Fayoum University
Faculty of Engineering
First Year Civil Eng. Department

Engineering Mathematics (2A) Final Exam.
Jan., 16, 2010
Time Allowed: 3 Hours

Attempt All Questions: Total Mark = 70

Q1. (a) Find $\lim\limits_{(x,y)\to(0,2)} \dfrac{\sin(xy)}{x}$ (4M)

(b) Find partial derivatives of the function and its total differential $\mathbf{u} = \mathbf{x}^{yz}$ (5M)

(c) Find the surface area of the region cut from the upper half of the sphere $x^2+y^2+z^2 = 9$ by the cylinder $x^2+y^2 = 9$. (5M)

Q2. (a) The material for the bottom of a rectangular box costs twice as much per square meter such as the material for sides and top. If the volume is fixed, find the relative dimensions that minimize the cost.

(b) Test for extermum the function $f(x, y) = 2 \sin x \sin y - 1$. (14M)

Q3. (a) Find the center of the mass of the region bounded by the curves $y = \sec x$, $y = 1/2$, $x = -\pi/4$ and $x = \pi/4$, where the density at any point (x, y) = $2y$.

(b) Evaluate the integral $\iint\limits_R (x^2 + 2y^2)dxdy$, where R is the region bounded by the curves $xy = 1$, $xy = 2$, $y = |x|$ **and** $y = 2x$. (14M)

Q4. (a) Find the volume of the region bounded by the paraboloid $z = x^2 + y^2$, the cylinder $x^2 + y^2 = 4$ and the xy plane.

(b) Find the total mass of the region bounded by the surfaces $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$. Assuming constant density. (14M)

Q5. (a) Evaluate $\oint\limits_C [(x^4 + 4)dx + xydy]$; C is the cardioid $r = 1 + \cos\theta$.

(b) Evaluate $\int\limits_{(0,0)}^{(1,\pi/2)} e^x[\sin y dx + \cos y dy]$, where C is the curve $x = \sin y$,

(14M)

My Best Wishes
Prof. Dr. Mohamed Eissa Sayed Ahmed

Model Answer of Mathematics 2 (a) Exam (23-1-2010)

First Year of Civil Engineering Department

Q1 (a) $\lim\limits_{(x,y)\to(0,2)} \dfrac{\sin(xy)}{y}$

$$\lim_{(x,y)\to(0,2)} \frac{\sin(xy)}{x} = \frac{0}{0},$$

Let the general path $y = xf(x)+2$, for any arbitrary function $f(x)$

$$\therefore \lim_{(x,y)\to(0,2)} \frac{\sin(xy)}{x} = \lim_{x\to 0} \frac{\sin(x^2 f(x)+2x)}{x} = \frac{0}{0},$$

$$= \lim_{x\to 0}\left[(xf(x)+2)\frac{\sin(x^2 f(x)+2x)}{x(xf(x)+2)}\right]$$

$$= \lim_{x\to 0}(xf(x)+2) \lim_{x(xf(x)+2)\to 0} \frac{\sin(x^2 f(x)+2x)}{x^2 f(x)+2x} = (2)(1) = 2$$

Q1 (b) $u=x^{yz}$,

$\dfrac{\partial u}{\partial x} = yz(x)^{yz-1} \to$ since y and z are constants, $\dfrac{\partial u}{\partial y} = z(x)^{yz}\ln x \to$ since x and z are constants,

$\dfrac{\partial u}{\partial z} = y(x)^{yz}\ln x \to$ since x and y are constants,

