Fayoum University
Faculty of Engineering
Dept. of Electrical Engineering
Second Year

Mathematics
First term of 2010/2011
Final Exam
Time : 3 hours

Model answer for the following **SIX** questions:

Question 1 [20 points]:

Consider Matrix $\mathbf{A} = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$

1.1 Compute $\mathbf{A}^2$, then show that the $|\mathbf{A}|$ equal to ± 1.**[5 points]**

1.2 Is Matrix $\mathbf{A}$ idempotent or involutary matrix? **[5 points]**

1.3 Determine $|\mathbf{A}|$.**[5 points]**

1.4 Find the cofactor C_{11} corresponding to the element a_{11}.**[5 points]**

Solution:

1.1

$$\mathbf{A}^2 = \frac{1}{4}\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} = \mathbf{I}$$

$$\because \left|\mathbf{A}^2\right| = |\mathbf{A}|^2 = |\mathbf{I}| = 1$$

$$\therefore |\mathbf{A}| = |\mathbf{I}| = \pm 1$$

1.2

$\because \mathbf{A}^2 = \mathbf{I}$

$\therefore$ Matrix $\mathbf{A}$ is involutary matrix.

$\because \mathbf{A}^3 = \mathbf{A}$

$\therefore$ Matrix $\mathbf{A}$ is idempotent of order 3.

1.3

$$\mathbf{A} = \begin{bmatrix} -0.5 & 0.5 & 0.5 & 0.5 \\ 0.5 & -0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & -0.5 & 0.5 \\ 0.5 & 0.5 & 0.5 & -0.5 \end{bmatrix} \underset{\substack{H_{13}\\H_{14}}}{\overset{H_{12}}{\sim}} \begin{bmatrix} -0.5 & 0.5 & 0.5 & 0.5 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$$\therefore |\mathbf{A}| = \begin{vmatrix} -0.5 & 0.5 & 0.5 & 0.5 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{vmatrix} = \frac{-1}{2}\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = \frac{-1}{2} * \left(-1 * (-1) + 1 * (1)\right)$$

$= -1$

1.4

$$C_{11} = M_{11} = \begin{vmatrix} -0.5 & 0.5 & 0.5 \\ 0.5 & -0.5 & 0.5 \\ 0.5 & 0.5 & -0.5 \end{vmatrix} = \begin{vmatrix} -0.5 & 0.5 & 0.5 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix}$$

$= -0.5 * -1$

$= 0.5$

Question 2 [10 points]:

2.1 The determinant of the 1000x1000 matrix **A** is 12. What is the determinant of $\left(-\mathbf{A}^{T}\right)$? (Careful; no credit for the wrong sign.) **[3 points]**

2.2 The vector $\underline{\mathbf{u}}$ is unit vector, what are all possible values of t which guarantee that the matrix $\mathbf{A} = \mathbf{I} + t\underline{\mathbf{u}}\underline{\mathbf{u}}^{T}$ is orthogonal? **[2 points]**

2.3 For which λ are the vectors $\begin{bmatrix} 11-\lambda \\ 18 \end{bmatrix}, \begin{bmatrix} 6 \\ -10-\lambda \end{bmatrix}$ not linearly dependent? **[5 points]**

Solution:

2.1

$$\because \left|\mathbf{A}\right| = 12$$

$$\because \left|-\mathbf{A}^{T}\right| = (-1)^{1000}\left|\mathbf{A}\right| = 12$$

2.2

$\because$ Matrix A is orthogonal.

