A Singular Value Decomposition Derivation in the Discrete Frequency Domain of Optimal Noncentro-Symmetric 2-D FIR Filters[1]

By

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ABSTRACT

A new derivation is presented for the least squares solution of the design problem of 2-D FIR filters by minimizing the Frobenius norm of the difference between the matrices of the ideal and actual frequency responses sampled at the points of a frequency grid. The mathematical approach is based on the singular value decomposition of two complex transformation matrices. Interestingly, the designed filter is proved to be zero-phase if the ideal filter is so without assuming any kind of symmetry.

I. INTRODUCTION

The least squares technique has been applied for designing 2-D FIR filters with quadrantally symmetric or antisymmetric frequency response [1] and has recently been extended to filters with only a centro-symmetric frequency response [2]. A new treatment of the special case of quadrantal symmetry or antisymmetry based on a singular value decomposition has been briefly presented in [3]. All the work reported in [1-3] is based on the assumption of having a real impulse response where the linear-phase condition implies a centro-symmetric frequency response amplitude.

The main objective of this paper is to present a new derivation for the least squares design problem of 2-D FIR filters which is based on the singular value decomposition of some transformation matrices rather than the classical technique based on setting the partial derivatives to zero. Moreover; the treatment will be for the general case where there are no assumptions of a centro-symmetric frequency response or even a real impulse response.

II. PROBLEM FORMULATION

The frequency response of a two-dimensional FIR filter is :

$$H(\omega_1,\omega_2) = \sum_{n_1=-N_1}^{N_1} \sum_{n_2=-N_2}^{N_2} h(n_1,n_2) e^{-j(\omega_1 n_1+\omega_2 n_2)} \tag{1}$$

The impulse response has a rectangular region of support centered at the origin and no assumptions are implied of quadrantal symmetry, quadrantal antisymmetry or centro-symmetry, i.e., $H(\omega_1,\omega_2)$ is not generally zero phase . Moreover $h(n_1,n_2)$ is not restricted to be real. The above equation can be compactly expressed as[3] :

$$H(\omega_1,\omega_2) = f^{+}(\omega_1) A g(\omega_2) \tag{2}$$

where A is the $(2N_1+1)$ x $(2N_2+1)$ impulse response matrix (the matrix of filter coefficients) :

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$$A = \begin{bmatrix} h(-N_1,-N_2) & \dots & h(-N_1,N_2) \\ . & \dots & . \\ . & \dots & . \\ h(N_1,-N_2) & \dots & h(N_1,N_2) \end{bmatrix} \tag{3}$$

and $f(\omega_1)$ and $g(\omega_2)$ are respectively $(2N_1 + 1)$- and $(2N_2 + 1)$-dimensional vectors defined by :

$$f(\omega_1) = \begin{bmatrix} Ju^*(\omega_1) \\ 1 \\ u(\omega_1) \end{bmatrix} \quad \text{and} \quad g(\omega_2) = \begin{bmatrix} Jv^*(\omega_2) \\ 1 \\ v(\omega_2) \end{bmatrix} . \tag{4}$$

In the above equation $u(\omega_1)$ and $v(\omega_2)$ are respectively N_1- and N_2-dimensional vectors defined by[4] :

$$u(\omega_1) = \begin{pmatrix} e^{j\omega_1} \\ . \\ . \\ e^{jN_1\omega_1} \end{pmatrix} \quad \text{and} \quad v(\omega_2) = \begin{pmatrix} e^{-j\omega_2} \\ . \\ . \\ e^{-jN_2\omega_2} \end{pmatrix} \tag{5}$$

and J is the contra-identity matrix of the proper order defined by :

$$J = \begin{pmatrix} & & & 1 \\ & & . & \\ & . & & \\ 1 & & & \end{pmatrix} . \tag{6}$$

Discretizing the continuous frequencies ω_1 and ω_2 by taking $2M_1$ and $2M_2$ samples of them respectively, we get :

$$\left(\omega_i\right)_{m_i} = -\pi + \frac{\pi}{M_i}\left(m_i - 1\right) \quad , m_i = 1 , \dots , 2M_i \quad , \quad i = 1, 2 . \tag{7}$$

Sampling the frequency response $H(\omega_1,\omega_2)$ at those discrete frequencies we get :

$$H_{m_1,m_2} = H\left(\left(\omega_1\right)_{m_1}, \left(\omega_2\right)_{m_2}\right) \tag{8}$$

which can be expressed using (2) as :

$$H_{m_1,m_2} = f_{m_1}^{+} A g_{m_2} \tag{9}$$

where the vectors f_{m1} and g_{m2} are obtained by sampling the vectors $f(\omega_1)$ and $g(\omega_2)$ of (4) at the discrete frequencies of (7) .

