads to a binomial expansion with negative power

$$\Rightarrow \sum_{x=0}^{\infty} f(x) = \; p^r \; (1 \; - q)^{-r} = p^r \; / \; p^r = 1$$

2-B) This is a geometric distribution:

$f(x) = p\, q^x \quad , \quad x = 0,\ 1,\ 2,\ 3,\ \ldots\ldots$, with p = 1/6 $\Rightarrow$ q = 5/6

$$\Rightarrow \; f(x) = \left(\frac{1}{6} \right) \left(\frac{5}{6} \right)^x \quad , \quad x = 0,\ 1,\ 2,\ 3,\ \ldots\ldots$$

$$\Rightarrow \; F(x) = 1 - q^{x+1} = 1 - \left(\frac{5}{6} \right)^{x+1} \quad , \quad x = 0,\ 1,\ 2,\ 3,\ \ldots\ldots$$

$$\Rightarrow \; P(X + 1 \le 10) = P(X \le 9) = F(9) = 1 - \left(\frac{5}{6} \right)^{10} \; = \; 0.838$$

2-C) Using the property: $\sum_{\text{all u}} \sum_{\text{all v}} f(u,v) = 1 \Rightarrow$ 0.25+0.25+0.25+a=1 , we find that :

a=0.25

Then, we may compute the marginal (pdf's) of u v as:

u	-5	-3	3	5
$f_U(u)$	0.25	0.25	0.5	0

v	0	1	2	3
$f_V(v)$	0.5	0	0.25	0.25

$$\mu_U = \sum_{all\ u} u\ \mathrm{f_U(u)} = -5*0.25-3*0.25+3*0.5+5*0 = -0.5$$

$$\mu_V = \sum_{all\ u} v\ \mathrm{f_V(v)} = 0*0.5+1*0+2*0.25+3*0.25 = 1.25$$

$$E(UV) = \sum_{\text{all u}} \sum_{\text{all v}} u\ v\ f(u,v) = -5*2*0.25+3*3*0.25 = -0.25$$

$$\Rightarrow COV(U,V) = E(UV) = \sum_{\text{all u}} \sum_{\text{all v}} u\ v\ f(u,v) - \mu_U \mu_V$$

=-0.25+0.5*1.25 = 0.375

$\sigma_U^2 = E(U^2) - \mu_U^2$ =12.75 and $\sigma_V^2 = E(V^2) - \mu_V^2$ = 1.688

$$\Rightarrow \rho(U,V) = \frac{COV(U,V)}{\sigma_U \quad \sigma_V} = \frac{0.375}{\sqrt{12.75}\ \sqrt{1.688}} = 0.08$$

3-A) $E(\sigma X^2 - \sigma\mu X) = \sigma E(X^2) - \sigma\mu E(X) = \sigma\left[E(X^2) - \mu^2\right] = \sigma\,\sigma^2 = \sigma^3$

3-B) $\mu_3' = E(Z^3) = \int_{-\infty}^{\infty} Z^3\ f(Z)\,dz = \int_{-\infty}^{\infty} Z^3 \exp(-z^2/2)\,dz$ (Since the function is odd)

$\Rightarrow \int_{-\infty}^{\infty} Z^3 \exp(-z^2/2)\,dz = 0 \quad \Rightarrow \mu_3' = 0$

3-C) $\mu = E[X] = \int_{-\infty}^{\infty} xf(x)\,dx = \int_{-\pi/2}^{\pi/2} x\cos(x)\,dx$ (Since the function is odd)

$\Rightarrow \int_{-\pi/2}^{\pi/2} x\cos(x)\,dx = 0 \Rightarrow \mu = 0$

To compute the median M, we may write:

$$\int_{-\infty}^{M} f(x)\,dx = \frac{1}{2} \quad \Rightarrow \quad \int_{-\pi/2}^{M} \cos(x)\,dx = \frac{1}{2} \Rightarrow \sin(x)\Big|_{-\pi/2}^{M} = 1 \quad \Rightarrow \sin(M) + \sin(\pi/2) = 1$$

$\Rightarrow M = 0$

4-A) $P\left(\overline{X} \le \mu + Z_{2\alpha}\sigma/\sqrt{n}\right) = P\left(\dfrac{\overline{X}-\mu}{\sigma/\sqrt{n}} \le Z_{2\alpha}\right)$