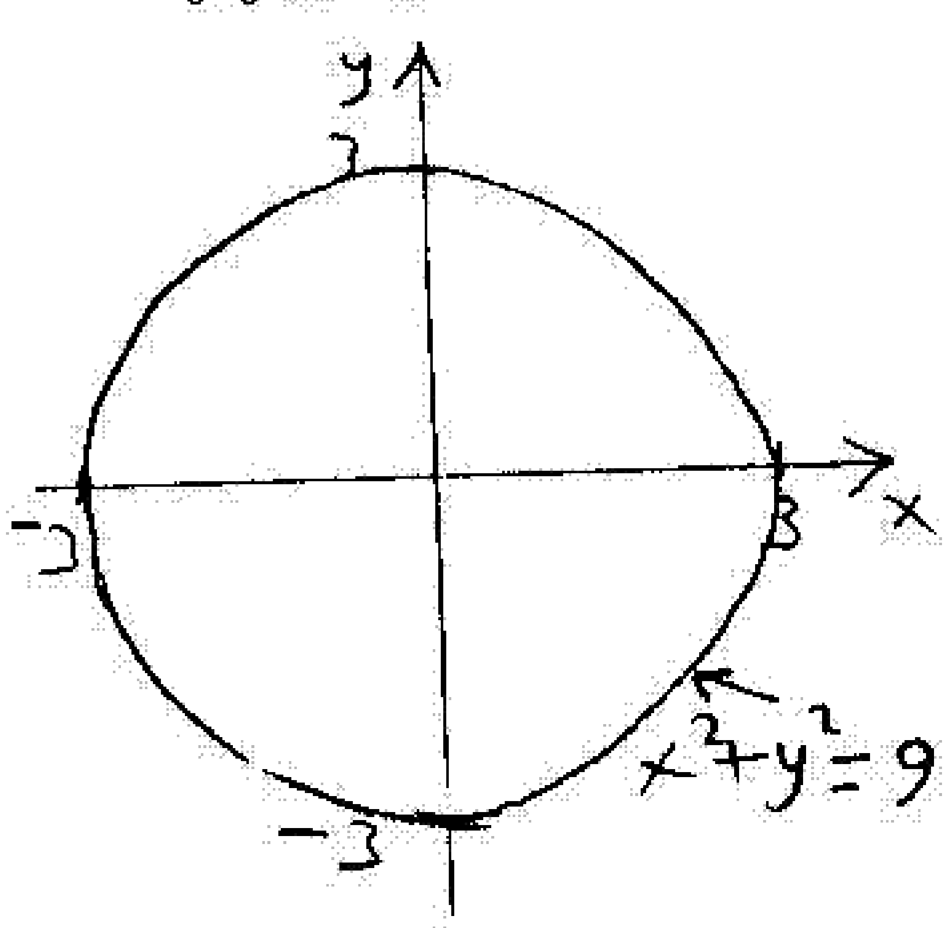
$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = yz(x)^{yz-1}dx + z(x)^{yz}\ln x dy + y(x)^{yz}\ln x dz$$

Q1 (c)

$\because F(x,y,z) = x^2+y^2+z^2-9=0,$

$\therefore \mathbf{n} = \dfrac{x}{3}\mathbf{i} + \dfrac{y}{3}\mathbf{j} + \dfrac{z}{3}\mathbf{k},$

$$S = \iint_{R_{xy}} \frac{3}{z}dA = \int_0^{2\pi}\int_0^3 \frac{3}{\sqrt{9-r^2}} r\,dr\,d\theta = 18\pi.$$



