Fayoum University
Faculty of Engineering

Subject : Mathematics — January,17,2010
Preparatory Year — Time : 3 Hours
Prof. Dr. Samy El Badawy Yehia — First Term Final Examination

Answer all the following questions. 20 marks each.

1) Let : $a_1, a_2, a_3, a_4, a_5, a_6, \ldots$ be the sequence defined by :

$$a_1 = 1, a_2 = 1 \text{ , and for } n \geq 3, a_n = a_{n-1} + a_{n-2}$$

which means that this series is the following one : 1,1,2,3,5,8,13,21,34,55,89,144,233,377,610,....
Prove that (using mathematical induction) :

a) For every $n \geq 1$ that each a_{3n} (i.e. $a_3, a_6, a_9, a_{12}, \ldots$) is even.

b) For every $n \geq 1$ that each a_{4n} ($i.e. a_4, a_8, a_{12}, a_{16}, \ldots$) is divisible by 3.

2) a) State both Roll's Theorem and the Mean Value Theorem. Prove the Mean Value Theorem.

b) Given :

$$f(x) = \begin{cases} 3 & if \quad x = 0 \\ -x^2 + 3x + a & if \quad 0 < x < 1 \\ mx + b & if \quad 1 \leq x \leq 2 \end{cases}$$

Find the values of a , b and m such that f(x) satisfies the hypothesis of the Mean Value Theorem on the interval [0 , 2]. Then find a number c on (0 , 2) that satisfies the conclusion of the theorem.

3) Using L'Hopital Rule find:

a) $\lim\limits_{h \to 0} \dfrac{f(x+h) + f(x-h) - 2f(x)}{h^2}$

b) $\lim\limits_{x \to 1} \left(\dfrac{x}{x-1} - \dfrac{1}{\ln x} \right)$

4) a) Prove that : $\sin^{-1}(\tanh x) = \tan^{-1}(\sinh x)$.

b) Find :

$$\frac{d}{dx}\left(\frac{1}{2}\tan^{-1} x + \frac{1}{4}\ln\frac{(x+1)^2}{x^2+1}\right)$$

(simplify your answer)

5) Sketch the following function :

$$y = \frac{x^2}{x^2+3}$$

Show :

a) Asymptotes
b)Local extreme.
c)Concavity and points of inflection.

6) Expand:

$$\frac{1}{1-5x+6x^2}$$

Put it in the form

$$a_1x+a_2x^2+a_3x^3+\ldots\ldots=\sum_{n=1}^{\infty}a_nx^n.$$

7) Prove the Following formula (Newton's Method for finding a root for differentiable equations) :

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$$

then use it to find $\sqrt{612}$ correct to two decimals starting with $x_0=10$.

GOOD LUCK

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ANSWERS

1) a) 1- For n=1 , $a_3 = 2$ which is even.

2-Assume true for n=k i.e. a_{3k} is even.

3- For n = k+1. Using the definition :

$$a_{3(k+1)} = a_{3k+3} = a_{3k+2} + a_{3k+1} = a_{3k+1} + a_{3k} + a_{3k+1} = 2a_{3k+1} + a_{3k}.$$

both the last two terms are even which proves it is true for n = k+1.

b) 1- For n = 1 , $a_4 = 3$ which is divisible by 3.

2- Assume true for n =k i.e. a_{4k} is divisible by 3.

3- for n=k+1. Using the definition :

$$a_{4(k+1)} = a_{4k+4} = a_{4k+3} + a_{4k+2} = a_{4k+2} + a_{4k+1} + a_{4k+2} = a_{4k+1} + 2a_{4k+2} = a_{4k+1} + 2(a_{4k+1} + a_{4k}) =$$
$$= 3a_{4k+1} + a_{4k}.$$

both the last two terms are divisible by 3 which proves it for n=k+1.

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2) $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(-x^2 + 3x + a) = a = f(0) = 3.$ Then a = 3.

$f'^-(1) = -2(1) + 3 = f'^+(1) = m.$ Then m = 1.

$\lim_{x\to 1^-} f(x) = -1 + 3 + 3 = \lim_{x\to 1^+} f(x) = m + b = 1 + b.$ Then b = 4.