$$\therefore \mathbf{A}^{*T}\mathbf{A} = \mathbf{D}$$

$$\therefore \left(\mathbf{I} + t\underline{\mathbf{u}}\underline{\mathbf{u}}^{T}\right)^{*T}\left(\mathbf{I} + t\underline{\mathbf{u}}\underline{\mathbf{u}}^{T}\right) = \mathbf{I}$$

$$\therefore \left(\mathbf{I} + t^{*}\underline{\mathbf{u}}\underline{\mathbf{u}}^{T}\right)\left(\mathbf{I} + t\underline{\mathbf{u}}\underline{\mathbf{u}}^{T}\right) = \mathbf{I}$$

$$\therefore \mathbf{I} + \left(t^{*} + t\right)\underline{\mathbf{u}}\underline{\mathbf{u}}^{T} + t^{*}t\underline{\mathbf{u}}\underline{\mathbf{u}}^{T}\underline{\mathbf{u}}\underline{\mathbf{u}}^{T} = \mathbf{I}$$

$$\therefore \mathbf{I} + \left(2t + t^{2}\right)\underline{\mathbf{u}}\underline{\mathbf{u}}^{T} = \mathbf{I}$$

$$\therefore t = 0;\ \text{or}\ \ t = -2$$

2.3

$\because$ The vectors $\begin{bmatrix} 11-\lambda \\ 18 \end{bmatrix}, \begin{bmatrix} 6 \\ -10-\lambda \end{bmatrix}$ are not linearly dependent

$$\therefore \begin{vmatrix} 11-\lambda & 6 \\ 18 & -10-\lambda \end{vmatrix} = 0$$

$$\therefore (11-\lambda)(-10-\lambda)-108=0$$

$$\lambda^2-\lambda-218=0$$

$$\lambda_{1,2}=\frac{1\pm 3\sqrt{97}}{2}$$

<u>Question 3</u> [20 points]:

3.1 Forward elementary changes $\mathbf{A}\,\underline{\mathbf{x}}=\underline{\mathbf{b}}$ to a row reduced $\mathbf{R}\,\underline{\mathbf{x}}=\underline{\mathbf{d}}$: the complete solution is

$$\underline{\mathbf{x}}=\begin{bmatrix}4\\0\\0\end{bmatrix}+C_1\begin{bmatrix}2\\1\\0\end{bmatrix}+C_2\begin{bmatrix}5\\0\\1\end{bmatrix}$$

3.1.1 What is reduced matrix **R ,** and what is $\underline{\mathbf{d}}$? **[5 points]**

3.1.2 If the process of elimination subtracted three times of row1 from row 2, and then five times of row1 from row 3, from the system $\mathbf{A}\,\underline{\mathbf{x}}=\underline{\mathbf{b}}$ in order to get $\mathbf{R}\,\underline{\mathbf{x}}=\underline{\mathbf{d}}$; then find **A** , and $\underline{\mathbf{b}}$.**[5 points]**

3.2 Suppose $\mathbf{A}=\begin{bmatrix}0&1&2&2\\0&3&8&7\\0&0&4&2\end{bmatrix}$; Solve $\mathbf{A}\,\underline{\mathbf{x}}=\underline{\mathbf{0}}$. **[10 points]**

<u>Solution:</u>

3.1.1

$$\mathbf{R}=\begin{bmatrix}1&-2&-5\\0&0&0\\0&0&0\end{bmatrix}\qquad \underline{\mathbf{d}}=\begin{bmatrix}4\\0\\0\end{bmatrix}$$; Why?

3.1.2

$$[\mathbf{R} \mid \underline{\mathbf{d}}] = \left[\begin{array}{ccc|c} 1 & -2 & -5 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \underset{H_{13}(5)}{\overset{H_{12}(3)}{\sim}} \left[\begin{array}{ccc|c} 1 & -2 & -5 & 4 \\ -3 & 6 & 15 & 12 \\ -5 & 10 & -25 & 20 \end{array}\right]$$

$$\therefore \mathbf{A} = \begin{bmatrix} 1 & -2 & -5 \\ 3 & -6 & -15 \\ 5 & -10 & -25 \end{bmatrix} \quad ; \quad \underline{\mathbf{d}} = \begin{bmatrix} 4 \\ 12 \\ 20 \end{bmatrix}$$