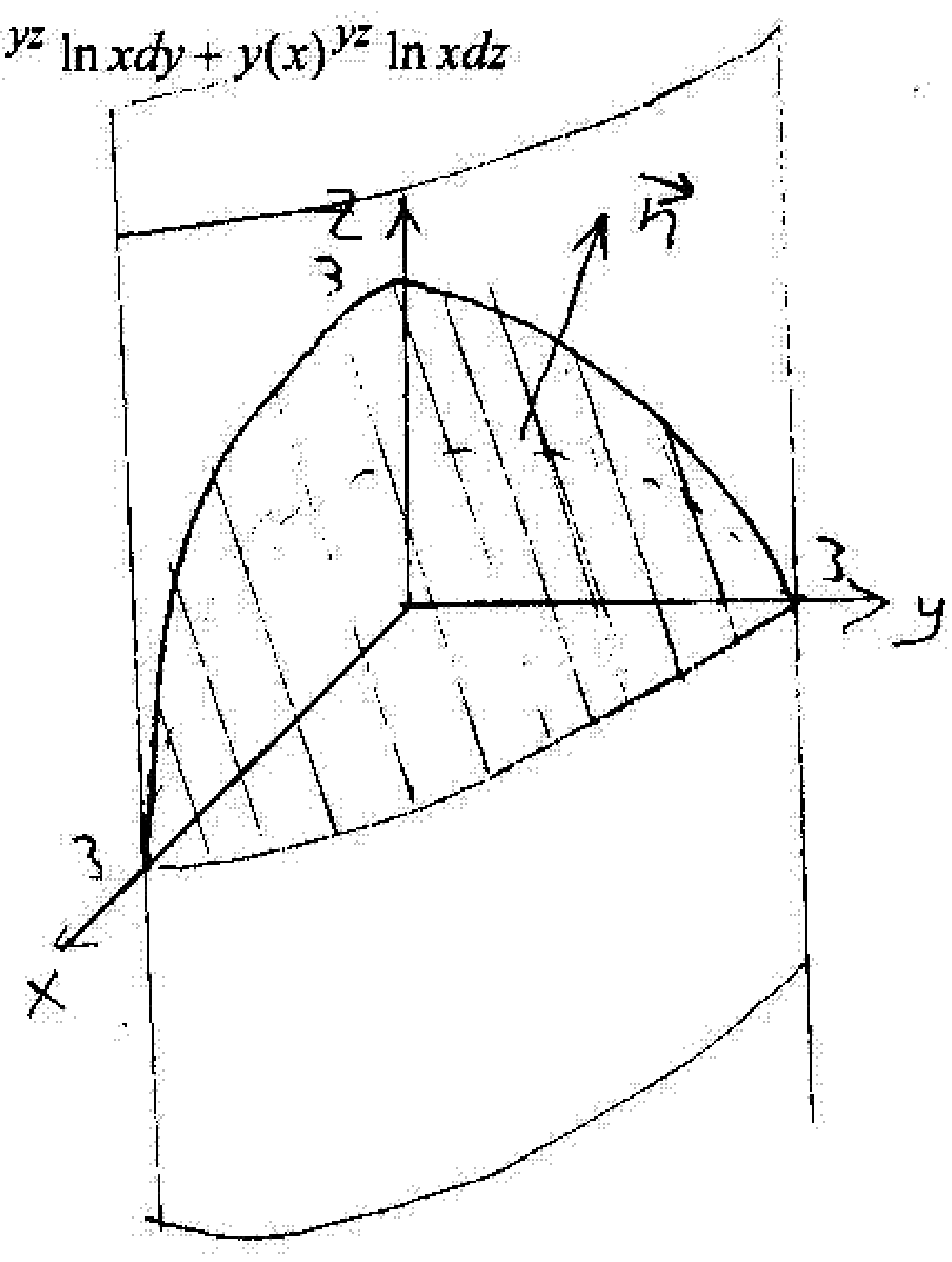


Fig. 1

Q2 (a).

$g(x,y,z)=xyz=V, \rightarrow x>0, y>0, z>0$

$f(x,y,z)=k(3xy+2xz+2yz)$

$\because \dfrac{f_x}{g_x}=\dfrac{f_y}{g_y}=\dfrac{f_z}{g_z} \rightarrow$ Lagrange Multiplier,

$\therefore \dfrac{k(3y+2z)}{yz}=\dfrac{k(3x+2z)}{xz}=\dfrac{k(2x+2y)}{xy}$

Solving the abdve eqns., we get $x=y$ and $z=3y/2=3x/2$,

$\therefore x(x)(3x/2)=\text{V}$, then $x=\sqrt[3]{\dfrac{2}{3}V}, y=\sqrt[3]{\dfrac{2}{3}V}$ and $z=\dfrac{3}{2}\sqrt[3]{\dfrac{2}{3}V}$.

Q2(b).

$f(x,y)=\sin x \sin y \ \ -1$

$f_x=\cos x \sin y=0 \rightarrow x=\pm(\dfrac{2n+1}{2})\pi$ or $y=\pm n\pi$ $(n=0,1,2,.....)$,

$f_x=\sin x \cos y=0 \rightarrow x=\pm n\pi$ or $y=\pm(\dfrac{2n+1}{2})\pi$ $(n=0,1,2,.....)$,

$\therefore$ CPs.: $(\pm(\dfrac{2n+1}{2})\pi, \pm(\dfrac{2n+1}{2})\pi)$, $(\pm n\pi, \pm n\pi)$

$f_{xx}=-\sin x \sin y, \ \ f_{yy}=-\sin x \sin y, \ \ f_{xy}=\cos x \cos y$

$\Delta=(\sin x \sin y)^2-(\cos x \cos y)^2$

$\therefore (\pm(\dfrac{2n+1}{2})\pi, \pm(\dfrac{2n+1}{2})\pi)$ are local maximum points and

$(\pm n\pi, \pm n\pi)$ are saddle points.

