Fayoum University
Faculty of Engineering
Dept. of Electrical Engineering (Communication Engineering Section)
Third Year

Mathematics
First Term of 2008/2009
Final Exam
Time: 3 hours

Answer the following **SEVEN** questions:

1) The joint probability density function of two random variables X and Y is given by:

$$f_{X,Y}(x,y)=\begin{cases} K\cos(x+y) & 0<x<0.25\pi \ , \ 0<y<0.25\pi \\ 0 & \text{otherwise} \end{cases}$$

a) Find the probability density function of X.

b) Find the constant K.

c) Find the best predictor $\hat{Y}$ of Y in the least mean squares sense.

(16 points)

2) A random process is defined by:

$$Z_t = X\frac{1}{1+t^4}$$

where X is a continuous random variable having the probability density function:

$$f_X(x)=\begin{cases} \dfrac{1}{a} & 0<x<a \\ 0 & \text{otherwise} \end{cases}$$

a) Find the mean and variance of X.
b) Find the autocorrelation function $R_Z(t_1,t_2)$ of Z_t.
c) Is the random process Z_t stationary in the wide sense? Why?
d) Is the random process Z_t stationary in the strict sense? Why?

(12 points)

3) Solve the partial differential equation:

$$\frac{\partial^2 u}{\partial x^2}=\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \qquad 0<x<\pi \quad , \quad t>0$$

subject to the boundary conditions: $u(0,t)=0 \quad , \quad u(\pi,t)=0 \qquad t>0$

and the initial conditions: $u(x,0)=f(x)=4\sin(2x)+13\sin(5x) \qquad 0<x<\pi$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0}=g(x)=3\sin(7x)+6\sin(10x) \qquad 0<x<\pi$$

(15 points)

4) a) Write the general solution of the following differential equations:

(i) $x^2y''+xy'+\left(81x^2-25\right)y=0$

(ii) $x^2y''+xy'-\left(49x^2+64\right)y=0$

(iii) $\left(1-x^2\right)y''-2xy'+\left(72-\dfrac{4}{1-x^2}\right)y=0$

b) What is the value of $\displaystyle\int_0^{\pi}\cos(17y-6\sin y)\,dy$?

(12 points)

5) Evaluate the following definite integrals:

a) $\int_0^\infty t^{-1.75}\left(1-e^{-t}\right)dt$ [Express your answer in terms of $\Gamma(0.25)$]

b) $\int_0^{0.5\pi}\left(\frac{1}{\sin^3 x}-\frac{1}{\sin^2 x}\right)^{\frac{1}{6}}\cos x\,dx$ [Express your answer in terms of $\Gamma\left(\frac{1}{6}\right)$ and $\Gamma\left(\frac{2}{3}\right)$]

(14 points)

6) Let $J_\nu(x)$ be Bessel function of the first kind of order ν. Evaluate the following integrals:

a) $\int_0^\infty J_\nu(bx)\,x^{\nu+1}\exp\left(-ax^2\right)dx \qquad a>0$

b) $\int_0^\infty J_\nu(bx)\,x^{\nu+3}\exp\left(-ax^2\right)dx \qquad a>0$

[Hint: $J_\nu(x)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,\Gamma(\nu+k+1)}\left(\frac{x}{2}\right)^{\nu+2k}$]

(16 points)

7) Let $P_n^m(x)$ be the associated Legendre function of the first kind. Prove that:

$$\sum_{n=0}^{\infty}P_{n+m}^{m}(x)\,t^n=\frac{(2m)!\left(1-x^2\right)^{0.5m}}{2^m\,m!\left(1-2xt+t^2\right)^{m+0.5}}$$

[Hint: The generating function of Legendre polynomials is: $w(x,t)=\left(1-2xt+t^2\right)^{-0.5}$]

(15 points)

Dept of EE (Communication) رياضيات
Third Year ... ثالثة كهرباء اتصالات
حل الامتحان النهائي

