Fayoum University	Department of Eng. Math.
Faculty of Engineering	16/1/2009
1st year Industrial Eng.	Time : 3 hours

Question 1: (35 marks)

1. If $w = e^{ax^2+by^2}$ and $w_{xy} - yw_x - xw_y + xyw = 0$, find values of **a** and **b**.
2. If the vector $u(x,y,z) = u_1(x,y,z)\,\underline{i} + u_2(x,y,z)\,\underline{j} + u_3(x,y,z)\,\underline{k}$ and the function F= f (x,y,z), Prove that
   a) $\nabla \bullet \nabla \times u = 0$ **b)** $\nabla \times \nabla F = \underline{0}$
3. Find the equations of the **tangent plane** and **normal line** to the ellipsoid $\frac{3}{4}x^2 + 3y^2 + z^2 = 6$ at the point (2,1,0).
4. Find the **extrema** of the function $F(x,y) = 3x^3 + y^2 - 9x + 4y$.
5. What is **the greatest area** a **rectangular** can have if the length of its **diagonal** equal **4.0** .

Question 2: (20 marks)

1. Evaluate the integral $\iint_R e^{-(x^2+y^2)}\ dx\ dy$ and R is the region bounded by the two circles $x^2 + y^2 = 1.0$ and $x^2 + y^2 = 4.0$.
2. Find the **volume** of the region bounded by parabolic cylinder $z = 4 - x^2$ and the planes x=0 , y=0, z=0 and y=6.0.
3. Find the **volume** bounded by $z = x^2 + y^2$ and plane z=4.0 (use **cylindrical coordinate**).

Question 3: (16 marks)

1. Evaluate by **Gauss's theorem** $\iint_S [x\,dy\,dz + y\,dz\,dx + z\,dx\,dy]$ where S is the surface bounded by cylinder $x^2 + y^2 = 9.0$ and the planes z=0 and z=3.0.
2. Prove that the integral $\int_{(1,1)}^{(3,2)} (ye^{xy}\,dx + xe^{xy}\,dy)$ is **path independent** and **evaluate** it.

Question 4: (16 marks)

- Test the following series:

a) $\sum_{n=1}^{\infty} \frac{1}{3^n + \sqrt{n^3}}$ **b)** $\sum_{n=1}^{\infty} \frac{5^n}{n^3}$

c) $\sum_{n=1}^{\infty} \frac{3^n}{\ln^n(2n+1)}$ **d)** $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$

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***Question* 1:** **(35 marks)**

3. If $w = e^{ax^2+by^2}$ and $w_{xy} - yw_x - xw_y + xyw = 0$, find values of **a** and **b**.

Solution:

$w = e^{ax^2+by^2}$

$w_x = 2axe^{ax^2+by^2}$, $w_y = 2bye^{ax^2+by^2}$, $w_{xy} = 4abxye^{ax^2+by^2}$

Substitute w, w_x, w_y and w_{xy} into $w_{xy} - yw_x - xw_y + xyw = 0$

We obtain

4ab – 2a - 2b + 1 = 0, then a=0.5 and b=0.5

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4. If the vector $u(x,y,z) = u_1(x,y,z)\,\underline{i} + u_2(x,y,z)\,\underline{j} + u_3(x,y,z)\,\underline{k}$ and the function F= f (x,y,z), Prove that

b) $\nabla \bullet \nabla \times u = 0$ **c)** $\nabla \times \nabla F = \underline{0}$

Solution:

a) $\nabla \bullet \nabla \times u = 0$

$$\nabla \bullet \nabla \times u = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u_1 & u_2 & u_3 \end{vmatrix} = \frac{\partial}{\partial x}\left[\frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z}\right] - \frac{\partial}{\partial y}\left[\frac{\partial u_3}{\partial x} - \frac{\partial u_1}{\partial z}\right] + \frac{\partial}{\partial z}\left[\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}\right] = 0$$

b) $\nabla \times \nabla F = \underline{0}$

$$\nabla \times \nabla F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix} = \left[\frac{\partial}{\partial y} f_z - \frac{\partial}{\partial z} f_y\right] i - \left[\frac{\partial}{\partial x} f_z - \frac{\partial}{\partial z} f_x\right] j + \left[\frac{\partial}{\partial x} f_y - \frac{\partial}{\partial y} f_x\right] k = \underline{0}$$

3. Find the equations of the **tangent plane** and **normal line** to the ellipsoid $\frac{3}{4}x^2 + 3y^2 + z^2 = 6$ at the point (2,1,0).

Solution:

$F = \frac{3}{4}x^2 + 3y^2 + z^2 - 6$

$F_x\big|_{(2,1,0)} = \frac{3}{2}x\Big|_{(2,1,0)} = 3 \quad ,F_y\big|_{(2,1,0)} = 6y\big|_{(2,1,0)} = 6 \quad ,F_z\big|_{(2,1,0)} = 2z\big|_{(2,1,0)} = 0$

<u>The equation of tangent plane:</u>

X+2y=4

<u>The equation of the normal lines:</u>

$\frac{x-2}{3} = \frac{y-1}{6}$

$x - 2 = 3t \quad , \quad y - 1 = 6t \quad , \quad z = 0$

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4. Find the **extrema** of the function $F(x, y) = 3x^3 + y^2 - 9x + 4y$.