3.2

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 2 & 2 \\ 0 & 3 & 8 & 7 \\ 0 & 0 & 4 & 2 \end{bmatrix} \overset{H_{12}(-3)}{\sim} \begin{bmatrix} 0 & 1 & 2 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 4 & 2 \end{bmatrix} \overset{H_{23}(-2)}{\sim} \begin{bmatrix} 0 & 1 & 2 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{Let} \quad \underline{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$\therefore x_2 + 2x_3 + 2x_4 = 0$

$\therefore 2x_3 + x_4 = 0$

Let $x_1 = c_1$

$x_3 = c_2$

$\therefore x_4 = -2c_2$

$\therefore x_2 = 2c_2$

$$\therefore \underline{\mathbf{x}} = \begin{bmatrix} c_1 \\ 2c_2 \\ c_2 \\ -2c_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 2 \\ 1 \\ -2 \end{bmatrix}$$

<u>Question 4</u> [20 points]:

The eigenvalues of a square nondiagonalizable nonderogatory matrix **A** of order n = 6 are given by $\lambda = 2,4,4,4,8,8$.

4.1 Find the characteristic polynomial of **A**. **[5 points]**

4.2 Find the minimal polynomial of **A**. **[5 points]**

4.3 Express $\exp(\mathbf{A}t)$ as a matrix polynomial in **A** and write down the equations required for evaluating its coefficients. (Don't solve these equations). **[5 points]**

4.4 Find the determinant of $\exp(\mathbf{A}t)$.**[5 points]**

<u>Solution:</u>

4.1

$$\phi(\lambda) = (\lambda-2)(\lambda-4)^3(\lambda-8)^2 = 0$$

4.2

$$m(\lambda) = (\lambda-2)(\lambda-4)(\lambda-8) = 0$$

4.3

$$\exp(\mathbf{A}t) = \alpha_0\mathbf{I} + \alpha_1\mathbf{A} + \alpha_2\mathbf{A}^2$$

$$\exp(\lambda t) = \alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2$$

$$\exp(2t) = \alpha_0 + 2\alpha_1 + 4\alpha_2$$
$$\exp(4t) = \alpha_0 + 4\alpha_1 + 16\alpha_2$$
$$\exp(8t) = \alpha_0 + 8\alpha_1 + 64\alpha_2$$

4.4

$$|\exp(\mathbf{A}t)| = \left|TD_{\exp(\lambda t)}T^{-1}\right| = \left|D_{\exp(\lambda t)}\right| = \exp(30t)$$

<u>Question 5</u> [15 points]:

Solve the following differential equations.

$$\frac{dx}{dt} = 3x - 4y$$
$$\frac{dy}{dt} = 2x - 3y$$

With initial conditions: $x(0) = 1;\ y(0) = 0$

<u>Solution:</u>

$$\because \dot{\underline{\mathbf{x}}} = \begin{bmatrix} 3 & -4 \\ 2 & -3 \end{bmatrix} \underline{\mathbf{x}}$$

where :

$$\underline{\mathbf{x}} = \begin{bmatrix} x \\ y \end{bmatrix} \quad ; \quad \text{and } \underline{\mathbf{x}}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\because \underline{\mathbf{x}}(t) = \exp(\mathbf{A}t)\underline{\mathbf{x}}(0)$$

$$\because \exp(\mathbf{A}t) = \begin{bmatrix} \cosh(t) + 3\sinh(t) & -4\sinh(t) \\ 2\sinh(t) & \cosh(t) - 3\sinh(t) \end{bmatrix}$$ **[10 points]**

$$\therefore \underline{\mathbf{x}}(t) = \begin{bmatrix} \cosh(t) + 3\sinh(t) \\ 2\sinh(t) \end{bmatrix}$$ **[3 points]**

$$\therefore x(t) = \cosh(t) + 3\sinh(t)$$
$$\therefore y(t) = 2\sinh(t)$$ **[2 points]**