Math
First Term 08/09
Final Exam
الفصل الأول
٢٠٠٩/٢٠٠٨

Solution

① ⓐ $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$

For $x < 0$ OR $x > 0.25\pi$:

$f_X(x) = 0$

For $0 \le x \le 0.25\pi$:

$$f_X(x) = \int_0^{\pi/4} K\cos(x+y)\,dy = K\sin(x+y)\Big|_0^{\frac{\pi}{4}}$$

$$= K\left[\sin\left(x+\tfrac{\pi}{4}\right) - \sin(x)\right]$$

$$= K\left[\sin x\cos\tfrac{\pi}{4} + \cos x\sin\tfrac{\pi}{4} - \sin x\right]$$

$$= K\left[\tfrac{1}{\sqrt{2}}(\sin x + \cos x) - \sin x\right]$$

$$= K\left[\tfrac{1}{\sqrt{2}}\cos x - \left(1-\tfrac{1}{\sqrt{2}}\right)\sin x\right]$$

Therefore

$$f_X(x) = \begin{cases} K\left[\frac{1}{\sqrt{2}}\cos x - \left(1-\frac{1}{\sqrt{2}}\right)\sin x\right] & 0 \le x \le \frac{\pi}{4} \\ 0 & \text{otherwise} \end{cases}$$

ⓑ $$1 = \int_{-\infty}^{\infty} f_X(x)\,dx = \int_0^{\frac{\pi}{4}} K\left[\tfrac{1}{\sqrt{2}}\cos x - \left(1-\tfrac{1}{\sqrt{2}}\right)\sin x\right]dx$$

$$= K\left[\tfrac{1}{\sqrt{2}}\sin x + \left(1-\tfrac{1}{\sqrt{2}}\right)\cos x\right]\Big|_0^{\frac{\pi}{4}}$$

$$= K\left[\tfrac{1}{\sqrt{2}}\left(\tfrac{1}{\sqrt{2}} - 0\right) + \left(1-\tfrac{1}{\sqrt{2}}\right)\left(\tfrac{1}{\sqrt{2}} - 1\right)\right]$$

$$= K\left[\tfrac{1}{2} - \left(1-\tfrac{1}{\sqrt{2}}\right)\left(1-\tfrac{1}{\sqrt{2}}\right)\right]$$

$$= K\left[\cancel{\tfrac{1}{2}} - \left(1 + \cancel{\tfrac{1}{2}} - \sqrt{2}\right)\right] = K\left[\sqrt{2} - 1\right]$$

$$\therefore K = \frac{1}{\sqrt{2}-1} = \frac{1}{\sqrt{2}-1}\left(\frac{\sqrt{2}+1}{\sqrt{2}+1}\right) = \frac{1}{(2-1)}\left(\sqrt{2}+1\right)$$

$K = 1+\sqrt{2} = 2.414$

(c) $\hat{Y} = E[Y|X]$

$$\hat{y} = E[Y|x] = \int_{-\infty}^{\infty} y\, f_{Y|X}(y|x)\, dy$$

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

$$f_{Y|X}(y|x) = \begin{cases} \dfrac{\cos(x+y)}{\sin\left(x+\frac{\pi}{4}\right) - \sin(x)} & 0 < x < \frac{\pi}{4},\ 0 < y < \frac{\pi}{4} \\ 0 & \text{otherwise} \end{cases}$$

For $0 < x < \frac{\pi}{4}$:

$$\hat{y} = \int_0^{\frac{\pi}{4}} y \frac{\cos(x+y)}{\sin\left(x+\frac{\pi}{4}\right) - \sin(x)}\, dy$$

$$= \frac{1}{\sin\left(x+\frac{\pi}{4}\right) - \sin x} \int_0^{\frac{\pi}{4}} y \cos(x+y)\, dy$$