Let $\underline{H}$ be the $2M_1 \times 2M_2$ matrix of the discretized frequency response :

$$\underline{H} = \begin{bmatrix} H_{m_1,m_2} \end{bmatrix} \tag{10}$$

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[3]The superscripts +, *, T denote the complex conjugate transpose, the complex conjugate and the transpose respectively.

[4]Notice that the exponents of the elements of vectors $u(\omega_1)$ and $v(\omega_2)$ are positive and negative respectively.

which can be expressed - using (9) - as :

$$\underline{H} = F^{+} A G \tag{11}$$

where F and G are respectively the $(2N_1 + 1) \times 2M_1$ and $(2N_2 + 1) \times 2M_2$ transformation matrices defined by :

$$F = \begin{pmatrix} f_1 & \dots & f_{2M_1} \end{pmatrix} \; , \tag{12}$$

$$G = \begin{pmatrix} g_1 & \dots & g_{2M_2} \end{pmatrix} \; . \tag{13}$$

Using (4), it is possible to express the above 2 matrices as :

$$F = \begin{pmatrix} JU^{*} \\ \mu^{T} \\ U \end{pmatrix} \text{ and } G = \begin{pmatrix} JV^{*} \\ \mu^{T} \\ V \end{pmatrix} \tag{14}$$

where the $N_1 \times 2M_1$ and $N_2 \times 2M_2$ matrices U and V are respectively defined by :

$$U = \begin{pmatrix} u_1 & u_2 & \dots & u_{2M_1} \end{pmatrix} \; , \tag{15a}$$

$$V = \begin{pmatrix} v_1 & v_2 & \mathrm{L} & v_{2M_2} \end{pmatrix} \tag{15b}$$

and μ is the summing vector of appropriate dimension defined by :

$$\mu = \begin{pmatrix} 1 & 1 & \dots & 1 \end{pmatrix}^{T} \; . \tag{16}$$

From (7) it can be shown that :

$$\left(\omega_i\right)_{2M_i - m_i} = -\left(\omega_i\right)_{m_i + 2} \quad , \text{ i = 1, 2 for any integer } m_i \tag{17}$$

and consequently it follows from (5) that :

$$u_{2M_1 - m_1} = u^{*}_{m_1 + 2} \; , \tag{18a}$$

$$v_{2M_2 - m_2} = v^{*}_{m_2 + 2} \; . \tag{18b}$$

Therefore the matrices U and V of (15) can be expressed as :

$$U = \begin{pmatrix} u_1 & U_a & u_{M_1 + 1} & U^{*}_{a} J \end{pmatrix} \; , \tag{19a}$$

$$V = \begin{pmatrix} v_1 & V_a & v_{M_2 + 1} & V^{*}_{a} J \end{pmatrix} \tag{19b}$$

where the $N_1 \times (M_1 - 1)$ and $N_2 \times (M_2 - 1)$ matrices U_a and V_a are defined by :

$$U_a = \begin{pmatrix} u_2 & \mathrm{L} & u_{M_1} \end{pmatrix} \; , \tag{20a}$$

$$V_a = \begin{pmatrix} v_2 & \cdots & v_{M_2} \end{pmatrix} \quad . \tag{20b}$$

From (5) and (7) it can be shown that :

$$u_1 = \begin{pmatrix} -1 & 1 & -1 & \cdots & (-1)^{N_1} \end{pmatrix}^T \quad , \tag{21a}$$

$$v_1 = \begin{pmatrix} -1 & 1 & -1 & \cdots & (-1)^{N_2} \end{pmatrix}^T \quad , \tag{21b}$$

$$u_{M_1+1} = \mu \quad , \tag{22a}$$

$$v_{M_2+1} = \mu \quad . \tag{22b}$$

Let $\underline{H}^0$ be the matrix of the ideal frequency response at the same discrete frequencies of (7) and let E be the Frobenius norm [4] of the matrix difference $\underline{H} - \underline{H}^0$:

$$E = \left\| \underline{H} - \underline{H}^\circ \right\| = \sqrt{\sum_{m_1=1}^{2M_1} \sum_{m_2=1}^{2M_2} \left| H_{m_1,m_2} - H^\circ_{m_1,m_2} \right|^2} \quad . \tag{23}$$

The impulse response matrix A of (11) will be derived next section by minimizing the above error criterion.

