Model Answer

Fayoum University
Faculty of Engineering
Dept. of Electrical Engineering
Second Year

Mathematics
First term of 2009/2010
Final Exam
Time : 3 hours

Model answer for the following **Five** questions:

Question 1 [15 points]

Mention True (T) or False (F), justify your answer

1.1 If **u**, **v**, **w** are three linearly dependent vectors then **w** must be expressed as linear combination of **u**, **v**. **[3 points]**

1.2 If Matrix **A** and **B** are commute, Matrix **B** and **C** are commute, then **A** and **C** are commute. **[4 points]**

1.3 There exists an invertible 3x3 matrix **A** such that A^2 is zero matrix. **[4 points]**

1.4 If two invertible matrices **A**, **B** commute then A^{-1}, B^{-1} must commute as well. **[4 points]**

Solution:

1.1 **T**;

1.2 **F**; in case that **B** = **I**; then **A I** = **I A**, and **I C** = **C I** for any **A** and **B** matrix, and in general there exists matrices **A** and **C** such that **A C** ≠ **C A**.

1.3 **F**; the statement will not be right if **A** = **Zero matrix**.

1.4 **T**; $\mathbf{A}^{-1}\mathbf{B}^{-1} = (\mathbf{BA})^{-1} = (\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$

Question 2 [15 points]:

2.1 Prove :

i. If **U** and **V** be unitary matrices then $\mathbf{U}^{-1}$ and **UV** are unitary. [3 points]:

ii. If **A** is a symmetric invertible matrix then $\mathbf{A}^{-1}$ is also symmetric [3 points]:

2.2 Prove that the set of vectors: $(1\ ,\ -1\ ,\ 4)^T, (2\ ,\ 0\ ,\ 3)^T, (1\ ,\ 1\ ,\ -6)^T$ is a set of linearly independent vectors, and then transform it to a set of orthonormal vectors. [9 points]:

Solution:

2.1

i Θ **U** is unitary matrix

$$\therefore \mathbf{U}\mathbf{U}^{*T} = \mathbf{I} \Rightarrow \mathbf{U}^{*T} = \mathbf{U}^{-1} \Rightarrow \mathbf{I} = \mathbf{U}^{-1}\left(\mathbf{U}^{*T}\right)^{-1} \Rightarrow \mathbf{U}^{-1}\left(\mathbf{U}^{-1}\right)^{*T} = \mathbf{I}$$

$\therefore \mathbf{U}^{-1}$ is also unitary

Θ **U** and **V** are unitary

$$\therefore \mathbf{U}\mathbf{U}^{*T} = \mathbf{I};\ \text{and}\ \mathbf{V}\mathbf{V}^{*T} = \mathbf{I}$$

$$\therefore (\mathbf{UV})(\mathbf{UV})^{*T} = \mathbf{U}\left(\mathbf{V}\mathbf{V}^{*T}\right)\mathbf{U}^{*T} = \mathbf{U}(\mathbf{I})\mathbf{U}^{*T} = \mathbf{U}\mathbf{U}^{*T} = \mathbf{I}$$

$\therefore$ **UV** is also unitary

ii Θ **A** is symmetric matrix

$$\therefore \mathbf{A} = \mathbf{A}^T \Rightarrow (\mathbf{A})^{-1} = \left(\mathbf{A}^T\right)^{-1} = \left(\mathbf{A}^{-1}\right)^T$$

$\therefore \mathbf{A}^{-1}$ is also symmetric

2.2

- arrange the three mentioned vectors in a matrix **A** such that:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 1 \\ 4 & 3 & -6 \end{bmatrix}$$

- Compute $|\mathbf{A}|$

$|\mathbf{A}| = 2(6\text{-}4)\text{+}3(1+1) = 10 \neq 0$

$\therefore (1\ ,\ -1\ ,\ 4)^T, (2\ ,\ 0\ ,\ 3)^T, (1\ ,\ 1\ ,\ -6)^T$ are linearly independent

- Apply Gram-Schmidt orthonormalization technique:

Let $\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ -6 \end{bmatrix}$

Let $\mathbf{v}_1 = \mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$

Let $\mathbf{v}_2 = \mathbf{u}_2 + \beta_1 \mathbf{v}_1$; where $\beta_1 = -\frac{\langle u_2, v_1 \rangle}{\|v_1\|^2} = -\frac{7}{9}$

$$\therefore \mathbf{v}_2 = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} - \frac{7}{9} \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{11}{9} \\ \frac{7}{9} \\ -\frac{1}{9} \end{bmatrix}$$

Let $\mathbf{v}_3 = \mathbf{u}_3 + \gamma_1 \mathbf{v}_1 + \gamma_2 \mathbf{v}_2$; where $\gamma_1 = -\frac{\langle u_3, v_1 \rangle}{\|v_1\|^2} = -\frac{4}{3}$; $\gamma_2 = -\frac{\langle u_3, v_2 \rangle}{\|v_2\|^2} = -\frac{24}{19}$

$$\therefore \mathbf{v}_3 = \begin{bmatrix} \frac{15}{19} \\ -\frac{25}{19} \\ -\frac{10}{19} \end{bmatrix}$$

The Orthonormal set of vectors: $\mathbf{w}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \begin{bmatrix} \frac{1}{\sqrt{18}} \\ -\frac{1}{\sqrt{18}} \\ \frac{4}{\sqrt{18}} \end{bmatrix}$; $\mathbf{w}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \begin{bmatrix} \frac{11\sqrt{19}}{57} \\ \frac{7\sqrt{19}}{57} \\ -\frac{\sqrt{19}}{57} \end{bmatrix}$;

$$\mathbf{w}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \begin{bmatrix} \frac{3}{\sqrt{38}} \\ -\frac{5}{\sqrt{38}} \\ -\frac{2}{\sqrt{38}} \end{bmatrix}$$