Consider

$$\int_0^{\frac{\pi}{4}} y \cos(x+y)\, dy = \int_0^{\frac{\pi}{4}} y\, d\left(\sin(x+y)\right)$$

$$= y \sin(x+y)\Big|_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \sin(x+y)\, dy$$

$$= \left[y \sin(x+y) + \cos(x+y)\right]\Big|_0^{\frac{\pi}{4}}$$

$$= \frac{\pi}{4} \sin\left(x+\frac{\pi}{4}\right) + \cos\left(x+\frac{\pi}{4}\right) - \cos(x)$$

$$= \frac{\pi}{4}\left[\sin x \cos\frac{\pi}{4} + \cos x \sin\frac{\pi}{4}\right] + \left[\cos x \cos\frac{\pi}{4} - \sin x \sin\frac{\pi}{4}\right] - \cos x$$

$$= \frac{\pi}{4\sqrt{2}}(\sin x + \cos x) + \frac{1}{\sqrt{2}}(\cos x - \sin x) - \cos x$$

$$= \left(\frac{\pi}{4\sqrt{2}} + \frac{1}{\sqrt{2}} - 1\right)\cos x + \left(\frac{\pi}{4\sqrt{2}} - \frac{1}{\sqrt{2}}\right)\sin x$$

$$\hat{y} = \frac{\left(\frac{\pi}{4\sqrt{2}} + \frac{1}{\sqrt{2}} - 1\right)\cos x + \left(\frac{\pi}{4\sqrt{2}} - \frac{1}{\sqrt{2}}\right)\sin x}{\frac{1}{\sqrt{2}}\cos x - \left(1 - \frac{1}{\sqrt{2}}\right)\sin x}$$

$$\hat{y} = \frac{\left(\frac{\pi}{4} + 1 - \sqrt{2}\right)\cos x + \left(\frac{\pi}{4} - 1\right)\sin x}{\cos x - (\sqrt{2} - 1)\sin x}$$

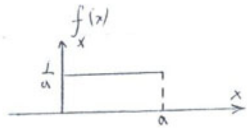
For $x < 0$ OR $x > \frac{\pi}{4}$:

$$\hat{y} = 0$$

Therefore

$$\hat{y} = \begin{cases} \dfrac{0.37118\cos x - 0.21460\sin x}{\cos x - 0.41421\sin x} & 0 < x < \frac{\pi}{4} \\ 0 & \text{otherwise} \end{cases}$$

(2) (a) $m = E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx$



$$= \int_0^a x \frac{1}{a}\,dx = \frac{1}{a}\left.\frac{x^2}{2}\right|_0^a = \frac{a}{2}$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_0^a x^2 \frac{1}{a}\,dx = \frac{1}{a}\left.\frac{x^3}{3}\right|_0^a = \frac{a^2}{3}$$

$$\sigma^2 = E[X^2] - m^2 = \frac{a^2}{3} - \frac{a^2}{4} = \frac{a^2}{12}$$

(b) $R_Z(t_1, t_2) = E[Z_{t_1} Z_{t_2}] = E\left[\left(\frac{X}{1+t_1^4}\right)\left(\frac{X}{1+t_2^4}\right)\right]$

$$= \frac{1}{(1+t_1^4)(1+t_2^4)} E[X^2] = \frac{a^2}{3}\frac{1}{(1+t_1^4)(1+t_2^4)}.$$

(c) $E[Z_t] = E\left[\frac{X}{1+t^4}\right] = \frac{1}{(1+t^4)} E[X] = \frac{a}{2}\cdot\frac{1}{(1+t^4)}$

Z_t is not stationary in the wide sense because $E[Z_t]$ is a function of t.

(d) Z_t is not stationary in the strict sense because it is not stationary in the wide sense.

