Fayoum University
Faculty of Engineering

Preparatory Year, Mathematics | Second Term Final Exam.

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3 Hours

ANSWERS

1) Find $\int y\,dx$ given that :

a) $y = x^2 \tan^{-1} x.$

$u = \tan^{-1} x \qquad dv = x^2$

$du = \dfrac{1}{1+x^2} \qquad v = \dfrac{x^3}{3}$

$$\int x^2 \tan^{-1} x\,dx = \frac{x^3 \tan^{-1} x}{3} - \frac{1}{3}\int \frac{x^3}{1+x^2}dx = \frac{x^3 \tan^{-1} x}{3} - \frac{1}{3}\int (x - \frac{x}{1+x^2})dx =$$

$$\frac{x^3 \tan^{-1} x}{3} - \frac{1}{3}[\frac{x^2}{2} - \frac{1}{2}\ln(1+x^2)] + C.$$

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b) $y = \dfrac{e^{2x} - e^x}{e^{2x} - 1}.$

$$\int \frac{e^{2x} - e^x}{e^{2x} - 1}dx = \int \frac{e^x (e^x - 1)}{(e^x - 1)(e^x + 1)}dx = \int \frac{e^x}{e^x + 1}dx = \ln(e^x + 1) + C.$$

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c) $y = \dfrac{e^{2x} - 1}{e^{2x} - e^x}.$

$$\int \frac{e^{2x} - 1}{e^{2x} - e^x}dx = \int \frac{(e^x - 1)(e^x + 1)}{e^x (e^x - 1)}dx = \int \frac{e^x + 1}{e^x}dx = \int (1 + e^{-x})dx = x - e^{-x} + C.$$

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d) $y = \dfrac{x^3}{x^2 + 4x + 3}.$

$$\int \frac{x^3}{x^2 + 4x + 3}dx = \int (x - 4 + \frac{-2}{x+1} + \frac{18}{x+3})dx = x^2 - 4x - 2\ln(x+1) + 18\ln(x+3) + C.$$

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e) $\int \cos(\tan^{-1} 2 + \tan^{-1} 3)dx. = x\cos(\tan^{-1} 2 + \tan^{-1} 3) + C.$

f) $y = e^{x+e^x}$. $\int e^{x+e^x}\,dx \;=\; e^{e^x} + C.$

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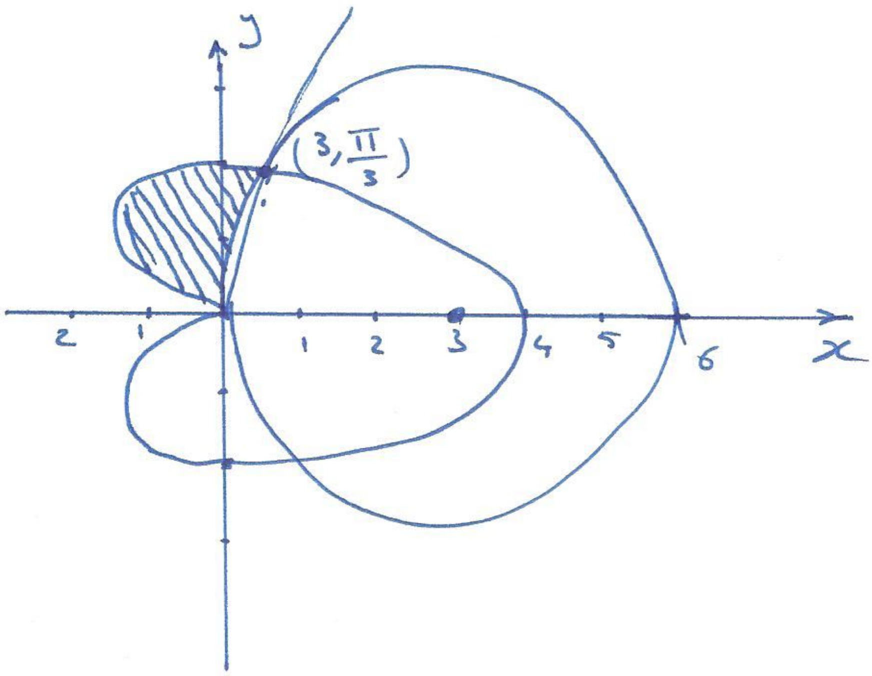
2) Find the area inside the curve $r = 2 + 2\cos\theta$ and outside the curve $r = 6\cos\theta$.



$$\text{Area} = \int_{\pi/3}^{\pi} \frac{1}{2}(2+2\cos\theta)^2\,d\theta - \int_{\pi/3}^{\pi/2} \frac{1}{2}(6\cos\theta)^2\,d\theta = \int_{\pi/3}^{\pi} (2+4\cos\theta+2\cos^2\theta)\,d\theta - \int_{\pi/3}^{\pi/2} (18\cos^2\theta)\,d\theta =$$

