Fayoum University
Faculty of Engineering
First Year Electrical Eng. Department

Engineering Mathematics (2A) Final Exam.
Jan., 14, 2009
Time Allowed: 3 Hours

Attempt All Questions:

Q1. Find z_{xx}, z_{yy} and z_{xy} for $z = f(x^2 y, x+y)$.

Q2. Test for extermum and saddle points the function $f(x,y) = \cosh x \sin y, \quad -\pi \le y \le \pi$

Q3. A topless, cylindrical metal tank is to be built to hold 512 π cubic meters of water. Determine the dimensions of the tank that will require the least amount of metal to build.

Q4. Find the integral $\int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{0}^{x^2+y^2} \ln(x^2+y^2)\, dz\,dy\,dx$

Q5. Find the center of the mass of the region inside both of the sphere $x^2+y^2+(z-2)^2 = 4$ and the cone $x^2+y^2 = 3z^2$ if the density of the filled material= $2z$.

Q6. Find the surface integral $\iint_S [(x^2+y^2)\mathbf{i} + 2xy\mathbf{j}] \bullet \mathbf{n}\,dS$, where S is the paraboloid $z= 9-x^2-y^2$ for $z \ge 0$.

Q7. Verify the divergence theorem for $\mathbf{A} = (x+y)\mathbf{i} + (y+z)\mathbf{j} + (x+z)\mathbf{k}$, and S is the surface bounds the region D given by $x^2+y^2-4 \le z \le 4-x^2-y^2,\ 0 \le x^2+y^2 \le 4.$

Q8. Evaluate $\oint_C [(4x-2y)dx + (2x-4y)dy]$; C is the circle $(x-2)^2+(y-2)^2 = 4$

Q9. Find the area of the region common to the cardioids $r = 1+\cos\theta$ and $r = 1-\cos\theta$.

Q10. Test the following series for convergent or divergent

$$\sum_{n=1}^{\infty} \frac{2^n}{n+1},\ \sum_{n=1}^{\infty} \frac{\ln n}{n\sqrt{n+1}},\ \sum_{n=2}^{\infty} (-1)^{n+1}\frac{1}{n \ln n}$$

Best Wishes
Prof. Mohamed Eissa Sayed Ahmed

Q1: $z = f(x^2 y, x+y)$

Let $u = x^2 y$ and $v = x+y$

$\therefore z = f(u,v)$, $\frac{\partial u}{\partial x} = 2xy$, $\frac{\partial v}{\partial x} = 1$

$\frac{\partial u}{\partial y} = x^2$ $\quad \frac{\partial v}{\partial y} = 1$

$$\therefore \frac{\partial z}{\partial x} = f_u \frac{\partial u}{\partial x} + f_v \frac{\partial u}{\partial x} = 2xy\, f_u + f_v$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left[2xy\, f_u + f_v\right] = 2y f_u + 2xy\left[f_{uu}\frac{\partial u}{\partial x} + f_{uv}\frac{\partial v}{\partial x}\right] + f_{uv}\frac{\partial u}{\partial x} + f_{vv}\frac{\partial v}{\partial x}$$

$$= 2y f_u + 4x^2y^2 f_{uu} + 2xy\, f_{uv} + 2xy\, f_{uv} + f_{vv}$$

3 $\boxed{z_{xx} = 2y\, f_y + 4x^2y^2 f_{uu} + 4xy\, f_{uv} + f_{vv}}$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left[2xy\, f_u + f_v\right] = 2x f_u + 2xy\left[f_{uu}\frac{\partial u}{\partial y} + f_{uv}\frac{\partial v}{\partial y}\right] + f_{uv}\frac{\partial u}{\partial y} + f_{vv}\frac{\partial v}{\partial y}$$

3 $\boxed{= 2x f_u + 2x^3 y\, f_{uu} + 2xy\, f_{uv} + x^2 f_{uv} + f_{vv}}$

$$\frac{\partial z}{\partial y} = f_u \frac{\partial u}{\partial y} + f_v \frac{\partial v}{\partial y} = x^2 f_u + f_v$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left[x^2 f_u + f_v\right] = x^2\left[f_{uu}\frac{\partial u}{\partial y} + f_{uv}\frac{\partial v}{\partial y}\right] \cancel{+ f_{uv}\frac{\partial u}{\partial y}} + f_{uv}\frac{\partial u}{\partial y} + f_{vv}\frac{\partial v}{\partial y}$$

3 $\boxed{\begin{aligned} z_{yy} &= x^4 f_{uu} + x^2 f_{uv} + x^2 f_{uv} + f_{vv} \\ &= x^4 + 2x^2 f_{uv} + f_{vv} \end{aligned}}$