Using the transformation $Z = \dfrac{\overline{X}-\mu}{\sigma/\sqrt{n}}$, we find that

$$P\left(\frac{\overline{X}-\mu}{\sigma/\sqrt{n}} \le Z_{2\alpha}\right) = P\left(Z \le Z_{2\alpha}\right) = \phi\left(Z_{2\alpha}\right) = 1-2\alpha$$

4-B) $\bar{x} = \dfrac{1}{n}\sum_{i=1}^{n} x_i = \dfrac{1}{10}\sum_{i=1}^{10} x_i = 25.5$, $s^2 = \dfrac{1}{(n-1)}\sum_{i=1}^{n}(x_i-\bar{x})^2 = \dfrac{1}{9}\sum_{i=1}^{10}(x_i-25.5)^2 = 10.5$

$\Rightarrow$ $s = 3.24$

σ is unknown and n is small and $X \sim N(\mu,\sigma)$ $\Rightarrow$ use t-distribution

$\Rightarrow$ $P\left(\overline{X} \le 26.417\right) = P\left(\dfrac{\overline{X}-\mu}{s/\sqrt{n}} \le \dfrac{26.417-25}{3.24/\sqrt{10}}\right)$ (Assuming $\mu = 25$)

$\Rightarrow$ $P\left(t_{9,\alpha} \le 1.383\right)$ = 1-0.1 = 0.9

(1-□) % confidence interval on μ is $\left(\overline{X} - t_{n-1,\alpha/2}\, s/\sqrt{n},\ \overline{X} + t_{n-1,\alpha/2}\, s/\sqrt{n}\right)$

$\Rightarrow$ 95 % confidence interval on μ is $\left(25.5 - t_{9,0.025}\, 3.24/\sqrt{10},\ 25.5 + t_{9,0.025}\, 3.24/\sqrt{10}\right)$

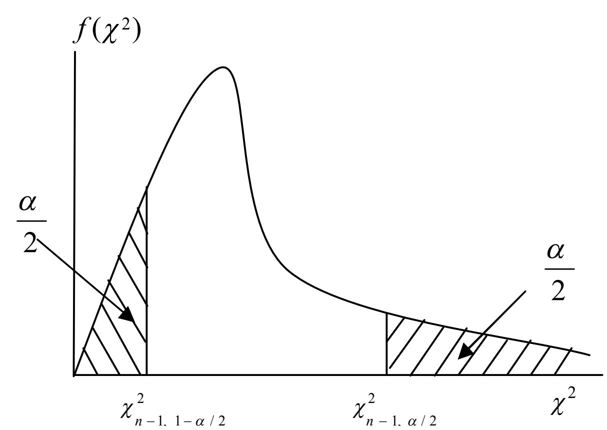
= $(\,22.182,\quad 27.818\,)$

(1-□) % confidence interval on σ^2 is $\left(\dfrac{(n-1)s^2}{\chi^2_{n-1,\,\alpha/2}}\ ,\ \dfrac{(n-1)s^2}{\chi^2_{n-1,\,1-\alpha/2}}\right)$

$\Rightarrow$ 95 % confidence interval on σ is

$$\left(3.24\sqrt{\frac{9}{\chi^2_{9,\ 0.025}}}\ ,\ s\sqrt{\frac{9}{\chi^2_{9,\ 0975}}}\right)$$

= $(\,2.23,\quad 5.92\,)$



5-A) σ is unknown and n is small and $X \sim N(\mu, \sigma)$ $\Rightarrow$ using t-distribution, we may write:

$H_0: \quad \mu = \mu_0$

$H_A: \quad \mu \leq \mu_0$

$P\left(t > -t_{n-1,\alpha}\right) = 1-\alpha$. □ $P\left(t\frac{s}{\sqrt{n}} > -t_{n-1,\alpha}\frac{s}{\sqrt{n}}\right)$ □ $P\left(\mu_0 + t\frac{s}{\sqrt{n}} > \mu_0 - t_{n-1,\alpha}\frac{s}{\sqrt{n}}\right) = P\left(\overline{X} > C\right)$

Where, C is the required criteria $\Rightarrow \quad C = \mu_0 - t_{n-1,\alpha}\frac{s}{\sqrt{n}}$

