Model Answer for Civil Exam (Jan. 15, 2011)

Q1 (a).

$$F(x,y,z)=y^2+z^2-4$$

$$\frac{\partial F}{\partial x}=0,\ \frac{\partial f}{\partial y}=2y,\ \frac{\partial f}{\partial z}=2z,\ \therefore \frac{\partial F}{\partial x}|(2,2,-2)=0,\ \frac{\partial f}{\partial y}|(2,2,-2)=4,\ \frac{\partial f}{\partial z}|(2,2,-2)=-4.$$

The tangent plane equation is $(0)(x-2)+4(y-2)-4(z+2)=0 \Rightarrow y-z=4.$

The normal line equation is $\frac{x-2}{0}=\frac{y-2}{4}=\frac{z+2}{-4}=t \Rightarrow x=2, y=4t+2, z=-4t-2.$

Q1(b).

$$z=f(x,y)=x^3+y^3-9xy+27$$

$$f_x=3x^2-9y=0 \Rightarrow\ (1),\ \ f_y=3y^2-9x=0 \Rightarrow\ \ (2)\ \therefore y^3=x^3\ \ \therefore y=x,$$

$$\therefore x^2-3x=0, \Rightarrow x=0,\ 3 \Rightarrow CPS:(0,0),\ (3,3).$$

x	y	$f_{xx}=6x$	$f_{yy}=6y$	$f_{xy}=-9$	Δ	conclusion
0	0	0	0	-9	-81<0	Saddle point
3	3	18	18	-9	243>0	Local minimum point

Q1 (c).

$$\frac{dI}{dt}=\int_t^{t^2}\frac{\partial}{\partial t}\left(\frac{\sin tx}{x}\right)dx+\frac{\sin t^3}{t^2}\left(\frac{d}{dt}(t^2)\right)-\frac{\sin t^2}{t}\left(\frac{d}{dt}(t)\right)$$

$$=\int_t^{t^2}\cos(tx)dx+\frac{2\sin t^3}{t}-\frac{\sin t^2}{t}=\frac{\sin t^3}{t}-\frac{\sin t^2}{t}+\frac{2\sin t^3}{t}-\frac{\sin t^2}{t}$$

$$=3\frac{\sin t^3}{t}-2\frac{\sin t^2}{t}$$

Q2 (a)

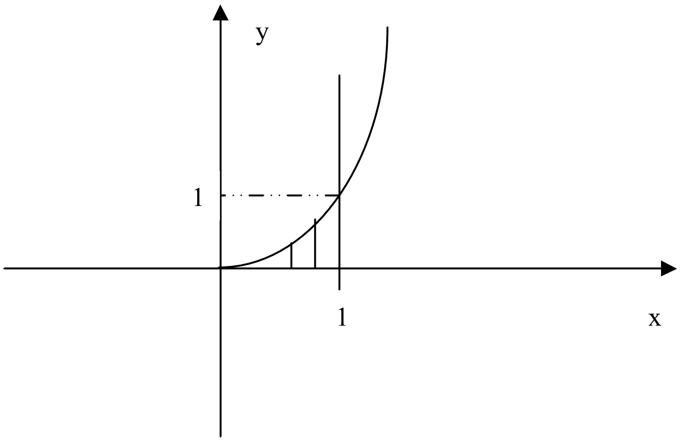

Fig. 1

From Fig. 1, we have

$$I = \int_0^1 \int_0^{x^3} \frac{2\pi \sin \pi x^2}{x^2} dy dx = 2\pi \int_0^1 x^3 \frac{\sin \pi x^2}{x^2} dx = 2\pi \int_0^1 x \sin \pi x^2 dx$$

$$= -\cos \pi x^2 \Big|_0^1 = 2$$

Q2 (b).