Q3(a). The density $\sigma = 2y$

$$M = \int_{-\pi/4}^{\pi/4}\Big[\int_{1/2}^{\sec x} 2y\,dy\Big]dx = \int_{-\pi/4}^{\pi/4}(y^2)\Big|_{1/2}^{\sec x} dx = \int_{-\pi/4}^{\pi/4}(\sec^2 x - \frac{1}{4})dx = 2(\tan x - \frac{1}{4}x)\Big|_0^{\pi/4} = 2(1-\frac{\pi}{16})$$

$$I_2 = \int_{-\pi/4}^{\pi/4}\Big[\int_{1/2}^{\sec x} 2y(y)\,dy\Big]dx = \frac{2}{3}\int_{-\pi/4}^{\pi/4}(y^3)\Big|_{1/2}^{\sec x} dx = \frac{2}{3}\int_{-\pi/4}^{\pi/4}(\sec^3 x - \frac{1}{8})dx = \frac{4}{3}\int_0^{\pi/4}(\sec^3 x - \frac{1}{8})dx$$

Since the gemetry and the density are symmetric about y-axis, then $\bar{x} = 0$, $\bar{y} = \dfrac{I_2}{M}$

Let $$I = \int_0^{\pi/4}\sec^3 x\,dx = \int_0^{\pi/4}\sec^2 x\sec x\,dx = (\tan x\sec x)\Big|_0^{\pi/4} - \int_0^{\pi/4}\tan x(\sec x\tan x)dx =$$

$$= \sqrt{2} - \int_0^{\pi/4}(\sec^3 x - \sec x)dx$$

$$\therefore 2I = \sqrt{2} + \int_0^{\pi/4}\sec x\,dx = \sqrt{2} + (\ln(\sec x + \tan x))\Big|_0^{\pi/4} = \sqrt{2} + \ln(\sqrt{2}+1) \to I = \frac{\sqrt{2}+\ln(\sqrt{2}+1)}{2}$$

$$\therefore I_2 = \frac{4}{3}\Big[\Big(\frac{\sqrt{2}+\ln(\sqrt{2}+1)}{2}\Big) - \frac{\pi}{32}\Big) \cong 1.4 \to \bar{y} = \frac{I_2}{M} \cong 0.87$$