III. Singular Value Decomposition Derivation

It will be assumed that the number of frequency samples in each direction is greater than or equal to the length of the impulse response in that direction , i.e. $2M_i \geq (2N_i + 1)$ for i = 1,2 .

Under the assumption that $2M_i \geq (2N_i + 1)$ for i = 1,2 the matrices F and G of (12) and (13) can be proved to have full row rank. Therefore the singular value decomposition of F and G can be expressed as [4] :

$$F = P_F \begin{pmatrix} \Sigma_F & 0 \end{pmatrix} Q_F^+ \tag{24}$$

$$G = P_G \begin{pmatrix} \Sigma_G & 0 \end{pmatrix} Q_G^+ \tag{25}$$

where P_F , Q_F , P_G and Q_G are unitary matrices of order $(2N_1+1)$, $2M_1$, $(2N_2+1)$ and $2M_2$ respectively; and Σ_F and Σ_G are real diagonal matrices of order $(2N_1+1)$ and $(2N_2+1)$ of the singular values of the matrices F and G respectively. The matrices Σ_F and Σ_G are nonsingular since the matrices F and G have full row rank.

In an attempt for finding the matrices P_F , Q_F and Σ_F of (24), the following $2M_1$-dimensional row vector will first be defined :

$$s(n) = \begin{pmatrix} e^{jn\left(\omega_1\right)_1} & e^{jn\left(\omega_1\right)_2} & \cdots & e^{jn\left(\omega_1\right)_{2M_1}} \end{pmatrix} . \tag{26}$$

Using (26) and (7) for the discrete frequencies, it is straightforward to show that s(n) has the following properties :

$$s(-n) = s^*(n) , \tag{27}$$

$$s\left(n + 2M_1\right) = s(n) . \tag{28}$$

Therefore using (15a) and (5), the $(2N_1+1)$ x $2M_1$ transformation matrix F of (14) can be expressed as :

$$F = \begin{pmatrix} s(-N_1) \\ \vdots \\ s(-1) \\ s(0) \\ s(1) \\ \vdots \\ s(N_1) \end{pmatrix} = \begin{pmatrix} s(2M_1 - N_1) \\ \vdots \\ s(2M_1 - 1) \\ s(0) \\ s(1) \\ \vdots \\ s(N_1) \end{pmatrix} . \tag{29}$$

From (26) and (7) it is possible to show that :

$$\begin{aligned} s(n_1)s^+(n_2) &= \sum_{m=1}^{2M_1} e^{j(n_1 - n_2)\left(\omega_1\right)_m} \\ &= (-1)^{n_1 - n_2} \sum_{m=1}^{2M_1} e^{j\frac{2\pi}{2M_1}(n_1 - n_2)(m-1)} \end{aligned}$$

$$s(n_1)s^+(n_2) = \begin{cases} (-1)^{(n_1 - n_2)}(2M_1) & \text{if } n_1 - n_2 = 2M_1 k, \text{ k integer} \\ 0 & \text{otherwise} \end{cases} \tag{30}$$

Since the integers n_1 and n_2 lie in the range $0 \le n_1 , n_2 \le 2M_1 - 1$, the above equation reduces to :

$$s(n_1)s^+(n_2) = 2M_1 \delta(n_1 - n_2) \tag{31}$$

where δ(n) is the Kronecker delta function. It follows from (29) and (31) that :

$$FF^+ = 2M_1 I_{(2N_1+1)} . \tag{32}$$

Since from (24) one gets :

$$FF^+ = P_F \Sigma_F^2 P_F^+ \tag{33}$$

the above two equations reveal that :

$$\Sigma_F = \sqrt{2M_1} I_{(2N_1+1)} , \tag{34}$$

$$P_F = I_{(2N_1+1)} . \tag{35}$$

Substituting (34) and (35) into (24) to obtain :

$$F = \sqrt{2M_1}\begin{pmatrix} I_{(2N_1+1)} & 0 \end{pmatrix} Q_F^{+} \tag{36}$$

and partitioning Q_F as :

$$Q_F = \begin{pmatrix} Q_{F1} & Q_{F2} \end{pmatrix} \tag{37}$$

one gets :

$$Q_{F1} = \frac{1}{\sqrt{2M_1}} F^{+} \ . \tag{38}$$

In order to find the remaining ($2M_1$ - $2N_1$ - 1) columns of matrix Q_F , the following square matrix of order $2M_1$ is first formed :

$$T = \begin{pmatrix} F \\ F_2 \end{pmatrix} \tag{39}$$

where the rows of the matrix F_2 are given by the row vector s(n) of (26) for n = N_1+1, N_1+2 , ... , $2M_1$ - N_1 - 1 , i.e. the rows of F_2 and of matrix F of (29) are given by s(n), n = 0, 1 , ... , $2M_1$ - 1 . Consequently the orthogonality condition (31) implies that :

