Model Answer For FALL 2017

2nd Electrical department

Q1 "7 pts"

1.1

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 5 & 3 & 9 \\ 0 & 7 & 8 & 5 & 9 \\ 0 & 0 & 0 & 0 & 11 \\ 0 & 0 & 0 & 6 & 7 \end{bmatrix} \quad \begin{array}{c} H_{23} \\ H_{45} \end{array}$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 7 & 8 & 5 & 9 \\ 0 & 0 & 5 & 3 & 9 \\ 0 & 0 & 0 & 6 & 7 \\ 0 & 0 & 0 & 0 & 11 \end{bmatrix}$$

$$\therefore |A| = 1 * 7 * 5 * 6 * 11$$

$$\boxed{|A| = 2310}$$

1.2

1.2.1 "2 pts"

False ; for $A = I_3$ & $B = -I_3$

$\therefore \rho(A+B) = 0$

but $\rho(A) = 3$ & $\rho(B) = 3$

$\therefore \rho(A+B) \neq \rho(A) + \rho(B)$

1.2.2. "2 pts"

False ; as it is right for only real & symmetric Matrix

1.2.3 "2 pts"

True

1.2.4. "2 pts"

False ; as $A\underline{x} = \lambda\underline{x} \Rightarrow A^{r}\underline{x} = \lambda^{r}\underline{x} = \underline{0}$

Q2 "15 pts"

$$[A \,\vdots\, \underline{b}] = \left[\begin{array}{ccccc|c} 1 & 0 & -1 & 2 & 1 & 3 \\ 1 & 1 & 2 & 0 & 0 & 5 \\ 2 & -1 & 0 & 0 & 0 & 1 \\ -4 & 5 & 4 & 0 & 0 & r \end{array}\right] \quad \begin{array}{l} H_{12}(-1) \\ H_{13}(-2) \\ H_{14}(4) \end{array}$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & -1 & 2 & 1 & 3 \\ 0 & 1 & 3 & -2 & -1 & 2 \\ 0 & -1 & 2 & -4 & -2 & -5 \\ 0 & 5 & 0 & 8 & 4 & r+12 \end{array}\right] \quad \begin{array}{l} H_{23}(1) \\ H_{24}(-5) \end{array}$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & -1 & 2 & 1 & 3 \\ 0 & 1 & 3 & -2 & -1 & 2 \\ 0 & 0 & 5 & -6 & -3 & -3 \\ 0 & 0 & -15 & 18 & 9 & r+2 \end{array}\right] \quad \begin{array}{l} H_{33}(\frac{1}{5}) \\ H_{31}(1) \\ H_{32}(-3) \\ H_{34}(15) \end{array}$$

$$\left[\begin{array}{ccc|cc|c} 1 & 0 & 0 & 4/5 & 2/5 & 12/5 \\ 0 & 1 & 0 & 8/5 & 4/5 & 19/5 \\ 0 & 0 & 1 & -6/5 & -3/5 & -3/5 \\ \hline 0 & 0 & 0 & 0 & 0 & r-7 \end{array}\right]$$

For having a solution $\Longrightarrow r = 7$ "infinite solution"

$$\underline{x} = c_1 \begin{bmatrix} 4/5 \\ 8/5 \\ -6/5 \\ -1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2/5 \\ 4/5 \\ -3/5 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 12/5 \\ 19/5 \\ -3/5 \\ 0 \\ 0 \end{bmatrix}$$

For $\underline{b} = \underline{0}$

$$\underline{x} = c_1 \begin{bmatrix} 4/5 \\ 8/5 \\ -6/5 \end{bmatrix} + c_2 \begin{bmatrix} 2/5 \\ 4/5 \\ -3/5 \end{bmatrix}$$

Q3 "20 pts"

3.1 "10 pts"