(3) We use the separation of variables method

$$u(x,t) = F(x)\, G(t)$$

$$\therefore \frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}$$

$$F'' G = \frac{1}{c^2} F \ddot{G}$$

$$\frac{F''}{F} = \frac{1}{c^2} \frac{\ddot{G}}{G} = \text{constant} = -p^2$$

$$0 = u(0, t) = F(0)\, G(t) \implies F(0) = 0$$

$$0 = u(\pi, t) = F(\pi)\, G(t) \implies F(\pi) = 0$$

$$F'' + p^2 F = 0$$

$$\therefore F(x) = A \cos(p x) + B \sin(p x)$$

$$0 = F(0) = A$$

$$\therefore F(x) = B \sin(p x)$$

$$0 = F(\pi) = B \sin(p \pi)$$

$\therefore \sin(p\pi) = 0 \quad \therefore p = n$ an integer

In order to get nontrivial independent solutions, one takes $n = 1, 2, 3, \ldots$

$$F_n(x) = B \sin(n x) \qquad , n = 1, 2, 3, \ldots$$

$$\ddot{G} + (pc)^2 G = 0$$

$$G_n(t) = C \cos(nct) + D \sin(nct)$$

$$u_n(x,t) = F_n(x)\, G_n(t)$$

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

$$u(x,t) = \sum_{n=1}^{\infty} \left[B_n \cos(nct) + B_n^* \sin(nct) \right] \sin(nx).$$

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin(nx)$$
$$= f(x) = 4\sin(2x) + 13\sin(5x) \qquad 0 < x < \pi$$

$$\therefore B_n = \begin{cases} 4 & n = 2 \\ 13 & n = 5 \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty}\left[-nc B_n \sin(nct) + nc B_n^{*} \cos(nct)\right]\sin(nx)$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \sum_{n=1}^{\infty} nc\, B_n^{*} \sin(nx)$$

$$= g(x) = 3\sin(7x) + 6\sin(10x) \qquad 0 < x < \pi$$

$$\therefore nc B_n^{*} = \begin{cases} 3 & n = 7 \\ 6 & n = 10 \\ 0 & \text{otherwise} \end{cases}$$

$$\therefore B_n^{*} = \begin{cases} \frac{3}{7c} & n = 7 \\ \frac{3}{5c} & n = 10 \\ 0 & \text{otherwise} \end{cases}$$

$$\therefore u(x,t) = 4\cos(2ct)\sin(2x) + 13\cos(5ct)\sin(5x)$$
$$+ \frac{3}{7c}\sin(7ct)\sin(7x) + \frac{3}{5c}\sin(10ct)\sin(10x)$$

4 a (i) Bessel differential Eq:

$$x^2 y'' + x y' + (\lambda^2 x^2 - \nu^2) y = 0$$

has the solution:

$$y = A J_\nu(\lambda x) + B Y_\nu(\lambda x)$$

$$\therefore y = A J_5(9x) + B Y_5(9x).$$

(ii) The modified Bessel differential Eq:

$$x^2 y'' + x y' - (\lambda^2 x^2 + \nu^2) y = 0$$

has the solution:

$$y = A\, I_\nu(\lambda x) + B\, K_\nu(\lambda x).$$

$$\therefore y = A\, I_8(7x) + B\, K_8(7x).$$

(iii) Take associated Legendre differential equation:

$$(1-x^2) y'' - 2x\, y' + \left[n(n+1) - \frac{m^2}{1-x^2}\right] y = 0$$

has the solution:

$$y = A\, P_n^m(x) + B\, Q_n^m(x)$$

$$n(n+1) = 72$$

$$\therefore n^2 + n - 72 = 0$$

$$(n+9)(n-8) = 0$$

$$\therefore n = -9,\ 8$$

Since $n \geq 0$, Take $n = 8$

$$m^2 = 4 \qquad \therefore m = 2$$

$$\therefore y = A\, P_8^2(x) + B\, Q_8^2(x).$$

(b) $$J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(n\varphi - x\sin\varphi)\, d\varphi$$