We accept H_0 if $\overline{X} > C$ and reject H_0 if $\overline{X} \leq C$

5-B) The given data: μ_0 =36 , μ_A=34 , σ =1.52 α =0.05 β =0.1

The required sample size is

$$n = \left(\frac{\sigma\,(Z_\alpha + Z_\beta)}{\mu_0 - \mu_A}\right)^2 = \left(\frac{1.52\,(1.645 + 1.28)}{36-34}\right)^2 \cong 5$$

$H_0: \quad \mu = 36$

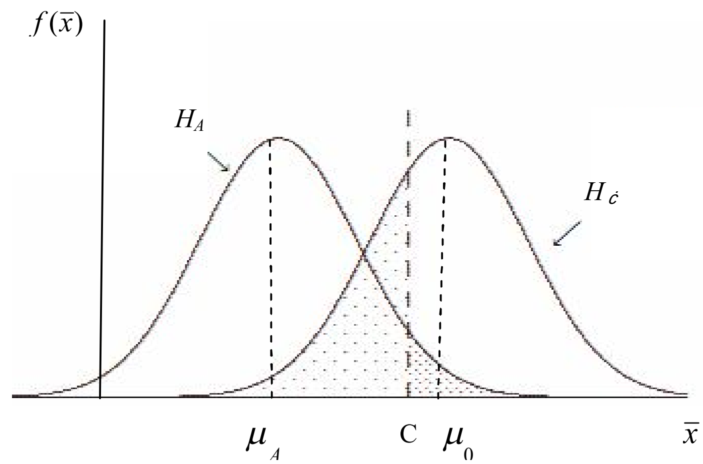
$H_A: \quad \mu \leq 36$

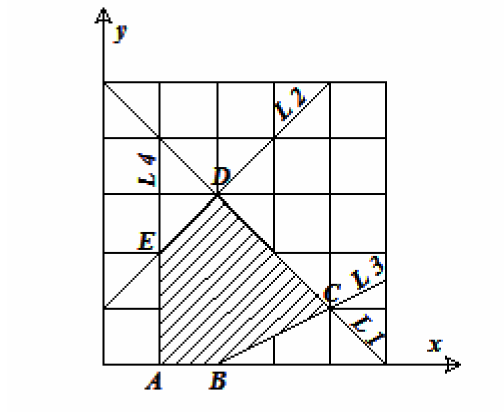
The required criteria, $C = \mu_0 - z_\alpha \frac{\sigma}{\sqrt{n}}$

$= 36 - 1.645\frac{1.52}{\sqrt{5}}$ =34.88

Since $\bar{x} = 35 \Rightarrow \bar{x} > C$

$\Rightarrow$ Decision : We accept H_0 .





6-A) A=(1,0) , B=(2,0)

C is the intersection of the two lines L1 and L3 $\Rightarrow$ C=(4,1)

D is the intersection of the two lines L1 and L2 $\Rightarrow$ D=(2,3)

E is the intersection of the two lines L2 and L4 $\Rightarrow$ E=(1,2)

We may construct the following table

Point	A(1,0)	B(2,0)	C(4,1)	D(2,3)	E(1,2)
Z=12x-4y	12	24	44	12	4

$\Rightarrow$ z_{min} =4 at point E = (1,2)

6-B) The objective function is $z = 10x + 5y$

$\Rightarrow z - 10x - 5y = 0$ ....(1)

The constraints are : $-x + y \leq 1$, $x + y \leq 2$, Where, $x \geq 0$, $y \geq 0$.

Using the *simplex method*, the constraints are written in the form:

$-x + y + s_1 = 1$ ....(2)

$x + y + s_2 = 2$ ....(3)

Using equations (1, 2 and 3), we may construct the following table to compute the z_{max}

pivot column ↓ (x)

	z	x	y	s_1	s_2	Solution	Ratio
z	1	-10	-5	0	0	0	----
s_1	0	-1	1	1	0	1	-1
s_2	0	1	1	0	1	2	2
z	1	0	5	0	10	20	----
s_1	0	0	2	1	1	3	----
x	0	1	1	0	1	2	----

← pivot row (s_2)

Note: Stop calculations in case of all coefficients in z are non-zero

$\Rightarrow$ z is maximum $\Rightarrow$ z_{max} = 20

مع دعــواتي بالتفـــوق **د / كــارم محمــود**