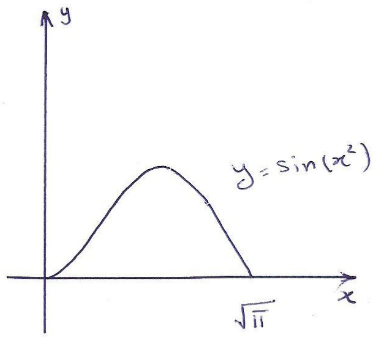
$$= \int_{\pi/3}^{\pi} (3+4\cos\theta+\cos 2\theta)\,d\theta - \int_{\pi/3}^{\pi/2} (9+9\cos 2\theta)\,d\theta = [3\theta + 4\sin\theta + \frac{\sin 2\theta}{2}]_{\pi/3}^{\pi} - (9\theta + \frac{9\sin 2\theta}{2})_{\pi/3}^{\pi/2} =$$

$$= \left(3\pi + 0 + 0) - (\pi + 4\sin \pi/3 + \frac{\sin 2\pi/3}{2}) \right) \text{ - } (\frac{9\pi}{2} + 0) + \left(3\pi + \frac{9\sin 2\pi/3}{2} \right) =$$

$$= (2\pi - \frac{9\sqrt{3}}{4}) \text{ - } \frac{3\pi}{2} + \frac{9\sqrt{3}}{4} = \frac{\pi}{2}.$$

Total area = $2(\frac{\pi}{2}.) = \pi.$

3) The region bounded by the curve $y = \sin x^2$ and the x-axis , $0 \le x \le \sqrt{\pi}$ is revolved about the y-axis. Sketch this curve and find the volume of the resulting solid.



Shell method :
Thickness dx
Radius x
Height y = $\sin x$

$$\text{Volume} = \int_{o}^{\sqrt{\pi}} 2\pi x \sin x^2 dx = 2\pi \left[\frac{-1}{2} \cos x^2 \right]_{x=0}^{x=\sqrt{\pi}} = -\pi(-1+1) = 2\pi.$$

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4) Find the equation of the circle cutting orthogonally the three circles :

$$x^2 + y^2 - 2x + 3y - 7 = 0,$$
$$x^2 + y^2 + 5x - 5y + 9 = 0,$$
$$x^2 + y^2 + 7x - 9y + 29 = 0.$$

The three radical axes are :

$$7x - 8y + 16 = 0,$$
$$2x - 4y + 20 = 0,$$
$$9x - 12y + 36 = 0.$$

These three axes intersect in the point (8 , 9). The radius is the tangent to any circle which is $\sqrt{149}$. Then the equation of the required circle is :

$$(x-8)^2+(y-9)^2=149$$
$$x^2+y^2-16x-18y-4=0.$$

5) Find the value of c such that the straight line y = mx + c touches the parabola $y^2=4px.$

Solving to get points of intersections :

$(mx+c)^2=4px$ Then $m^2x^2+(2cm-4p)x+c^2=0.$

To be a tangent the quadratic equation must have two equal roots or the discriminator vanishes. So

$$(2cm-4p)^2-4c^2m^2=0.$$
i.e. $$4c^2m^2-16cmp+16p^2-4c^2m^2=0$$
OR $$16p(p-cm)=0.$$
Then either p = 0 (cannot be) OR $c=\frac{p}{m}.$

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6) Discuss and sketch the graph of the conic section :

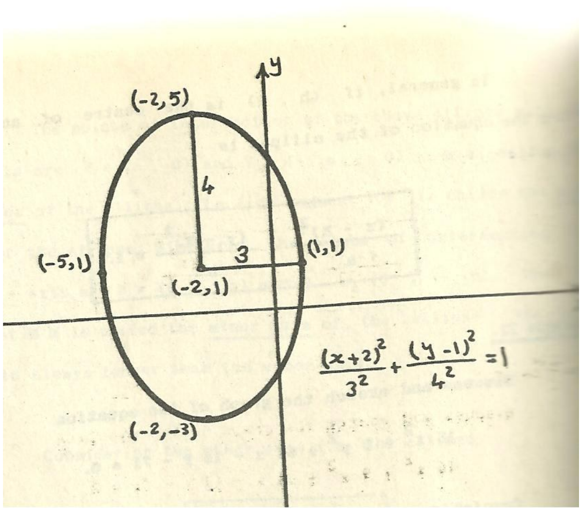
$$16x^2+9y^2+64x-18y-71=0.$$

Show the axes, centre, vertices and foci of this conic section.

The equation of the ellipse is :

$$\frac{(x+2)^2}{3^2}+\frac{(y-1)^2}{4^2}=1.$$

$c^2=4^2-3^2=7.$



Major axis = 4(vertical) , minor axis = 3.

the foci are (- 2 , 1 $\pm\sqrt{7}$).