$$\int_0^\pi \cos(17y - 6\sin y)\, dy = \pi\, J_{17}(6).$$

(5) (a) $$I = \int_0^\infty t^{-1.75}(1-e^{-t})\, dt = \frac{1}{-0.75}\int_0^\infty (1-e^{-t})\, d\, t^{-0.75}$$

$$= -\frac{4}{3}\left[(1-e^{-t}) t^{-0.75} - \int t^{-0.75} e^{-t}\, dt\right]\Big|_0^\infty$$

$$= -\frac{4}{3}(1-e^{-t})\, t^{-0.75}\Big|_0^\infty + \frac{4}{3}\int_0^\infty t^{-0.75} e^{-t}\, dt$$

$$\lim_{t\to\infty} \frac{1-e^{-t}}{t^{0.75}} = \frac{1}{\infty} = 0$$

$$\lim_{t\to 0} \frac{(1-e^{-t})}{t^{0.75}} = \lim_{t\to 0} \frac{e^{-t}}{0.75\, t^{-0.25}} = \frac{(1)(0)}{0.75} = 0$$

$$\because \Gamma(x) = \int_0^\infty e^{-t}\, t^{x-1}\, dt \qquad Re\,(x) > 0$$

$$\therefore I = \frac{4}{3}\, \Gamma(0.25)$$

(b) Let $y = \sin x \qquad \therefore dy = \cos x\, dx$

$$I \equiv \int_0^{\frac{\pi}{2}} \left(\frac{1}{\sin^3 x} - \frac{1}{\sin^2 x}\right)^{\frac{1}{6}} \cos x\, dx = \int_0^1 \left(\frac{1}{y^3} - \frac{1}{y^2}\right)^{\frac{1}{6}} dy$$

$$= \int_0^1 \left(\frac{1-y}{y^3}\right)^{\frac{1}{6}} dy = \int_0^1 y^{-\frac{1}{2}} (1-y)^{\frac{1}{6}}\, dy$$

$$\because \beta(p, q) = \int_0^1 t^{p-1} (1-t)^{q-1}\, dt \qquad p > 0,\ q > 0$$

$$\therefore I = \beta\left(\frac{1}{2}, \frac{7}{6}\right)$$

$$\because \beta(p, q) = \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}$$

$$\therefore I = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{7}{6}\right)}{\Gamma\left(\frac{10}{6}\right)}$$

$$\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \quad \& \quad \Gamma(x+1) = x\,\Gamma(x)$$

$$\therefore I = \sqrt{\pi}\ \frac{1}{6}\, \Gamma\left(\frac{1}{6}\right) \frac{1}{\frac{2}{3}\,\Gamma\left(\frac{2}{3}\right)} = \frac{\sqrt{\pi}}{4}\, \frac{\Gamma\left(\frac{1}{6}\right)}{\Gamma\left(\frac{2}{3}\right)}$$

6) a) $$f(a, b) \equiv \int_0^\infty J_\nu(bx)\, x^{\nu+1}\, e^{-ax^2}\, dx$$

$$f = \int_0^\infty x^{\nu+1} e^{-ax^2} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(\nu+k+1)} \left(\frac{bx}{2}\right)^{\nu+2k} dx$$

$$f = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(\nu+k+1)} \left(\frac{b}{2}\right)^{\nu+2k} \int_0^{\infty} x^{2k+2\nu+1} \, e^{-ax^2} \, dx$$