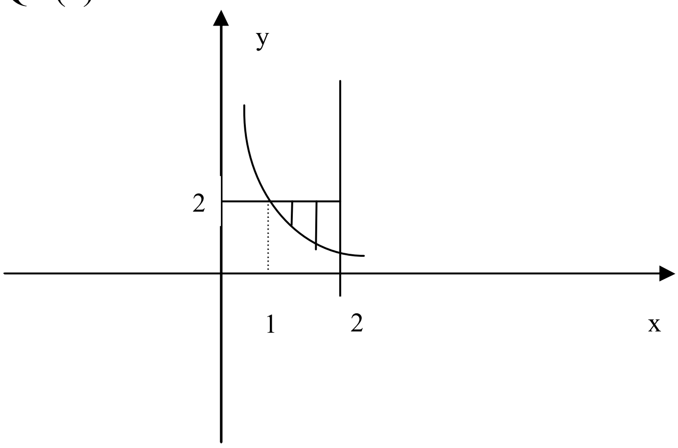

Fig. 2

σ=*ky*

$$M = \iint_R \sigma dA = k\int_1^2 \int_{2/y}^2 y\,dx\,dy = k$$

$$I_1 = \iint_R \sigma x dA = k\int_1^2 \int_{2/y}^2 yx\,dx\,dy = 2k\int_1^2 (y - \frac{1}{y})dy = k(y^2 - 2\ln y)\Big|_1^2 = k(3 - 2\ln 2)$$

Volu

$$I_2 = \iint_R \sigma y dA = k\int_1^2 \int_{2/y}^2 y^2\,dx\,dy = 2k\int_1^2 (y^2 - y)dy = k(\frac{5}{3})$$

$$\therefore \bar{x} = \frac{I_1}{M} = (3 - 2\ln 2) \cong 1.614, \quad \bar{y} = \frac{I_2}{M} = \frac{5}{3} \cong 1.667$$

Q2(c).

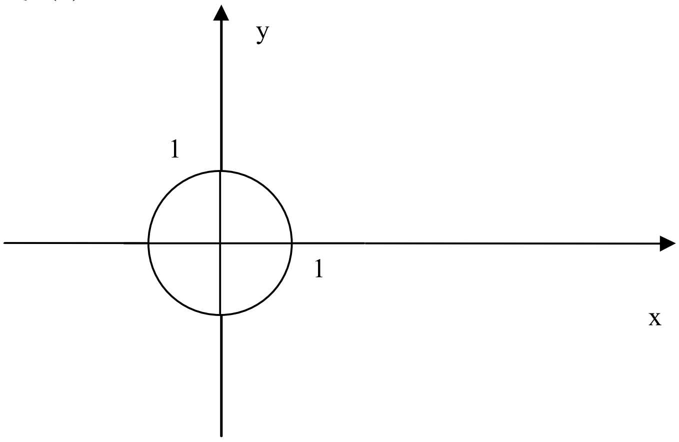

Fig. 3

$$Volume = \iint_R z dA = \iint_R (2 - x - y)dA = \int_0^{2\pi}\int_0^1 (2 - r\cos\theta - r\sin\theta) r dr d\theta$$

$$= \int_0^{2\pi} (r^2 - \frac{r^3}{3}(\cos\theta + \sin\theta)\Big|_0^1 d\theta = \int_0^{2\pi} (1 - \frac{1}{3}(\cos\theta + \sin\theta)d\theta = 2\pi$$

Q3(a). I= $\int_0^{2/3} \int_y^{2-2y} (x + 2y)\, e^{(y-x)} dy\, dx$

Let $u=x+2y$ and $v=y-x$,

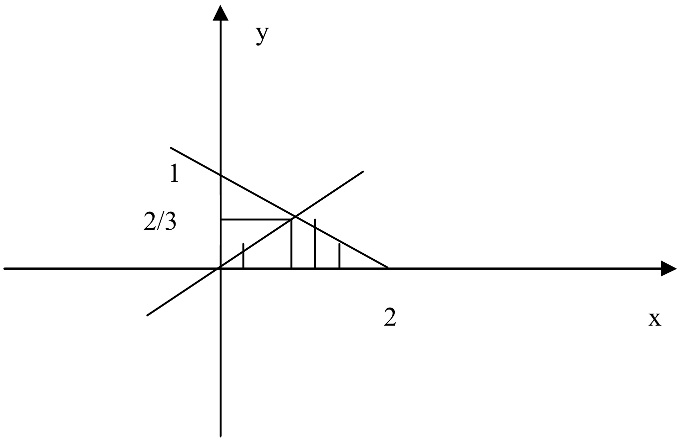

Fig. 4(a)