Then

$$f(x) = \begin{cases} 3 & if \quad x = 0 \\ -x^2 + 3x + 3 & if \quad 0 < x < 1 \\ x + 4 & if \quad 1 \le x \le 2 \end{cases} \quad \text{Then} \quad f'(x) = \begin{cases} -2x + 3 & if \quad 0 < x \le 1 \\ 1 & if \quad 1 \le x < 2 \end{cases}$$

$\frac{f(b) - f(a)}{b - a} = \frac{6-3}{2-0} = \frac{3}{2}.$ Then f'(3/4) = 3/2 then c = 3/4 in the open interval (0 , 2).

3) a)

$$\lim_{h\to o}\frac{f(x+h)+f(x-h)-2f(x)}{h^2}=\frac{f(x)+f(x)-2f(x)}{0}=\frac{0}{0}=\lim_{h\to 0}\frac{f'(x+h)-f'(x-h)}{2h}=$$

$$=\frac{f(x)-f(x)}{0}=\frac{0}{0}=\lim_{h\to 0}\frac{f''(x+h)+f''(x-h)}{2}=\frac{f''(x)+f''(x)}{2}=f''(x).$$

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b)

$$\lim_{x\to 1}(\frac{x}{x-1}-\frac{1}{\ln x})=\infty-\infty=\lim_{x\to 1}(\frac{x\ln x-x+1}{(x-1)\ln x})=\frac{0}{0}=\lim_{x\to 1}(\frac{x(1/x)+\ln x-1}{(x-1)(1/x)+\ln x})=$$

$$=\lim_{x\to 1}(\frac{\ln x}{(x-1)(1/x)+\ln x})=\lim_{x\to 1}(\frac{x\ln x}{x-1+x\ln x})=\frac{0}{0}=\lim_{x\to 1}(\frac{1+\ln x}{1+1+\ln x})=\frac{1}{2}.$$

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4) a) L.H.S. $=\Theta=\sin^{-1}(\tanh x)$. Then $\sin\Theta=\tanh x$. Then :

$$\tan\Theta=\frac{\sin\Theta}{\cos\Theta}=\frac{\sin\Theta}{\sqrt{1-\sin^2\Theta}}=\frac{\tanh x}{\sqrt{1-\tanh^2 x}}=\frac{\tanh x}{\sec hx}=\frac{\sinh x/\cosh x}{1/\cosh x}=\sinh x.$$

Then :

$$\Theta=\tan^{-1}(\sinh x)=\text{R.H.S.}$$

b) $$\frac{d}{dx}\left[\frac{1}{2}\tan^{-1}x+\frac{1}{4}(2\ln(x+1)-\ln(x^2+1))\right]=\frac{1}{2(x^2+1)}+\frac{1}{2(x+1)}-\frac{x}{2(x^2+1)}=$$