Let $\quad a x^2 = y \quad \therefore \quad x = \sqrt{\frac{y}{a}}$

$$dx = \frac{1}{\sqrt{a}} \cdot \frac{1}{2\sqrt{y}} \, dy = \frac{1}{2\sqrt{ay}} \, dy$$

$$f = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(\nu+k+1)} \left(\frac{b}{2}\right)^{\nu+2k} \int_0^{\infty} \left(\frac{y}{a}\right)^{k+\nu+\frac{1}{2}} e^{-y} \frac{1}{2\sqrt{ay}} \, dy$$

$$f = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(\nu+k+1)} \, \frac{b^{\nu+2k}}{2^{\nu+2k+1}} \, \frac{1}{a^{k+\nu+1}} \int_0^{\infty} y^{k+\nu} \, e^{-y} \, dy$$

$$\because \Gamma(x) = \int_0^{\infty} e^{-y} \, y^{x-1} \, dy$$

$$\therefore f = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\cancel{\Gamma(\nu+k+1)}} \, \frac{b^{\nu+2k}}{2^{\nu+2k+1}} \, \frac{1}{a^{k+\nu+1}} \, \cancel{\Gamma(\nu+k+1)}.$$

$$= \frac{b^{\nu}}{2^{\nu+1} \, a^{\nu+1}} \sum_{k=0}^{\infty} \frac{\left(-\frac{b^2}{4a}\right)^k}{k!}$$

$$\because e^{\lambda} = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}$$

$$\therefore f(a,b) = \frac{b^{\nu}}{(2a)^{\nu+1}} \exp\left(-\frac{b^2}{4a}\right).$$

(b) $\quad g(a,b) \equiv \int_0^{\infty} J_\nu(bx) \, x^{\nu+3} \, e^{-ax^2} \, dx \qquad a>0$

$$\because f(a,b) = \frac{b^{\nu}}{2^{\nu+1}} \cdot a^{-(\nu+1)} \cdot e^{-\frac{b^2}{4a}} = \frac{1}{2a}\left(\frac{b}{2a}\right)^{\nu} e^{-\frac{b^2}{4a}}$$

Differentiating both sides of the result (a) w.r.t a, one gets:

$$\frac{\partial f}{\partial a} = -\int_0^{\infty} J_\nu(bx) \, x^{\nu+3} \, e^{-ax^2} \, dx$$

$$\frac{\partial f}{\partial a} = -g \qquad \text{——— (1)}$$

$$\frac{\partial f}{\partial a} = \frac{b^{\nu}}{2^{\nu+1}}\left[-(\nu+1)\,a^{-\nu-2} + a^{-(\nu+1)}\left(-\frac{b^2}{4}\right)\left(-\frac{1}{a^2}\right)\right] e^{-\frac{b^2}{4a}}$$

$$= \frac{b^{\nu}}{2^{\nu+1}}\left[-(\nu+1)\,a^{-\nu-2} + \frac{a^{-\nu-3}}{4}\,b^2\right] e^{-\frac{b^2}{4a}}$$

$$= -\frac{b^{\nu}}{2^{\nu+1}\,a^{\nu+3}}\left[(\nu+1)\,a - \frac{b^2}{4}\right] e^{-\frac{b^2}{4a}}$$

$$= -\frac{b^{\nu}}{2^{\nu+1}\,a^{\nu+2}}\left[(\nu+1) - \frac{b^2}{4a}\right] e^{-\frac{b^2}{4a}} \qquad \text{——— (2)}$$

From (1) and (2), one gets:

$$g(a,b) = \frac{1}{2a^2}\left(\frac{b}{2a}\right)^{\nu}\left[(\nu+1) - \frac{b^2}{4a}\right]\exp\left(-\frac{b^2}{4a}\right).$$

7) $$w(x,t) = \sum_{n=0}^{\infty} P_n(x)\, t^n = \left(1 - 2xt + t^2\right)^{-0.5}$$

$$\frac{\partial^m w}{\partial x^m} = \sum_{n=0}^{\infty} \frac{d^m}{dx^m}\left[P_n(x)\right] t^n$$

Since $P_n(x)$ is a polynomial of degree n, one gets:

$$\frac{d^m}{dx^m}\left[P_n(x)\right] = 0 \qquad m > n$$

$$\therefore \frac{\partial^m w}{\partial x^m} = \sum_{n=m}^{\infty} \frac{d^m}{dx^m}\left[P_n(x)\right] t^n$$

