Fayoum University
Faculty of Engineering

Preparatory Year — Time : 3 Hours.
Mathematics — June, 12, 2010.
Prof. Dr. Samy El Badawy Yehia — Second Term Final Exam.

1) Evaluate the following integrals:

a) $\int 2x^3\sqrt{x^2+1}\,dx,$ Put $x^2 = u-1$ then $2xdx = du$

$$= \int x^2(x^2+1)^{1/2}(2xdx) = \int (u-1)u^{\frac{1}{2}}du = \int (u^{\frac{3}{2}} - u^{\frac{1}{2}})du = \frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}} + C =$$

$$= \frac{2}{5}(x^2+1)^{\frac{5}{2}} - \frac{2}{3}(x^2+1)^{\frac{3}{2}} + C.$$

---

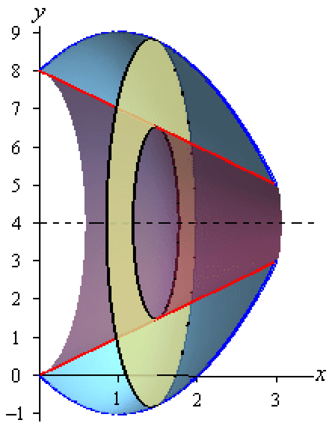
b) $\int \frac{\tan x}{\sec^4 x}dx = \int \frac{\sin x}{\cos x}\cos^4 x dx = \int \sin x \cos^3 x dx = \frac{-1}{4}\cos^4 x + C.$

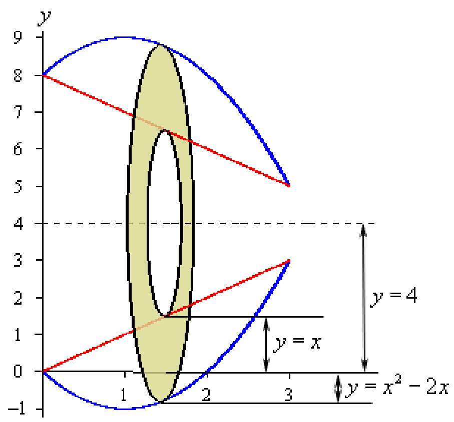
---

c) $\int \frac{1}{1+\sin x}dx = \int \frac{1}{1+\sin x}.\frac{1-\sin x}{1-\sin x}dx = \int \frac{1-\sin x}{1-\sin^2 x}dx = \int \frac{1-\sin x}{\cos^2 x}dx =$

$$= \int \frac{1}{\cos^2 x} - \frac{\sin x}{\cos^2 x}dx = \int \sec^2 x - \sec x \tan x dx = \tan x - \sec x + C.$$

2) Determine the volume of the solid obtained by rotating the region bounded by : $y = x^2 - 2x$ and $y = x$ about the line $y = 4$.





Thickness dx
Inner Radius 4-x
Outer Radius 4-(x^2-2x)

$$V = \pi\int_0^3 (4-x^2+2x)^2 - (4-x)^2 dx = \pi\int_0^3 (x^4 - 4x^3 - 5x^2 + 24x)dx =$$

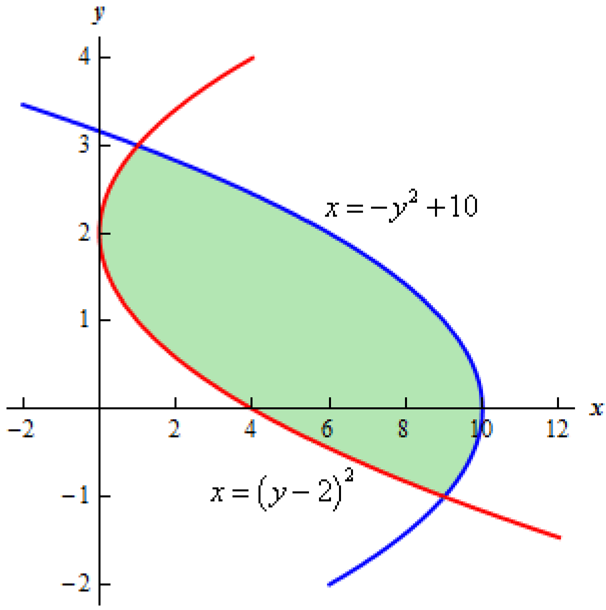
$$= \pi\left(\frac{1}{5}x^5 - x^4 - \frac{5}{3}x^3 + 12x\right)_0^3 = \frac{153\pi}{5}.$$