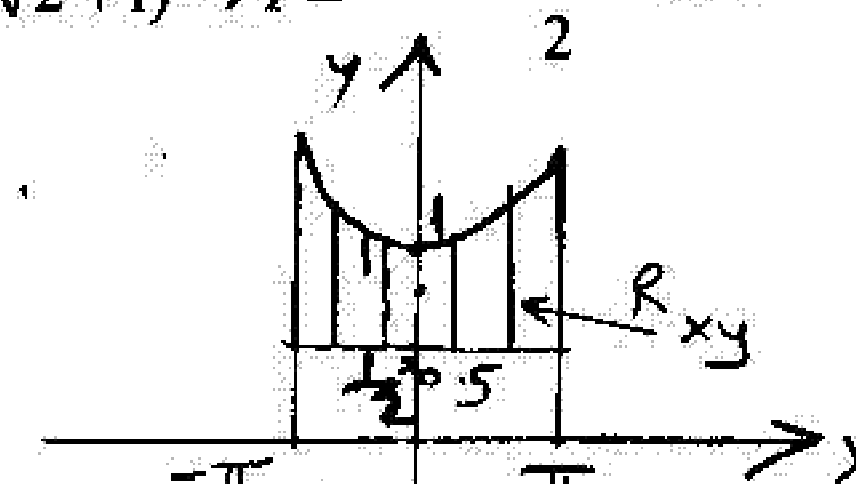


Fig. 2

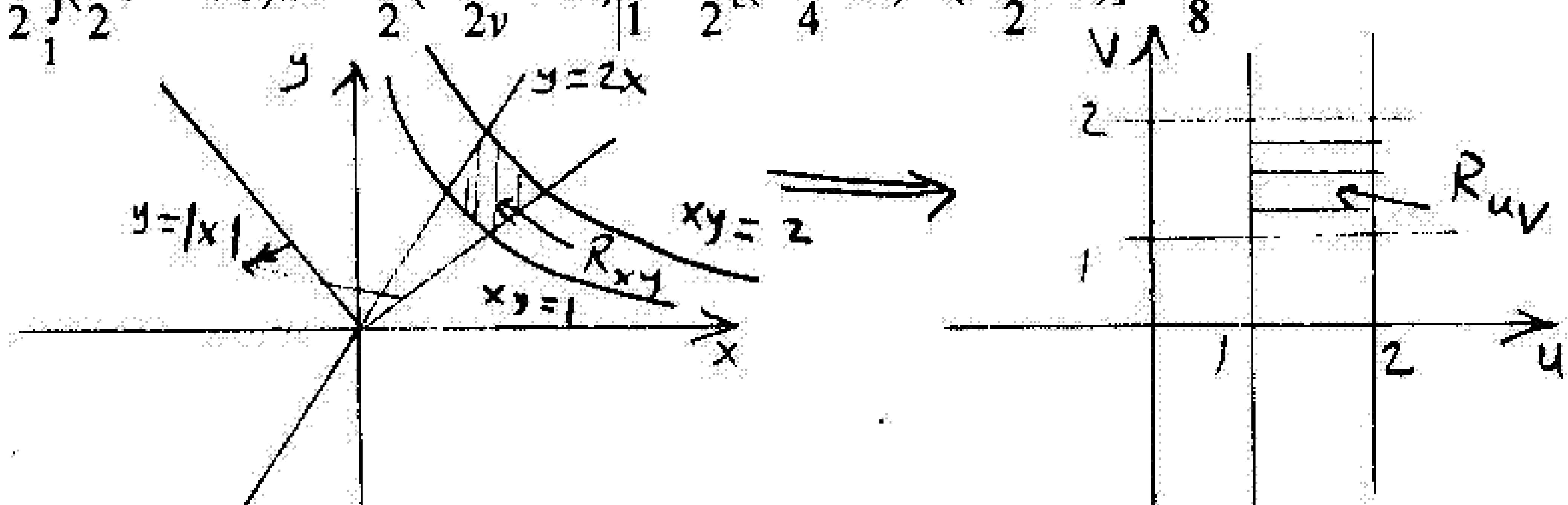
Q3 (b).

Let $u = xy$ and $v = y/x$,

$$\Delta = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial x}\end{vmatrix} = \begin{vmatrix} y & x\\ -\frac{y}{x^2} & \frac{1}{x}\end{vmatrix} = 2\frac{y}{x} = 2v, \therefore J = \frac{1}{\Delta} = \frac{1}{2v}$$

$$I = \int_1^2\int_1^2 \frac{1}{2v}\Big(\frac{u}{v} + 2uv\Big)du\,dv = \frac{1}{2}\int_1^2\int_1^2\Big(\frac{u}{v^2} + 2u\Big)du\,dv = \frac{1}{2}\int_1^2\Big(\frac{u^2}{2v^2} + u^2\Big)\Big|_1^2 dv$$

$$= \frac{1}{2}\int_1^2\Big(\frac{3}{2}v^{-2} + 3\Big)dv = \frac{1}{2}\Big(-\frac{3}{2v} + 3v\Big)\Big|_1^2 = \frac{1}{2}\Big[\Big(-\frac{3}{4} + 6\Big) - \Big(-\frac{3}{2} + 3\Big)\Big] = \frac{15}{8}$$



(a) (b)

Fig 3.

Q4 (a)

$$V=\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_{0}^{x^2+y^2} dz\,dx\,dy \Rightarrow \text{Using the cylinderical coordinates, then}$$

$$V=\int_0^{2\pi}\int_0^{2}\int_0^{r^2} r\,dz\,r\,dr\,d\theta = -\int_0^{2\pi}\int_0^{2} r^3\,dr\,d\theta = \left(\frac{r^4}{4}\right)\Big|_0^2 2\pi = 8\pi.$$