Let $n = m + k$

$$\therefore \frac{\partial^m w}{\partial x^m} = \sum_{k=0}^{\infty} \frac{d^m}{dx^m}\left[P_{k+m}(x)\right] t^{k+m}$$

$$\frac{\partial^m w}{\partial x^m} = t^m \sum_{n=0}^{\infty} \frac{d^m}{dx^m}\left[P_{n+m}(x)\right] t^n \qquad \text{——— (1)}$$

On the other hand, one gets:

$$\frac{\partial w}{\partial x} = \left(-\frac{1}{2}\right)\left(1-2xt+t^2\right)^{-1.5}(-2t)$$

$$\frac{\partial^2 w}{\partial x^2} = \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(1-2xt+t^2\right)^{-2.5}(-2t)^2$$

$$\frac{\partial^3 w}{\partial x^3} = \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(1-2xt+t^2\right)^{-3.5}(-2t)^3$$

$$\vdots$$

$$\frac{\partial^m w}{\partial x^m} = \frac{(1)(3)(5)\dots(2m-1)}{\cancel{2}^m}\left(1-2xt+t^2\right)^{-m-0.5}(\cancel{2}t)^m$$

Since

$$(1)(3)(5)\dots(2m-1) = (1)(3)(5)\dots(2m-1)\cdot\frac{(2)(4)(6)\dots(2m)}{(2)(4)(6)\dots(2m)}.$$

$$= \frac{(2m)!}{2^m\, m!}$$

$$\therefore \frac{\partial^m w}{\partial x^m} = \frac{(2m)!}{2^m\, m!}\,\frac{1}{\left(1-2xt+t^2\right)^{m+0.5}}\,t^m \quad \text{--- (2)}$$

Equating (1) and (2), one gets:

$$\sum_{n=0}^{\infty}\frac{d^m}{dx^m}\left[P_{n+m}(x)\right]t^n = \frac{(2m)!}{2^m\, m!\left(1-2xt+t^2\right)^{m+0.5}}$$

Since

$$P_n^m(x) = \left(1-x^2\right)^{\frac{m}{2}}\frac{d^m}{dx^m}\left[P_n(x)\right]$$

multiplying both sides by $\left(1-x^2\right)^{\frac{m}{2}}$, one gets:

$$\sum_{n=0}^{\infty}P_{n+m}^{m}(x)\,t^n = \frac{(2m)!\left(1-x^2\right)^{0.5m}}{2^m\, m!\left(1-2xt+t^2\right)^{(m+0.5)}}.$$

Alternative Solutions

(6) (b)
$$g \equiv \int_0^\infty J_\nu(bx)\, x^{\nu+3} e^{-ax^2}\, dx$$
$$= \int_0^\infty x^{\nu+3} e^{-ax^2} \sum_{k=0}^\infty \frac{(-1)^k}{k!\,\Gamma(\nu+k+1)} \left(\frac{bx}{2}\right)^{\nu+2k} dx$$
$$= \sum_{k=0}^\infty \frac{(-1)^k}{k!\,\Gamma(\nu+k+1)} \left(\frac{b}{2}\right)^{\nu+2k} \int_0^\infty e^{-ax^2} x^{2\nu+2k+3}\, dx$$