$$=\frac{x+1+x^2+1-x(x+1)}{2(x^2+1)(x+1)}=\frac{1}{(x+1)(x^2+1)}.$$

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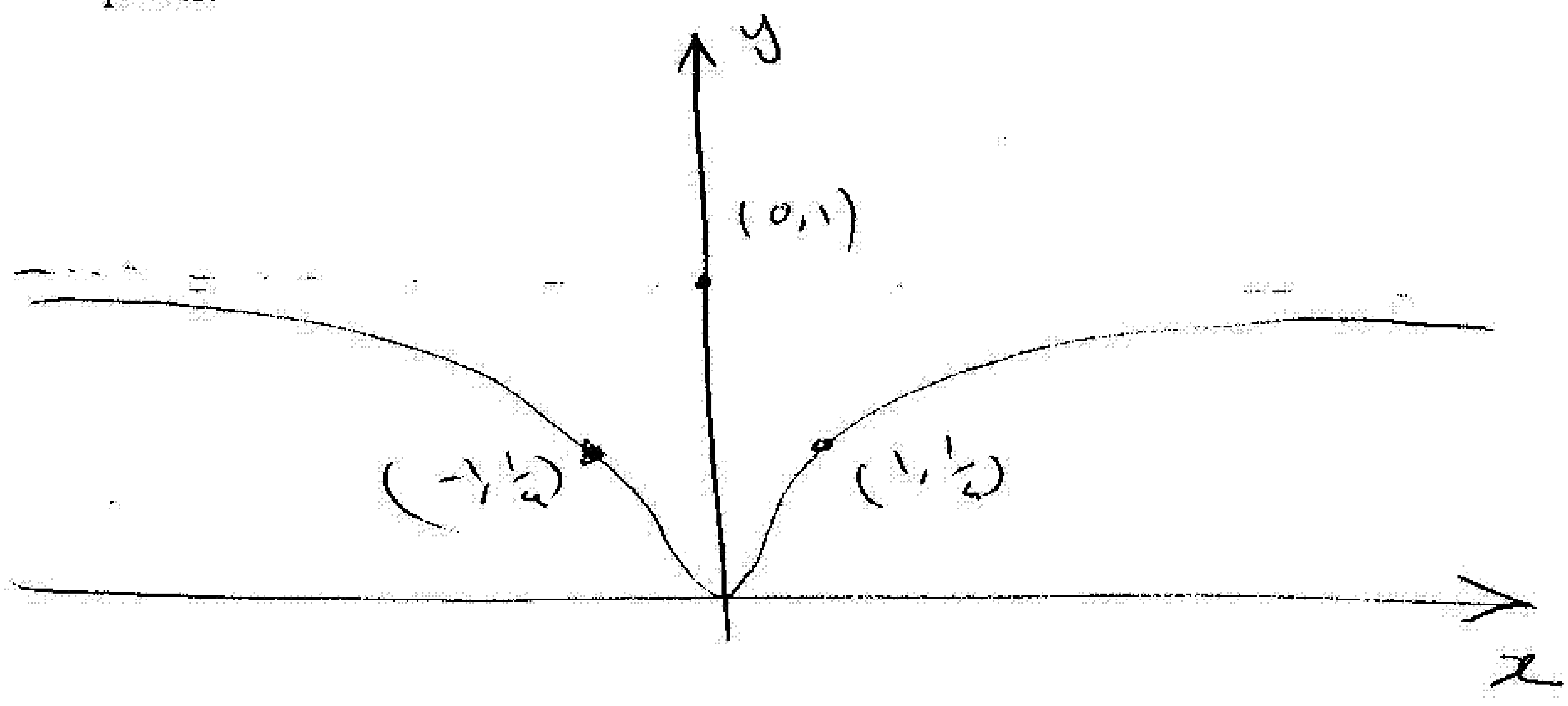
5) a) No vertical Asymptote. Horizontal asymptote y = 1.

b) y' = $\frac{6x}{(x^2+3)^2}$, y" = $\frac{18(1-x^2)}{(x^2+3)^2}$. The (0 , 0) is the only critical point and it is a LOCAL MINIMUM.

c) Concave up for $-1\le x\le 1$. Concave down for $x\le -1$ and for $x\ge 1$. (-1 , ¼) and (1 , ¼) are inflection points.



6) $\frac{1}{1-5x+6x^2}=\frac{3}{1-3x}+\frac{-2}{1-2x}=3(1-3x)^{-1}-2(1-2x)^{-1}=$

$=3\left[1+(3x)+(3x)^2+(3x)^3+(3x)^4+....\right]-2\left[1+(2x)+(2x)^2+(2x)^3+(2x)^4+......\right].$

Then the coefficient of $x^n is (3^{n+1}-2^{n+1})$ for n=0,1,2,3,4,... Then :

$$\frac{1}{1-5x+6x^2}=(3^1-2^1)+(3^2-2^2)x+(3^3-2^3)x^2+(3^4-2^4)x^3+(3^5-2^5)x^4+......=$$