<u>Solution</u>:

$F(x, y) = 3x^3 + y^2 - 9x + 4y$

$F_x = 9x^2 - 9 \qquad , \quad F_y = 2y + 4$

$F_{xx} = 18x \qquad , \quad F_{yy} = 2 \qquad , \quad F_{xy} = 0$

The critical points are (1,-2) and (-1,-2)

$\Delta = 36x$

point	Fxx	Fyy	Δ	conclusion
(1,-2)	+ve	+ve	+ve	Min. point
(-1,-2)	-ve	+ve	-ve	Saddle point

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5. What is **the greatest area** a **rectangular** can have if the length of its **diagonal** equal **4.0** .

<u>Solution</u>:

g(x,y) = (Diagonal length)^2 = $x^2 + y^2$ =16

f(x,y) = area of rectangular = x y

$\frac{f_x}{g_x} = \frac{f_y}{g_y} = \lambda$,

$\frac{y}{2x} = \frac{x}{2y} = \lambda$ we get x = y then x^2=8

the greatest area = 8

***<u>Question 2</u>*:**

<u>(20 marks)</u>

4. **4.** Evaluate the integral $\iint_R e^{-(x^2+y^2)}\ dx\ dy$ and R is the region bounded by the two circles $x^2+y^2 = 1.0$ and $x^2+y^2 = 4.0$.

Solution:

$$\iint_R e^{-(x^2+y^2)}\ dx\ dy = \int_0^{2\pi}\int_1^2 e^{-r^2} r\ dr\ d\theta = \pi\left(e^{-1} - e^{-4}\right)$$

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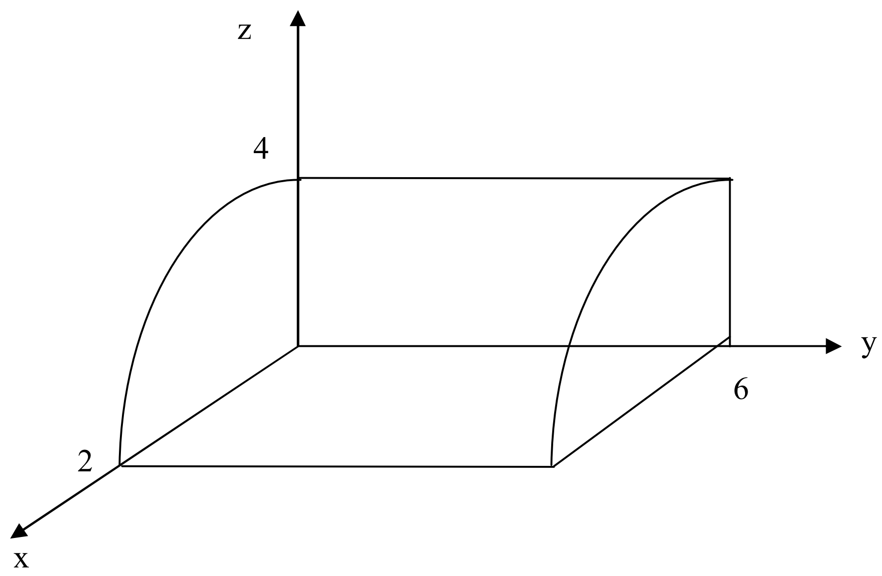
5. Find the **volume** of the region bounded by parabolic cylinder $z = 4 - x^2$ and the planes x=0 , y=0, z=0 and y=6.0.

Solution:



$$\text{Volume} = \int_0^6\int_0^2\int_0^{4-x^2} dz\,dx\,dy = \int_0^6\int_0^2 (4 - x^2)\,dx\,dy = 32$$