Question 3 [15 points]:

Consider the system of linear equations:

$$x + y + z = 1$$
$$2x + z = 2$$
$$-x + y + az = b$$

3.1 Study the solution in case of:

i) $a = 1, b = -1$

ii) $a = 0, b = 1$

iii) $a = 0, b = -1$

3.2 Solve the system for $a = b = 1$.

Solution:

3.1 If we eliminate the augmented matrix we will get:

$$[A|b] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 2 \\ -1 & 1 & a & b \end{array}\right] \underset{\substack{H_{13}(1) \\ H_{23}(1)}}{\overset{H_{12}(-2)}{\sim}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -2 & -1 & 0 \\ 0 & 0 & a & b+1 \end{array}\right]$$

i) Exactly one solution: The matrix is invertible.

ii) No solutions: Get a row of zeroes in the matrix with no zero in the augmented column.

iii) Infinitely many solutions:

3.2 Using back substitution we get $x = [0 \quad -1 \quad 2]$

Question 4 [20 points]:

Let **A** be the matrix

$$\mathbf{A} = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$

4.1 Find the characteristic equation, and then find the eigenvalues of the matrix **A**.

4.2 Find the eigenvectors of the matrix **A**.

4.3 Is the matrix **A** diagonalizable? Why?

4.4 Find the minimal polynomial of matrix **A**.

4.5 Find the matrix Exp (**At**).

Solution:

4.1

$$\Theta \; |\mathbf{A} - \lambda \mathbf{I}| = 0$$

$$\therefore \begin{vmatrix} 4-\lambda & 2 & 2 \\ 2 & 4-\lambda & 2 \\ 2 & 2 & 4-\lambda \end{vmatrix} = 0$$

$\therefore \lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$

$\therefore \lambda_i = 2, 2, 8 \quad \forall\, i = 1, 2, 3$ Prove?

4.2

For $\lambda = 2$

$(\mathbf{A} - 2\mathbf{I})\mathbf{x} = \mathbf{0}$

$$\therefore \mathbf{x} = c_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$ Prove?

$$\therefore \mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} ; \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

For $\lambda = 8$

$(\mathbf{A} - 8\mathbf{I})\mathbf{x} = \mathbf{0}$

$$\therefore \mathbf{x} = c_3 \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$$

$$\therefore \mathbf{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

4.3

$\Theta\ M_{\lambda=2} = m_{\lambda=2} = 2$

ΘCMd4≠≠TibA$_{\lambda=8}$w€l, ; M, .C;, +TA Cx3TC-4ATdA04:: .C;, +TA Cx3TC-4ATdA0xGA4

Matrix is diagonalizable

4.4 $m(\lambda) = (\lambda - 2)(\lambda - 8) = 0$

4.5

$$\Theta\ e^{At} = \alpha_0 \mathbf{I} + \alpha_1 \mathbf{A}$$

$$\Theta\ e^{\lambda t} = \alpha_0 + \alpha_1 \lambda$$

$$\therefore e^{2t} = \alpha_0 + 2\alpha_1$$

$$\therefore e^{8t} = \alpha_0 + 8\alpha_1$$

$$\therefore \alpha_0 = \frac{4e^{2t} - e^{8t}}{3} \ ; \ \alpha_1 = \frac{-e^{2t} + e^{8t}}{6}$$

$$\therefore e^{At} = \begin{bmatrix} \frac{1}{3}e^{8t} + \frac{2}{3}e^{2t} & \frac{1}{3}e^{8t} - \frac{1}{3}e^{2t} & \frac{1}{3}e^{8t} - \frac{1}{3}e^{2t} \\ \frac{1}{3}e^{8t} - \frac{1}{3}e^{2t} & \frac{1}{3}e^{8t} + \frac{2}{3}e^{2t} & \frac{1}{3}e^{8t} - \frac{1}{3}e^{2t} \\ \frac{1}{3}e^{8t} - \frac{1}{3}e^{2t} & \frac{1}{3}e^{8t} - \frac{1}{3}e^{2t} & \frac{1}{3}e^{8t} + \frac{2}{3}e^{2t} \end{bmatrix}$$

<u>Question 5</u> [20 points]:

Convert the differential equations:

$$\frac{d^3 y}{d t^3} + \frac{d^2 y}{d t^2} = 0$$

With initial conditions: $y(0) = y'(0) = y''(0) = 1$

Into a system of three first order differential equations and solve the resulting state equation.

<u>Solution:</u>

$$\text{Let}\quad y'(t) = x(t)$$
$$y''(t) = x'(t) = z(t)$$
$$y'''(t) = x''(t) = z'(t)$$

$$\therefore x'(t) = z(t)$$
$$y'(t) = x(t)$$
$$z'(t) = -z(t)$$

$$\therefore \begin{bmatrix} x'(t) \\ y'(t) \\ z'(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

$$\Theta\ \mathbf{w}(t) = e^{At}\mathbf{w}(0)$$

$$\Theta\ e^{At} = \begin{bmatrix} 1 & 0 & 1-e^{-t} \\ t & 1 & t+e^{-t}-1 \\ 0 & 0 & 2t+e^{-t} \end{bmatrix}$$ Prove?

$$\therefore \mathbf{w}(t) = \begin{bmatrix} 2-e^{t} \\ 2t+e^{t} \\ e^{t} \end{bmatrix}$$

(Good Luck)

Dr. Eng. Waleed ahmed