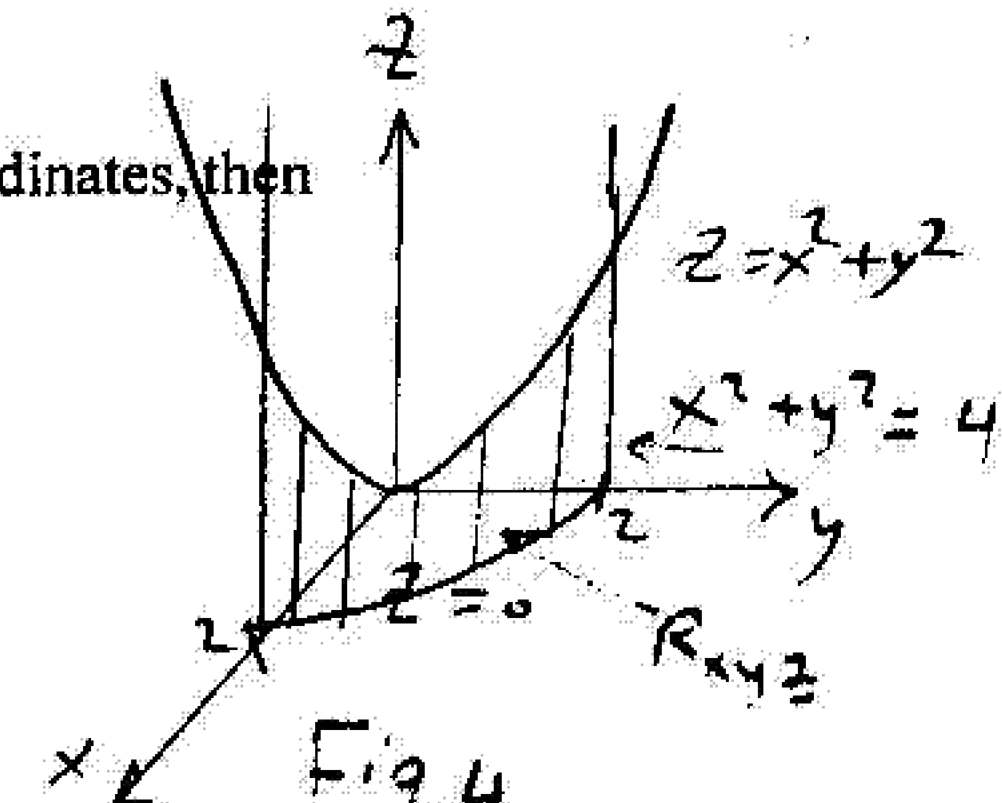


Fig. 4

Q4 (b).

$$\text{Mass}= M = k\iint_{R_{xy}}\int_{x^2+3y^2}^{8-x^2-y^2} dz\; dA = k\iint_{R_{xy}}(8-2x^2-4y^2)\,dA,$$

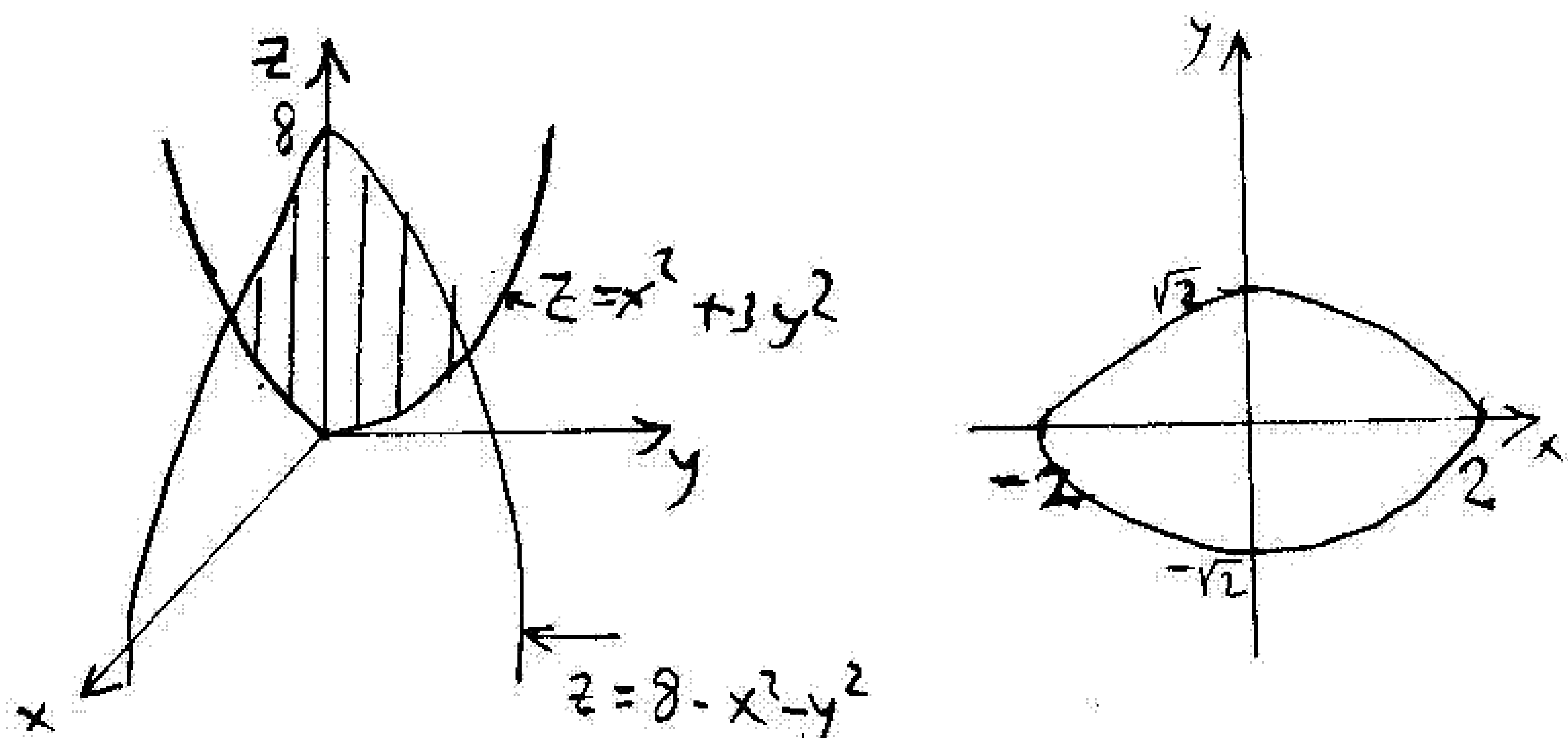
where R_{xy} is the elipose given by $\dfrac{x^2}{4}+\dfrac{y^2}{2}=1$