Q_2: $f(x,y) = \cosh x \sin y$, $-\pi \le y \le \pi$

(2) $\begin{cases} f_x = \sinh x \sin y = 0 \Rightarrow \sin y = 0 \Rightarrow y = 0 \text{ or } \sinh x = 0 \Rightarrow x = 0 \\ f_y = \cosh x \cos y = 0 \Rightarrow \cos y = 0 \Rightarrow y = \pm\frac{(2n+1)}{2}\pi \end{cases}$

(3) or $y = \pm\frac{\pi}{2}$ on $[-\pi, \pi]$

$\therefore$ C.P. are $(0, \frac{\pi}{2})$ and $(0, -\frac{\pi}{2})$

x	y	f_{xx} $\cosh x \sin y$	f_{yy} $-\cosh x \sin y$	f_{xy} $\sinh x \cos y$	Δ $f_{xx}f_{yy} - f_{xy}^2$	Conclusion
0	$\frac{\pi}{2}$	1	-1	0	$-1 < 0$	saddle point
0	$-\frac{\pi}{2}$	-1	1	0	$-1 < 0$	saddle point

(4)

Q_3: $V = 512\pi \Leftarrow$ given

$\therefore \pi r^2 h = 512\pi$ or

$g(r,h) = r^2 h = 512$ (1) $\Rightarrow$ (2)

Req. min Cost $= S = 2\pi r h + \pi r^2$

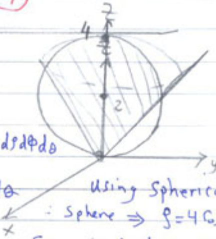
$\therefore f(r,h) = 2\pi r h + \pi r^2$ (2) $\Rightarrow$ (2)

Using Largange multipliers, $\frac{f_r}{g_r} = \frac{f_h}{g_h}$ (1)

$\therefore \frac{2\pi h + 2\pi r}{2rh} = \frac{2\pi r + 0}{r^2} \Rightarrow 2h = h + r \Rightarrow h = r$

$\therefore r^3 = 512 \Rightarrow r = (512)^{\frac{1}{3}} = 8$

$\therefore \boxed{r = h = 8 \text{ m}} \Rightarrow$ (4)



Q_5: $(\bar{x}, \bar{y}, \bar{z}) = ???$, $\sigma = 2z$

for symmetry $\boxed{\bar{x} = \bar{y} = 0}$ (2)

$M = \iiint_{R_{xyz}} 2z\, dv = 2\int_0^{2\pi}\int_0^{\pi/3}\int_0^{4\cos\phi} \rho^3 \cos\phi \sin\phi \, d\rho\, d\phi\, d\theta$

(3) $= \frac{1}{2}\int_0^{2\pi}\int_0^{\pi/3} (4\cos\phi)^4 \cos\phi \sin\phi \, d\phi\, d\theta$

$= \frac{256}{2} \int_0^{2\pi} \left(-\frac{\cos^6\phi}{6}\right)\Big|_0^{\frac{\pi}{3}} d\theta$

$= \frac{64}{3}(2\pi)\left[1 - \left(\frac{1}{2}\right)^6\right] = \left(\frac{64}{3}\right)\left(\frac{63}{64}\right)2\pi = \frac{126}{3}\pi$

$= 126\pi$

Using Spherical Cord.

$\therefore$ Sphere $\Rightarrow \rho = 4\cos\phi$

Cone: $\phi = \tan^{-1}\sqrt{3} = \frac{\pi}{3}$

(2)

$$I_3 = \iiint_{R_{xy}} z\,(2z)\,dV = 2\int_0^{2\pi}\int_0^{\frac{\pi}{3}}\int_0^{4\cos\phi} \rho^4 \cos^2\phi \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \frac{2}{5}\int_0^{2\pi}\int_0^{\frac{\pi}{3}} (4\cos\phi)^5 \sin\phi \cos^2\phi \, d\phi \, d\theta$$

$$= \frac{2(4)^5}{5}\int_0^{2\pi} \left(-\frac{\cos^8\phi}{8}\right)\Bigg|_0^{\frac{\pi}{3}} d\theta$$

$$= \frac{2(4)^5}{5(8)}\left[1-\left(\tfrac{1}{2}\right)^8\right](2\pi) = \frac{2(4)^5}{5(8)}\left(\frac{255}{256}\right)(2\pi)$$

$$= \frac{255(2\pi)}{5} = 102\pi$$

2

$$\therefore\ \boxed{\bar{z} = \frac{102\pi}{\frac{126\pi}{3}} \cong 2.43}$$

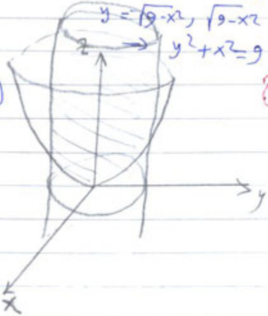
$$Q_4:\ I = \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_0^{x^2+y^2} \ln(x^2+y^2)\, dz\, dy\, dx$$

Using cylindrical coord.