$$TT^{+} = 2M_1 I_{2M_1} \tag{40}$$

which means that the matrix $\frac{1}{\sqrt{2M_1}} T$ is unitary. Equations (37) - (40) imply that matrix Q_F is given by :

$$Q_F = \frac{1}{\sqrt{2M_1}} T^{+} \ . \tag{41}$$

Therefore a complete singular value decomposition of the transformation matrix F is given by Eqs (36) and (41). By the same token the singular value decomposition of matrix G of (14) is given by :

$$G = \sqrt{2M_2}\begin{pmatrix} I_{(2N_2+1)} & 0 \end{pmatrix} Q_G^{+} \tag{42}$$

where the unitary matrix Q_G of order $2M_2$ is defined by :

$$Q_G = \frac{1}{\sqrt{2M_2}} W^{+} \tag{43}$$

and the square matrix W is defined by :

$$W = \begin{pmatrix} G \\ G_2 \end{pmatrix} \ . \tag{44}$$

In the above equation, the rows of matrix G_2 are formed from the $2M_2$-dimensional row vector[5] :

$$t(n) = \begin{pmatrix} e^{-jn(\omega_2)_1} & e^{-jn(\omega_2)_2} & \text{L} & e^{-jn(\omega_2)_{2M_2}} \end{pmatrix} \tag{45}$$

with n = N_2 + 1 , N_2 + 2 , ... , $2M_2$ - N_2 - 1 .

Utilizing the singular value decomposition of (36) and (42), the frequency response matrix $\underline{H}$ of (11) can be expressed as :

$$\begin{aligned}\underline{H} &= \sqrt{4M_1M_2}\,Q_F\begin{pmatrix} I_{(2N_1+1)} \\ 0\end{pmatrix} A\begin{pmatrix} I_{(2N_2+1)} & 0\end{pmatrix} Q_G^+ \\ &= \sqrt{4M_1M_2}\,Q_F\begin{pmatrix} A & 0 \\ 0 & 0\end{pmatrix} Q_G^+\end{aligned} \tag{46}$$

Since the Frobenius norm is invariant under pre- or postmultiplication by a unitary matrix, the error criterion of (23) can be expressed using the above equation as :

$$E = \left\| \sqrt{4M_1M_2}\begin{pmatrix} A & 0 \\ 0 & 0\end{pmatrix} - Q_F^+\,\underline{H}^0 Q_G \right\| . \tag{47}$$

Partitioning the second matrix appearing in the above equation conformably with the first one to get :

$$Q_F^+\,\underline{H}^0 Q_G = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22}\end{pmatrix} \tag{48}$$

the error criterion reduces to :

$$E = \left\| \begin{pmatrix} \sqrt{4M_1M_2}\,A - B_{11} & -B_{12} \\ -B_{21} & -B_{22}\end{pmatrix} \right\| . \tag{49}$$

Since the dependence of E on matrix A is restricted to the submatrix in the (1,1) location of the matrix appearing on the R.H.S. of the above equation, and calling to remembrance the definition of the Frobenius norm, one concludes that the minimum value of E is attained at :

$$A = \frac{1}{\sqrt{4M_1M_2}} B_{11} . \tag{50}$$

By definition (48) of B_{11} , one gets :

$$A = \frac{1}{\sqrt{4M_1M_2}}\begin{pmatrix} I_{(2N_1+1)} & 0\end{pmatrix} Q_F^+\,\underline{H}^0 Q_G \begin{pmatrix} I_{(2N_2+1)} \\ 0\end{pmatrix} \tag{51}$$

and utilizing (41), (39), (43) and (44) one obtains :

$$A = \frac{1}{4M_1M_2} F\,\underline{H}^0 G^+ . \tag{52}$$

This is the main result of this paper ; namely the above matrix A is the unique optimal solution in the least squares sense of the approximation problem inherent in the design of 2-D FIR filters. The corresponding frequency response obtained by substituting (52) into (2) is :

$$H(\omega_1,\omega_2) = \frac{1}{4M_1M_2} f^+(\omega_1) F\,\underline{H}^0 G^+ g(\omega_2) . \tag{53}$$

From (4) and (14), one obtains :

$$\begin{aligned} f^+(\omega_1)F &= u^T(\omega_1)U^* + \mu^T + u^+(\omega_1)U \\ &= \mu^T + 2\,\mathrm{Re}\{u^+(\omega_1)U\} .\end{aligned} \tag{54}$$

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[5] The negative sign is used for the exponents of the elements of vector t(n) in (45) in order to be consistent with vector v(ω_2) defined in (5).