Using the transformation $x=2r\cos\theta, y=\sqrt{2}r\sin\theta$

$$J=\frac{\partial(x,y)}{\partial(r,\theta)}=2\sqrt{2}r$$

$$\therefore M = 2\sqrt{2}k\int_0^{2\pi}\int_0^{1}(8\text{-}8r^2\cos^2\theta-8r^2\sin^2\theta)\,r\,dr\,d\theta$$

$$=16\sqrt{2}k\int_0^{2\pi}\int_0^{1}(1\text{-}r^2)\,r\,dr\,d\theta = 16\sqrt{2}\left(\frac{r^2}{2}-\frac{r^4}{4}\right)\Big|_0^1(2\pi)=8\sqrt{2}k\,\pi.$$



(a) (b)

Fig. 5

Q5 (a).

$$I = \oint_C y^2(x^2+4)dx + xydy$$

$$= \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dA = \iint_R (y)dA \Rightarrow \text{ using polar coordinates}$$

$$\therefore I \Rightarrow \int_0^{2\pi}\int_0^{1+\cos\theta} r^2 \sin\theta dr d\theta = \int_0^{2\pi} \left(\frac{r^3}{3}\right)\Bigg|_0^{1+\cos\theta} \sin\theta d\theta$$

$$= \frac{1}{3}\int_0^{2\pi} (1+\cos\theta)^3 \sin\theta d\theta = -\frac{1}{12}(1+\cos\theta)^4\Bigg|_0^{2\pi} = 0.$$