$z=0$ & $z = x^2+y^2 \Rightarrow$ Paraboloid

$y = \sqrt{9-x^2}$, $\sqrt{9-x^2}$ $\rightarrow$ $y^2+x^2=9 \Rightarrow$ Cylinder

2



$$2 \Rightarrow I = \int_0^{2\pi}\int_0^{3}\int_0^{r^2} \ln r^2 \,(r\,dz\,dr\,d\theta)$$

$$= 2\int_0^{2\pi}\int_0^{3} r^3 \ln r \, dr\, d\theta$$

3

$$= 2\int_0^{2\pi}\left[\frac{r^4}{4}\ln r\Big|_0^3 - \int_0^3 \frac{r^4}{4}\frac{dr}{r}\right] d\theta$$

$$= 2\int_0^{2\pi}\left[\frac{81}{4}\ln 3 - \frac{r^4}{16}\Big|_0^3\right] d\theta$$

$$= 2\left[\frac{81}{4}\ln 3 - \frac{81}{16}\right] 2\pi = \frac{162}{16}\left[4\ln 3 - 1\right] 2\pi \cong 58.74\,\pi$$

2

Q_6: $I = \iint_S [(x^2+y^2)\, i + 2xy\, j] \cdot \vec{n}\, dS$

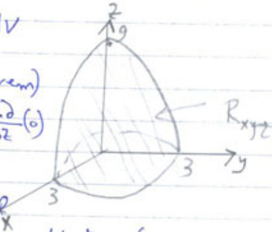
$$I = \iint_S \vec{A} \cdot \bar{n}\, dS = \iiint_{R_{xyz}} \nabla \cdot A\, dV$$

$\Rightarrow$ (Gauss's theorem)

3

$$\nabla \cdot A = \frac{\partial}{\partial x}(x^2+y^2) + \frac{\partial}{\partial y}(2xy) + \frac{\partial}{\partial z}(0)$$

$$= 2x + 2x = 4x$$



$$\therefore I = \iiint_{R_{xyz}} 4x\, dV$$

$\Rightarrow$ Using ~~Polar~~ Cylindrical coordinates (r, θ, z)

3

$$\Rightarrow \therefore I = 4 \int_0^{2\pi}\int_0^{3}\int_0^{9-r^2} r\cos\theta\, (r\, dz\, dr\, d\theta)$$

3

$$= 4 \int_0^{2\pi}\int_0^{3} r^2(9-r^2)\cos\theta\, dr\, d\theta$$

$$= 4 \left[\int_0^{3} r^2(9-r^2)\, dr\right]\left[\int_0^{2\pi} \cos\theta\, d\theta\right]$$

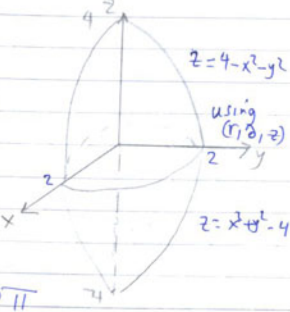
$$= \quad '' \quad [\sin\theta]_0^{2\pi} = 0$$

Q_7: $\vec{A} = (x+y)\, i + (y+z)\, j + (x+z)\, k$

$$\iint_S \vec{A} \cdot \vec{n}\, dS = \iiint_{R_{xyz}} \nabla \cdot A\, dV$$

$$\nabla \cdot A = \frac{\partial}{\partial x}(x+y) + \frac{\partial}{\partial y}(y+z) + \frac{\partial}{\partial z}(x+z)$$

$$= 1 + 1 + 1 = 3$$



$$R.H.S. = 3\int_0^{2\pi}\int_0^{2}\int_{r^2-4}^{4-r^2} r\, dz\, dr\, d\theta$$

5

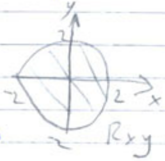
$$= 3\int_0^{2\pi}\int_0^{2} r(8-2r^2)\, dr\, d\theta$$

$$= 3\left[\frac{(8-2r^2)^2}{2(-4)}\right]_0^2 (2\pi) = 48\pi$$

$$\vec{n} = \frac{2xi+2yj+k}{\sqrt{1+4(x^2+y^2)}}, \quad \vec{A}\cdot\vec{n} = \frac{2x(x+y)+2y(y+z)+x+z}{\sqrt{1+4(x^2+y^2)}}$$