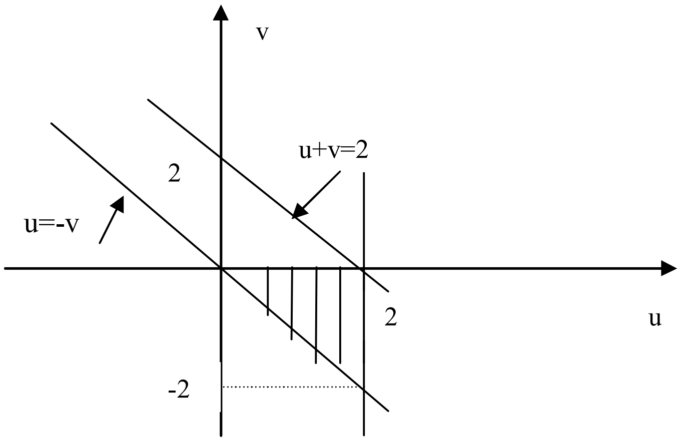

Fig. 4 (b)

$$\therefore \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} = 3, \ \therefore \mathrm{J} = \frac{1}{3}$$

$$I = \frac{1}{3}\int_0^2 \int_{-u}^0 ue^v dv du = \frac{1}{3}\int_0^2 u(1-e^{-u})du = \frac{1}{3}\left(\frac{u^2}{2} + ue^{-u} + e^{-u}\right)\Big|_0^2 = \frac{1}{3}(1+3e^{-2})$$

Q3(b).

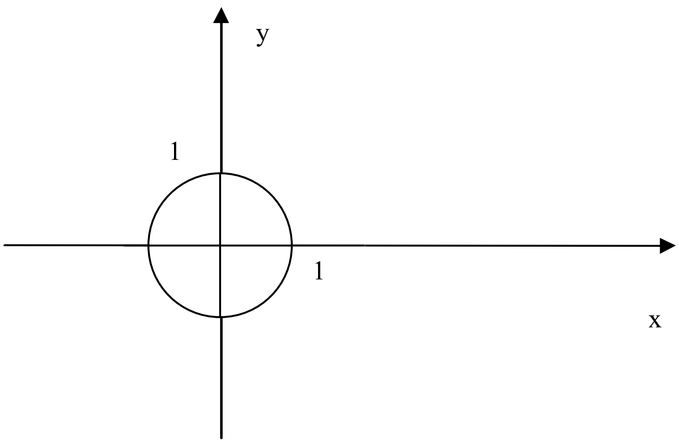

Fig. 5

$$F(x,y,z)=x^2+y^2+z^2-4=0$$

$$\mathbf{n}=\frac{\nabla F}{\|\nabla F\|}=\frac{2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k}}{2\sqrt{x^2+y^2+z^2}}=(2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k})/4=(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})/2$$

$$\therefore S=\iint_R \frac{2}{z}dA=\iint_R \frac{2}{\sqrt{4-x^2-y^2}}dA=\int_0^{2\pi}\int_0^1 \frac{2}{\sqrt{4-r^2}}rdrd\theta=4\pi(2-\sqrt{3})$$

Q4. (a)

$$f(x,y)=2x\sin y+4e^x$$

$$\therefore \nabla f=(2\sin y+4e^x)\mathbf{i}+(2x\cos y)\mathbf{j}+(0)\mathbf{k}$$

$$Work=\int_C \nabla f\bullet d\mathbf{r}=\int_A^B(2\sin y+4e^x)dx+(2x\cos y)dy$$

$$\because p=2\sin y+4e^x,\therefore \frac{\partial p}{\partial y}=2\cos y, Q=2x\cos y,\therefore \frac{\partial Q}{\partial x}=2\cos y,$$

$$\frac{\partial p}{\partial y}=\frac{\partial Q}{\partial x}=2\cos y\Rightarrow \text{ the line integral is path indepent.}$$

$$\therefore \mathrm{I}=\int_1^2(2\sin 1+4e^x)dx+\int_1^8 4\cos y dy=4(e^2-e)+4\sin 8-2\sin 1=18.7$$

4(c)

$$V=\int_0^{2\pi}\int_0^1\int_{r^2}^{(r^2+1)/2} rdzdrd\theta=\frac{1}{2}\int_0^{2\pi}\int_0^1(1-r^2)rdrd\theta=\frac{\pi}{4}$$