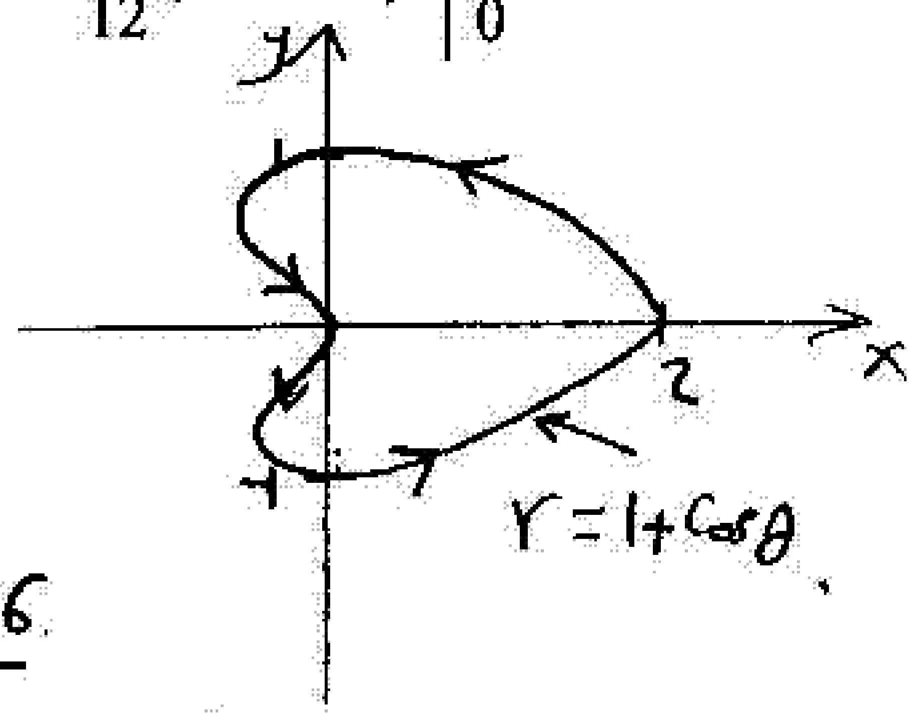


Fig. 6

Q5(b).

$$I = \int_{(0,0)}^{(1,\pi/2)} e^x \sin y dx + e^x \cos y dy, \quad \because \frac{\partial P}{\partial y} = e^x \cos y, \ \frac{\partial Q}{\partial x} = e^x \cos y,$$

Since, $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$, and are continuous, then The line integral is path independent, therefore

$$I = \int_{C_1} e^x \sin y dx + e^x \cos y dy + \int_{C_2} e^x \sin y dx + e^x \cos y dy$$

$C_1 : y = 0, x \rightarrow 0 \text{ to } 1, C_2 : x = 1, y \rightarrow 0 \text{ to } \pi/2,$

$$I = 0 + \int_0^{\pi/2} e(\cos y)dy = e(\sin y)\Bigg|_0^{\pi/2} = e.$$

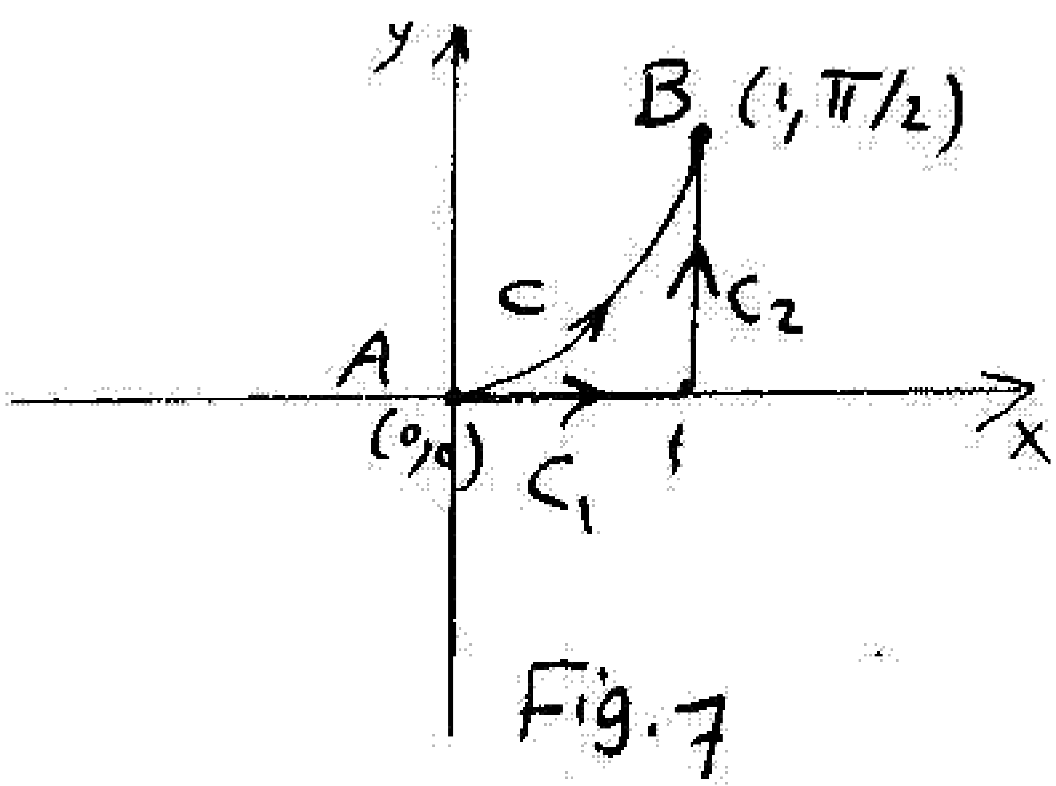


Fig. 7