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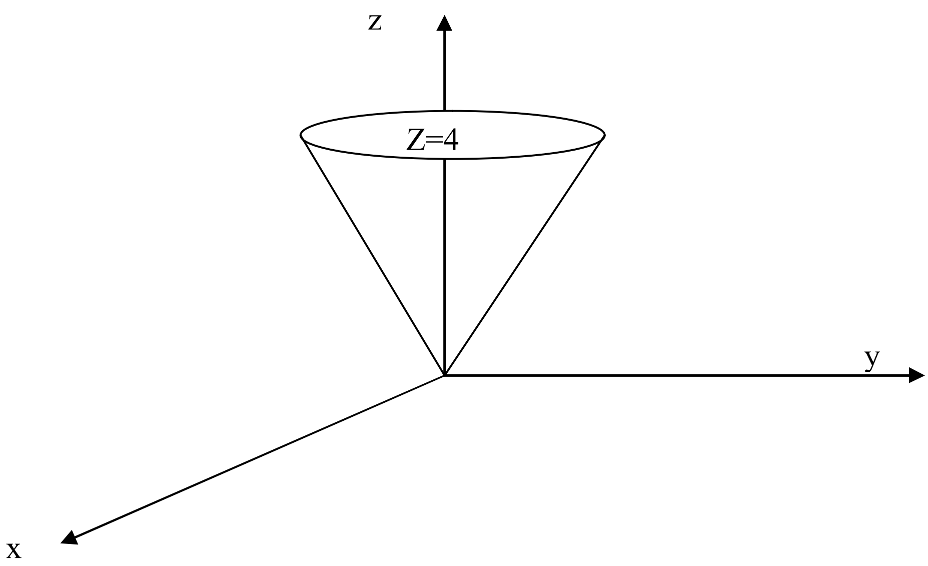
3. Find the **volume** bounded by $z = x^2 + y^2$ and plane z=4.0 (use **cylindrical coordinate.**

Solution:



Volume= $\int_0^{2\pi}\int_0^{2}\int_{r^2}^{4} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^{2} r(4-r^2)\,dr\,d\theta = 8\pi$

***Question* 3:** **(16 marks)**

3. Evaluate by **Gauss's theorem** $\iint_S [x\,dy\,dz + y\,dz\,dx + z\,dx\,dy]$ where S is the surface bounded by cylinder $x^2 + y^2 = 9.0$ and the planes z=0 and z=3.0.

Solution:

$F = x\,i + y\,j + z\,k$, $\nabla \bullet F = 3$

$I = \int_0^{2\pi}\int_0^{3}\int_0^{3} 3r\;dz\,dr\;d\theta = 81\pi$

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4. Prove that the integral $\int_{(1,1)}^{(3,2)} (ye^{xy}\,dx + xe^{xy}\,dy)$ is **path independent** and **evaluate** it.

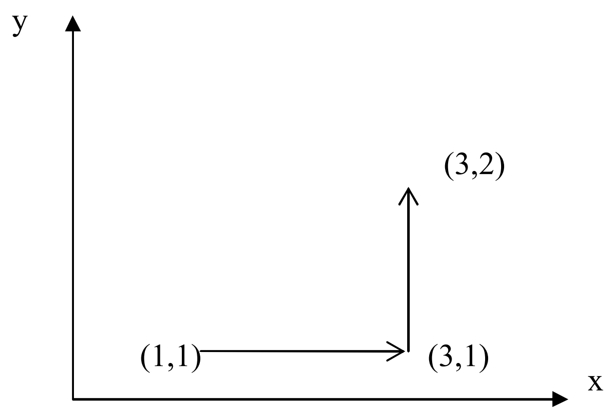
Solution

$P = ye^{xy}$, $Q = x\,e^{xy}$

$\frac{\partial P}{\partial y} = e^{xy}(1+xy)$, $\frac{\partial Q}{\partial x} = e^{xy}(1+xy)$

since

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = e^{xy}(1+xy)$$ then the integrals is independent of path.



Along y=1 , dy = 0

$$I_1 = \int_{x=1}^{x=3} e^x dx = e^3 - e$$

Along x=3 , dx = 0

$$I_2 = \int_{y=1}^{y=2} 3\ e^{3y}\ dx = e^6 - e^3 \quad \text{then ,}$$

$$I = I_1 + I_2 = e^6 - e$$

***Question* 4:** **(<u>16</u> marks)**

- <u>Test the following series:</u>

e) $\sum_{n=1}^{\infty} \frac{1}{3^n + \sqrt{n^3}}$

We have $\frac{1}{3^n} > \frac{1}{3^n + \sqrt{n^3}}$ and $\sum_{n=1} \frac{1}{3^n}$ is a geometric series and converges, then

By comparison test the series $\sum_{n=1}^{\infty} \frac{1}{3^n + \sqrt{n^3}}$ converges.

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f) $\sum_{n=1}^{\infty} \frac{5^n}{n^3}$

By ratio test

$$\lim_{n\to\infty} \frac{5^{n+1} n^3}{5^n n^{n+1}} = \lim_{n\to\infty} 5\left(\frac{n}{n+1}\right)^3 = 5 > 1 \text{ then the series diverges.}$$

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g) $\sum_{n=1}^{\infty} \frac{3^n}{\ln^n (2n+1)}$

By Cauchy test

$$\lim_{n\to\infty} \sqrt[n]{\frac{3^n}{\ln^n (2n+1)}} = \lim_{n\to\infty} \frac{3}{\ln(2n+1)} = 0 < 1$$, then the series converges.

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h) $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$

By ratio test

$$\lim_{n\to\infty} \frac{((n+1)!)^2}{(2n+2)!} \frac{(2n!)}{(n)^2 !} = \lim_{n\to\infty} \frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{1}{4} < 1,$$ then the series converges.