Therefore the row vector $f^{+}(\omega_1)F$ is real and the same holds for the column vector $G^{+}g(\omega_2)$ which implies - based on (53) - that the frequency response $H(\omega_1,\omega_2)$ of the designed filter will be zero-phase for a zero-phase ideal frequency response $\underline{H}^0$. Any wonder about this interesting result of getting a zero-phase frequency response in the general case of no centro-symmetry should be removed by remembering that the impulse response $h(n_1,n_2)$ and consequently matrix A have not been restricted to be real.

It should be noticed that the application of (52) does not require the singular value decomposition (svd) of any matrix. The svd has been needed only for the derivation.

IV. SIMULATION RESULTS

A general-shaped frequency response is shown in Fig. 1 where the right triangular passband (PB) lies completely in the first quadrant and defined by the 3 vertices (a_1,b_1), (a_2,b_2) and (a_2,b_1) . The transition band (TB) has a fixed width c and can overlap with the second and fourth quadrant. The phase of this frequency response is specified to be zero so that the resulting filter to be designed by the method derived in Section III will also be zero-phase. Figure 2 depicts the ideal frequency response which has a unity magnitude over the passband, a zero magnitude over the stopband (SB), and a linearly varying magnitude over the transition band. The following values are used for the parameters : $a_1 = 0.6\pi$, $b_1 = 0.05\pi$, $a_2 = 0$, $b_2 = 0.5\pi$ and $c = 0.1\pi$. The frequency response is sampled over a grid of $2M_1$ x $2M_2$ points where $M_1 = M_2 = 40$. The support of the impulse response of the filter which has $(2N_1+1)$ x $(2N_2+1)$ points will be assumed to be square , i.e. $N_1 = N_2 = N$. Table 1 summarizes the simulation results for 3 designed filters with N = 14, 22 and 39 . It provides the maximum absolute error in the passband, the maximum absolute error in the stopband, and the sum of the squares of the errors of the frequency response at the same points of the grid used for specifying the ideal frequency response, i.e. the square of the minimum value of criterion E of (23) . The table also lists the CPU time required for computing both the impulse response matrix A and the frequency response samples of the designed filter. Figures 3 and 4 portrays both the perspective and contour plots of the frequency response of the designed filter for the case of N = 22 .

V. CONCLUSION

A new derivation is presented for the least squares solution of the design problem of 2-D FIR filters in the discrete frequency domain for the general case of a noncentro-symmetric frequency response and complex impulse response. The derivation is based on the singular value decomposition of two complex transformation matrices. Interestingly the application of the final result does not require the singular value decomposition of any matrix. Moreover it has been proved that the designed filter will be zero-phase if the ideal filter is so despite being noncentro-symmetric. This surprising result has been attained by allowing the impulse response to be complex.

References

[1] M.O. Ahmad and J.D. Wang, "An analytical least square solution to the design problem of 2-D FIR filters with quadrantally symmetric or antisymmetric frequency response," *IEEE Transactions on Circuits and Systems*, vol. CAS-36, no. 7, pp. 968-979, July 1989.

[2] W.-P. Zhu, M.O. Ahmad and M.N.S. Swamy, "An analytical approach for obtaining a closed-form solution to the least-square design problem of 2-D zero-phase FIR filters," *IEEE Transactions on Circuits and Systems, Part II : Analog and Digital Signal Processing*, vol. 41, no. 12, pp. 796-807, December 1994.

[3] J.Y. Chang and W.H. Wang, "A new 2-D zero phase digital filter design with quadrantally symmetric or antisymmetric properties," Proceedings of the *IEEE International Symposium on Circuits and Systems*, London, England, May 30 - June 2, 1994, vol. 2, pp. 585 - 588.

[4] G.W. Stewart, *Introduction to Matrix Computations*, New York, N.Y. : Academic Press, 1973.

Figure Captions

Fig. 1 : Frequency response specifications.

Fig. 2 : The ideal frequency response .

Fig. 3 : A perspective plot of the actual frequency response for N = 22.

Fig. 4 : A contour plot of the actual frequency response for N = 22.

Table Caption

Table 1 : Simulation Results

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N	Max error in passband	Max error in stopband	$E^2_{\min}$	Computer CPU time in seconds
14	0.0997	0.1155	1.0967	2.86
22	0.0769	0.0694	0.4632	4.89
39	0.0019	0.0049	0.0051	11.8