---

3) Find the shown area between the two curves $x = -y^2 + 10$ and $x = (y-2)^2$ by integrating :

a) with respect to y,

b)with respect to x (set up an integral only).



a) $A=\int_{-1}^{3}\left[(-y^2+10)-(y-2)^2\right]dy = \int_{-1}^{3}(-2y^2+4y+6)dy = \left(-\frac{2}{3}y^3+2y^2+6y\right)_{-1}^{3}$

$=\frac{64}{4}$.

b) $A = \int_{0}^{1}\left[(\sqrt{x}+2)-(-\sqrt{x}+2)\right]dx + \int_{1}^{9}[(\sqrt{-x+10})-(-\sqrt{x}+2)]dx +$

$+\int_{9}^{10}\left[(\sqrt{-x+10})-(-\sqrt{-x+10})\right]dx$.

---

4) Prove that the straight lines joining the origin to the points of intersection of the straight line 3x + 4y = 24 with the circle $(x-4)^2+(y-3)^2=c^2$ are at right angles if c = ± 5.

The equation of the required line pair is:

$$(x-4(\frac{3x+4y}{24}))^2+(y-3(\frac{3x+4y}{24}))^2=c^2(\frac{3x+4y}{24})^2$$

then :

$$(\frac{12x-16y}{24})^2+\left(\frac{-9x+12y}{24}\right)^2=c^2(\frac{3x+4y}{24})^2$$

which gives :

$$\left[\frac{3x-4y}{6}\right]^2+\left[\frac{-3x+4y}{8}\right]^2=c^2\left[\frac{3x+4y}{24}\right]^2$$

then :

$$\frac{16(3x-4y)^2+9(3x-4y)^2}{24^2}=c^2\frac{(3x+4y)^2}{24^2}$$

then :

$$25(3x-4y)^2=c^2(3x+4y)^2$$

then the two lines are :

5(3x-4y)=±c(3x+4y) with slopes $\frac{15+3c}{20-4c}$ and $\frac{15-3c}{20+4c}$ the are perpendicular if :

$$\frac{(15+3c)(15-3c)}{(20-4c)(20+4c)} = -1$$

then :

$$-(400-16c^2) = (225-9c^2)$$

then :

$$-400+16c^2 = 225-9c^2 \text{ then } 25c^2 = 625 \text{ which gives c=±5.}$$

---

5) Find the value of λ that the plane x – y + z = λ is tangent to the circle $(x-2)^2 + (y+1)^2 + (z-1)^2 = 9$. Find the parametric equation of the straight line joining the centre of this circle to the point of tangency of this circle to this plane.

The condition is :

$$\frac{2-(-1)+1-\lambda}{\sqrt{1+1+1}} = \pm 3 \quad \text{then} \quad 4-\lambda = \pm 3\sqrt{3} \quad \text{OR} \quad \lambda = 4 \pm 3\sqrt{3}$$

The equation of the required line is :

$$\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \\ z &= z_0 + ct \end{aligned} \qquad \text{Then} \qquad \begin{aligned} x &= 2+t \\ y &= -1-t \\ z &= 1+t \end{aligned}$$

---

6) Find the equation of the tangent and the normal at any point $(\frac{c}{t}, ct)$ of the rectangular hyperbola $xy = c^2$.

By differentiating the equation of the hyperbola :

$x\frac{dy}{dx} + y = 0$, then $\frac{dy}{dx} = \frac{-y}{x}$ which gives the slopes of the tangent and the normal at the point $(\frac{c}{t}, ct)$:

$m_{\tan gent} = \frac{-(ct)}{c/t} = -t^2 = \frac{y-ct}{x-(c/t)}$ then the equation of the tangent is :

$$t^2 x + y = 2ct.$$

and the equation of the normal :

$m_{normal} = \frac{1}{t^2} = \frac{y-ct}{x-(c/t)}$ then the equation normal is :

$$x - t^2 y = \frac{c}{t} - ct^3.$$