$$=1+5x+19x^2+65x^3+211x^4+.....=\sum_{n=0}^{\infty}(3^{n+1}-2^{n+1})x^n.$$

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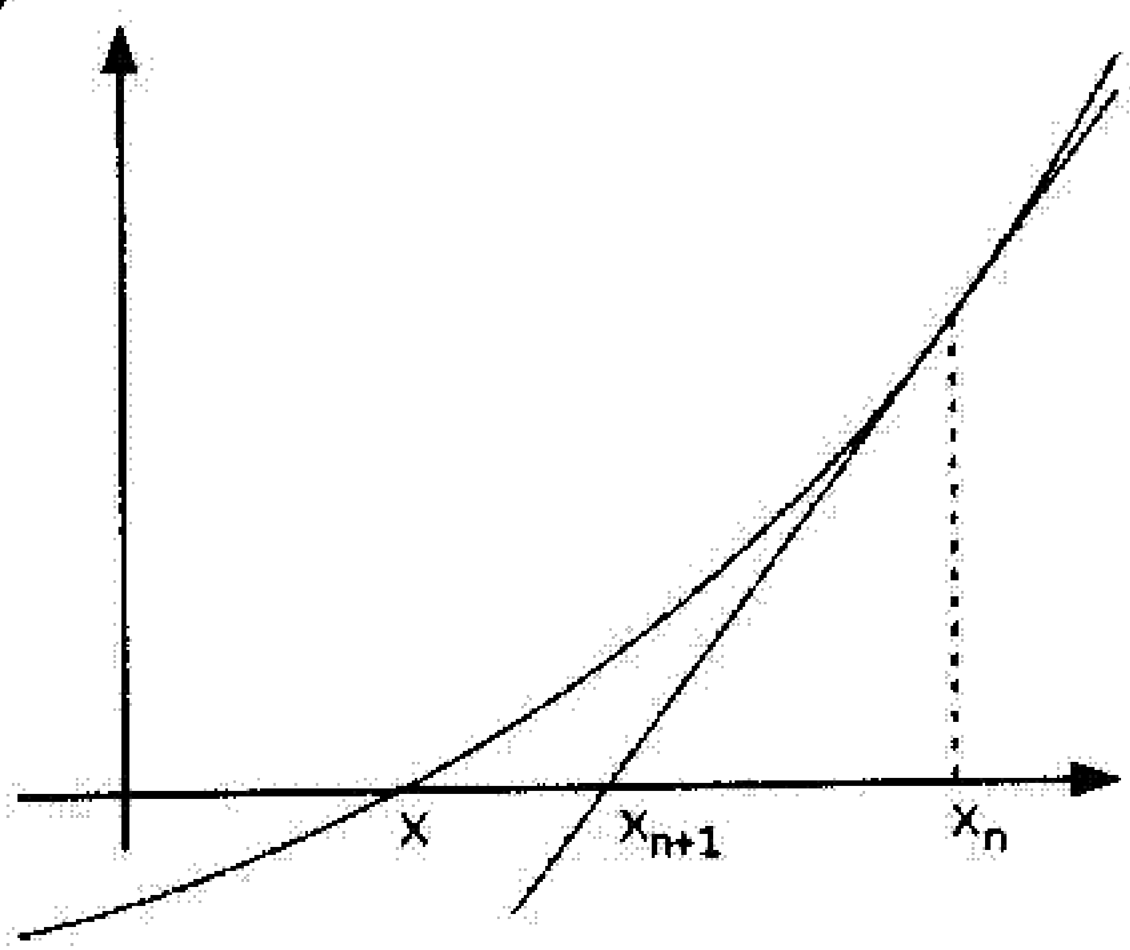
7)



$f'(x_n)=\frac{y-f(x_n)}{x-x_n}$. It passes through the point $(x_{n+1},0)$ then $f'(x_n)=\frac{0-f(x_n)}{x_{n+1}-x_n}$.

Which gives

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}.$$

$$\begin{aligned}
x_1 &= x_0-\frac{f(x_0)}{f'(x_0)} &=& \quad 10-\frac{10^2-612}{2\cdot 10} &=& \quad 35.6\\
x_2 &= x_1-\frac{f(x_1)}{f'(x_1)} &=& \quad 35.6-\frac{35.6^2-612}{2\cdot 35.6} &=& \quad \underline{2}6.3955056\\
x_3 &= \quad \vdots &=& \quad \vdots &=& \quad \underline{24.7}906355\\
x_4 &= \quad \vdots &=& \quad \vdots &=& \quad \underline{24.7386}883\\
x_5 &= \quad \vdots &=& \quad \vdots &=& \quad \underline{24.7386338}
\end{aligned}$$