3.1.1. "5 pts"

$$\text{Let } A_2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\therefore P(\lambda) = |A_2 - \lambda I| = 0 \Rightarrow \begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix} = 0$$

$$\therefore P(\lambda) = (a_{11}-\lambda)(a_{22}-\lambda) - a_{12}a_{21} = 0$$

$$= \lambda^2 - \lambda(a_{11}+a_{22}) + (a_{11}a_{22} - a_{12}a_{21}) = 0$$

$$\boxed{\therefore P(\lambda) = \lambda^2 - Tr(A)\lambda + |A|} \quad \#$$

3.1.2 "5 pts"

$\because$ A is singular Matrix

$$\therefore |A| = 0$$

$$\therefore P(\lambda) = \lambda(\lambda - Tr(A)) = 0$$

3.2 "10pts"

3.2.1. "5pts" $\because$ A is similar To B

then they shared the same eigenvalues

then

$$Tr(A) = Tr(B)$$

as $Tr(A) = \sum_{i=1}^{n} \lambda_i$

3.2.2. "5pts"

$\because$ A is similar to B

$$\therefore A = P^{-1} B P$$

$$\therefore |A| = |P^{-1} B P| = |P^{-1}||B||P|$$

$$= |P^{-1} P| |B|$$

$$= |B| \quad \#$$

Q4 "20 pts"

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

4.1 "15 pts"

$$\therefore \; P(\lambda) = |A - \lambda I| = 0$$

$$\begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = 0$$

$$\therefore \; -\lambda(\lambda^2 - 1) - (-\lambda - 1) + (1 + \lambda) = 0$$

$$\therefore \; -\lambda(\lambda-1)(\lambda+1) + (\lambda+1) + (\lambda+1) = 0$$

$$\therefore \; (\lambda+1)\left[-\lambda(\lambda-1) + 2\right] = 0$$

$$(\lambda+1)(\lambda^2 - \lambda - 2) = 0$$

$$(\lambda+1)(\lambda-2)(\lambda+1) = 0$$

$\lambda_1 = -1$; $\lambda_2 = -1$; $\lambda_3 = 2$ "5pts"

For $\lambda = -1$

$$\therefore (A + I)\,\underline{x} = \underline{0}$$

$$\therefore \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \underset{H_{13}(-1)}{\overset{H_{12}(-1)}{\sim}} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \underline{x} = c_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\therefore \underline{x}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \; ; \quad \underline{x}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

"5 pts"

For $\lambda = 2$

$$\therefore (A - 2I)\,\underline{x} = \underline{0}$$

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \underset{\substack{H_{12}(-1) \\ H_{13}(-1)}}{\overset{H_{11}(-\frac{1}{2})}{\sim}} \begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & -3/2 & \frac{3}{2} \\ 0 & 3/2 & -\frac{3}{2} \end{bmatrix} \overset{H_{23}(-1)}{\sim}$$

$$\begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & -3/2 & \frac{3}{2} \\ 0 & \cdots & \cdots \end{bmatrix} \underset{H_{21}(\frac{1}{2})}{\overset{H_{22}(\frac{-2}{3})}{\sim}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ \cdots & \cdots & \cdots \end{bmatrix}$$

$$\therefore \underline{x} = c \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$$

$$\therefore \underline{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Eigenvalues :-

$$\lambda_1 = -1 \quad ; \quad \lambda_2 = -1 \quad ; \quad \lambda_3 = 2$$

Eigenvectors :-

$$\underline{x}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad ; \quad \underline{x}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad ; \quad \underline{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

4.2.

apply Gram-schmedit for $\underline{x}_1$, $\underline{x}_2$

$$\therefore \underline{y}_1 = \underline{x}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$\underline{y}_2 = \underline{x}_2 + \alpha \underline{y}_1 \quad ; \quad \alpha = - \frac{\langle \underline{x}_2, \underline{y}_1 \rangle}{\| \underline{y}_1 \|^2}$$

$$\therefore \alpha = - \frac{1}{}$$

$$\therefore \underline{y}_2 = \begin{bmatrix} 1/2 \\ 1/2 \\ -1 \end{bmatrix}$$

$$\therefore P = \left[\frac{\underline{y}_1}{\|\underline{y}_1\|} \;\vdots\; \frac{\underline{y}_2}{\|\underline{y}_2\|} \;\vdots\; \frac{\underline{X}_3}{\|\underline{X}_3\|} \right]$$