Let $y = ax^2 \quad \therefore \quad x = \sqrt{\frac{y}{a}}$
$$dx = \frac{1}{\sqrt{a}} \frac{1}{2\sqrt{y}}\, dy = \frac{1}{2\sqrt{ay}}\, dy$$
$$I \equiv \int_0^\infty e^{-ax^2} x^{2\nu+2k+3}\, dx = \int_0^\infty e^{-y} \left(\frac{y}{a}\right)^{\nu+k+\frac{3}{2}} \frac{1}{2\sqrt{ay}}\, dy$$
$$= \frac{1}{2\, a^{\nu+k+2}} \int_0^\infty e^{-y} y^{\nu+k+1}\, dy = \frac{1}{2\, a^{\nu+k+2}} \Gamma(\nu+k+2)$$
$$g = \sum_{k=0}^\infty \frac{(-1)^k}{k!\,\Gamma(\nu+k+1)} \left(\frac{b}{2}\right)^{\nu+2k} \cdot \frac{1}{2\, a^{\nu+k+2}} \Gamma(\nu+k+2)$$
$$= \sum_{k=0}^\infty \frac{(-1)^k}{k!} \frac{b^{\nu+2k}}{2^{\nu+2k+1}} \frac{1}{a^{\nu+k+2}} (\nu+k+1)$$
$$= \frac{b^\nu}{2^{\nu+1} a^{\nu+2}} \sum_{k=0}^\infty \frac{\left(\frac{-b^2}{4a}\right)^k}{k!} \left[(\nu+1) + k\right].$$
$$= \frac{b^\nu}{2^{\nu+1} a^{\nu+2}} \left\{ (\nu+1) \underbrace{\sum_{k=0}^\infty \frac{\left(-\frac{b^2}{4a}\right)^k}{k!}}_{\exp\left(-\frac{b^2}{4a}\right)} + \underbrace{\sum_{k=1}^\infty \frac{\left(-\frac{b^2}{4a}\right)^k}{(k-1)!}}_{\left(-\frac{b^2}{4a}\right) \sum_{k=1}^\infty \frac{\left(\frac{-b^2}{4a}\right)^{(k-1)}}{(k-1)!}} \right\}$$
$$\left(-\frac{b^2}{4a}\right) \exp\left(-\frac{b^2}{4a}\right)$$

$$g = \frac{b^{\nu}}{2^{\nu+1} a^{\nu+2}} \left[(\nu+1) - \frac{b^2}{4a} \right] \exp\left(-\frac{b^2}{4a} \right).$$

Grading

1) 16 a) 4 (formula : 1 , effort : 1 , ANS : 2)
b) 4 (formula : 1 , effort : 1 , ANS : 2)
c) 8 (three formulae : 3 , $f_{Y|X}$: 1 , integration : 1 , effort : 1 , ANS : 2)

2) 12 a) 4 (three formulae : 1.5 , effort 1 , three ANS : 1.5)
b) 4 (formula : 1 , effort : 1 , ANS : 2)
c) 2 (ANS : 1 , Reason : 1)
d) 2 (ANS : 1 , Reason : 1)

3) 15 Separation of variables : 2
$F(0) = F(\pi) = 0$: 2
$F_n(x) = B \sin(nx)$: 2
$G_n(t)$: 2
B_n : 2
B_n^* : 2
effort : 1
$u(x,t)$: 2

4) 12 a) 9 (i) 3 (J & Y : 1 , order : 1 , arguments : 1)
(ii) 3 (I & K : 1 , order : 1 , arguments : 1)
(iii) 3 (associated functions : 1 , n : 1 , m : 1)
b) 3 (formula : 1 , effort : 1 , ANS : 1)

5) 14 a) 7 (integration by parts : 2 , two limits : 1 , def of Γ : 1 , effort : 1 , ANS : 2)
b) 7 (substitution : 1 , def of β : 1 , relation between β, Γ : 1 , functional relation : 1 , $\Gamma(\frac{1}{2})$: 1 , effort : 1 , ANS : 1)

⑥ 16 ⓐ 8 (substitution : 1 , def of Γ : 1 , exponential : 2 , effort : 2 , ANS : 2)

ⓑ 8 (Differentiate RHS : 3 , differentiate integral : 1 , effort : 2 , ANS : 2)

⑦ 15 Differentiate $\sum P_n x^n t^n$: 2
Recognize lower limit of sum : 2
Change of index : 2
Differentiate $w(x,t)$ m times : 2
Expression for $\frac{\partial^m w}{\partial x^m}$: 3
Effort : 2
ANS : 2