(4)

$$L.H.S. = 2\iint_{R_{xy}} \frac{[2(x^2+y^2)+2xy+2yz+x+z]}{\cancel{\sqrt{}}} \cancel{\sqrt{}}\, dx\,dy$$



$$= 2\int_0^{2\pi}\int_0^2 [2r^2 + 2r^2\cos\theta\sin\theta + 2r\sin\theta(4-r^2) + r\cos\theta + 4 - r^2]\, r\,dr\,d\theta$$

$$= 2\int_0^{2\pi}\int_0^2 [r^3 + 4r + r^3\sin 2\theta + (8r^2-2r^4)\sin\theta + r^2\cos\theta]\,dr\,d\theta$$

$$= 2\left\{\left[\frac{1}{4}r^4 + 2r^2\right]_0^2 (2\pi) + 0 + 0 + 0\right\}$$

$$= 4\pi[4+8] = 48\pi$$

$\therefore$ L.H.S. = R.H.S.

Q8: $I = \oint_C [(4x-2y)\,dx + (2x-y)\,dy]$

$$= \iint_{R_{xy}} \left[\frac{\partial(2x-y)}{\partial x} - \frac{\partial}{\partial y}(4x-2y)\right] dx\,dy$$

using Green's theorem

$\boxed{9}$

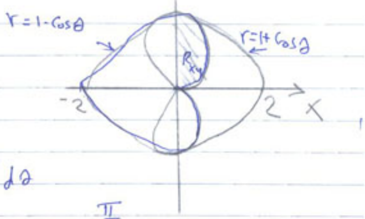
$$= \iint_{R_{xy}} [2+2]\,dx\,dy$$

$$= 4\iint_{R_{xy}} dxy = 4(\text{Area of the Circle})$$

$$= 4(2)^2\pi = 16\pi^2$$

Q9:

(3)
$$A = 4 R_{xy} = 4\int_0^{\frac{\pi}{2}}\int_0^{1-\cos\theta} r\,dr\,d\theta$$



(6)
$$= 4\int_0^{\frac{\pi}{2}}\left[\frac{r^2}{2}\right]_0^{1-\cos\theta} d\theta$$
$$= 2\int_0^{\frac{\pi}{2}}(1-\cos\theta)^2\,d\theta = 2\int_0^{\frac{\pi}{2}}\left(1-2\cos\theta+\tfrac{1}{2}(1+\cos 2\theta)\right]d\theta$$
$$= 2\int_0^{\frac{\pi}{2}}\left[\frac{3}{2}-2\cos\theta+\frac{1}{2}\cos 2\theta\right)d\theta$$
$$= 2\left[\frac{3}{2}(\pi/2) - 2\sin\theta\Big)_0^{\frac{\pi}{2}} + \frac{1}{4}\sin 2\theta\Big)_0^{\frac{\pi}{2}}\right]$$
$$= 2\left[\frac{3\pi}{4}-2\right] = \frac{3\pi}{2}-4 \cong \underline{0.7124}$$

Q10: $\sum_{n=1}^{\infty}\frac{2^n}{n+1}$ ⇒ Using ratio test,

$$\therefore \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{2^{n+1}/(n+2)}{2^n/(n+1)} = 2\lim_{n\to\infty}\frac{n+1}{n+2} = 2(1) = 2 > 1$$

$\boxed{\therefore \text{the Inf. series is divergent}}$ (3)

$$\sum_{n\to\infty}\frac{\ln n}{n\sqrt{n+1}} \le \sum_{n=1}^{\infty}\frac{\ln n}{n^{3/2}} \Rightarrow \text{Convergent by integral test}$$

Convergent ⇐ $\left[\int_1^{\infty} x^{-3/2}\ln x\,dx = \frac{x^{-\frac{1}{2}}}{(\frac{1}{2})}\ln x\Big]_1^{\infty} - 2\int_1^{\infty}x^{-\frac{3}{2}}dx\right.$ (3)

$$= 2\lim_{x\to\infty}\frac{\ln x}{\sqrt{x}} + 4\,x^{-\frac{1}{2}}\Big]_1^{\infty} = \text{Const}$$
 0

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n\ln n}\ ,\quad \sum_{n=2}^{\infty}\frac{1}{n\ln n} \Rightarrow \text{using Integral test}$$

$$\therefore \int_2^{\infty}\frac{\frac{dx}{x}}{\ln x} = \ln\ln x\Big]_2^{\infty} = \infty \Rightarrow \text{divergent}$$ (3)

Convergent by Liebnitz test since $\lim_{n\to\infty}\frac{1}{n\ln n} = 0$ and its decreasing, then the series is conditionally conv