$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

"4 pts"

$$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

"1 pt"

Q5

$$\ddot{y}(t) - \dot{y}(t) - 2y(t) = e^{2t}$$

Let

$$\dot{y}(t) = x(t)$$

$$\therefore \quad \dot{x}(t) - x(t) - 2y(t) = e^{2t}$$

$$\therefore \quad \underline{\dot{x}}(t) = \begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} x(t) + 2y(t) + e^{2t} \\ x(t) \end{bmatrix}$$

$$\therefore \underline{\dot{x}}(t) = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} e^{2t} \\ 0 \end{bmatrix} \quad \text{"5 pts"}$$

$$\therefore \quad \underline{\dot{x}}(t) = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} e^{2t} \\ 0 \end{bmatrix} = A\,\underline{x}(t) + \underline{y}(t)$$

$$\therefore \quad x(t) = e^{At} \int_0^t e^{-A\tau} y(\tau)\, d\tau$$

* Calculation of e^{At}

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \; ; \quad \text{use}$$

$$\therefore |A - \lambda I| = 0 \Rightarrow \therefore \begin{vmatrix} 1-\lambda & 2 \\ 1 & -\lambda \end{vmatrix} = 0$$

$$\therefore -\lambda(1-\lambda) - 2 = 0 \Rightarrow \lambda^2 - \lambda - 2 = 0 \Rightarrow (\lambda-2)(\lambda+1) = 0$$

$$\therefore \lambda_1 = 2 \; ; \; \lambda_2 = -1$$

$$\therefore e^{At} = \alpha_0 I + \alpha_1 A$$

$$e^{\lambda t} = \alpha_0 + \alpha_1 \lambda \Rightarrow \begin{aligned} e^{2t} &= \alpha_0 + 2\alpha_1 \\ e^{-t} &= \alpha_0 - \alpha_1 \end{aligned}$$

$$\therefore \alpha_0 = \frac{1}{3}\left[e^{2t} + 2e^{-t}\right]$$

$$\therefore \alpha_1 = \frac{1}{3}\left[e^{2t} - e^{-t}\right]$$

$$\therefore e^{At} = \frac{1}{3}\left[e^{2t} + 2e^{-t}\right]\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{3}\left[e^{2t} - e^{-t}\right]\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$$

$$\therefore e^{At} = \begin{bmatrix} \frac{1}{3}(e^{-t}+2e^{2t}) & \frac{2}{3}(e^{2t}-e^{-t}) \\ \frac{1}{3}(e^{2t}-e^{-t}) & \frac{1}{3}(e^{2t}+2e^{-t}) \end{bmatrix}$$ "5 pts"

$$\therefore e^{At} y(t) = \begin{bmatrix} \frac{1}{3}(e^{t}+2e^{4t}) \\ \frac{1}{3}(e^{4t}-e^{t}) \end{bmatrix}$$

$$\therefore \int_0^t e^{A\tau} y(\tau)\, d\tau = \begin{bmatrix} \frac{1}{3}(e^{t}+\frac{e^{4t}}{2}) - \frac{1}{2} \\ \frac{1}{3}(\frac{e^{4t}}{4}-e^{t}) - \frac{1}{4} \end{bmatrix}$$

$$\therefore \underline{x}(t) = e^{At} \begin{bmatrix} \frac{1}{3}(e^{t}+\frac{e^{4t}}{2}) - \frac{1}{2} \\ \frac{1}{3}(\frac{e^{4t}}{4}-e^{t}) - \frac{1}{4} \end{bmatrix}$$ Continue... ?

"5 pts"

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