Fractional Discrete Fourier Transform of Type IV Based on the Eigenanalysis of a Nearly Tridiagonal Matrix

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ABSTRACT

A fully-fledged definition for the fractional discrete Fourier transform of type IV (FDFT-IV) is presented and shown to outperform the simple definition of the FDFT-IV which is proved to be just a linear combination of the signal, its DFT-IV and their flipped versions. This definition heavily depends on the availability of orthonormal eigenvectors of the DFT-IV matrix **G**. An eigenanalysis is performed of a nearly tridiagonal matrix **S** which commutes with matrix **G**. An involutary unitary matrix **P** is defined and used for performing a similarity transformation that reduces **S** to a block diagonal form where the two diagonal blocks are exactly tridiagonal matrices. Moreover the elements of those two diagonal blocks are derived in order to circumvent the need for performing the two matrix multiplications involved in the similarity transformation. Orthonormal even and odd symmetric eigenvectors for **S** are generated – in terms of the eigenvectors of the two diagonal blocks – and proved to always be eigenvectors of **G** irrespective of the multiplicities of the eigenvalues of **S**. The relevance of the method contributed here is manifested in the case of a repeated eigenvalue of **S** with multiplicity 2 where a direct application of a general eigenanalysis procedure in any software package will not produce a pair of even and odd symmetric eigenvectors corresponding to this repeated eigenvalue. *It should be mentioned that the almost tridiagonal matrix* ***S*** *which commutes with the DFT-IV matrix* ***G*** *being dealt with here is distinct from matrix* ***S*** *which commutes with the DFT matrix* ***F*** *dealt with in a previous paper [7].*

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This work was financially supported by the Science and Technology Development Fund (STDF), Egypt under Grant No. 2091 Basic and Applied Research.

Index terms: Discrete Fourier transform of type IV (DFT-IV), fractional discrete Fourier transform of type IV (FDFT-IV), discrete Hartley transform, even and odd symmetric eigenvectors, a nearly tridiagonal matrix.

I. Introduction

Both the discrete Fourier transform (DFT) and discrete Hartley transform (DHT) have been generalized and four types have been defined [1]. Types I and IV are of prime importance since the time and frequency indices enter symmetrically in their definitions. Type I is the regular discrete transform and type IV is the main concern of the present paper. The fractionalization of any discrete transform necessitates a detailed eigenanalysis of the transform matrix. McClellan and Parks [2] followed by Dickinson and Steiglitz [3] were among the first digital signal processing researchers to perform an eigenanalysis of the DFT matrix **F**. Although Santhanam and McClellan [4] were pioneers in presenting a fractional DFT, it was Pei et al. [5] who presented a fully-fledged definition of the fractional transform. Candan, Kutay and Ozaktas [6] carried out an elegant study of the eigenvectors of a nearly tridiagonal matrix **S** which commutes with the DFT matrix **F** arriving at a common set of eigenvectors of both matrices. Their work has been recently put on a more rigorous foundation leading to some explicit expressions by Hanna, Seif and Ahmed [7]. Pei et al. [8] looked at the eigenvectors of **S** as only initial eigenvectors of matrix **F** and arrived at final superior ones in the sense of better approximating the Hermite-Gaussian functions. Other techniques for deriving orthonormal eigenvectors of matrix **F** have also been suggested [9-11].

Turning to the DFT-IV, Tseng [12] obtained eigenvectors of its kernel matrix **G** by finding those of an almost tridiagonal matrix **S** which commutes with matrix **G**. However he did not treat the case of repeated eigenvalues of **S** where the corresponding eigenvectors – as obtained by a general eigenanalysis procedure in any mathematical software package – are not guaranteed to be eigenvectors of **G**. Tseng next used his results in developing a simple definition of the fractional

DFT-IV which can be shown to be just a linear combination of four terms; namely the signal, its DFT-IV and their flipped versions.

The main objective of the present paper is the development of a computationally efficient technique which is guaranteed to generate a common set of orthonormal eigenvectors of both the nearly tridiagonal matrix **S** defined in [12] and the kernel matrix **G** of the DFT-IV even in the degenerate case when matrix **S** has a repeated eigenvalue. More specifically matrix **S** will be reduced to a block diagonal form by means of a similarity transformation defined in terms of an elementary involutary matrix **P**. The two diagonal blocks resulting from applying the similarity transformation will be shown to be unreduced tridiagonal matrices and *explicit expressions will be derived for their elements*. Even and odd symmetric eigenvectors of **S** will be systematically generated and rigorously proved to always be eigenvectors of matrix **G** *irrespective of the multiplicities of the eigenvalues of matrix* **S**. Orthonormal eigenvectors of **G** that approximate the Hermite-Gaussian functions better than those of **S** are generated using some advanced techniques. A fully-fledged definition of the Fractional Discrete Fourier Transform of type IV (FDFT-IV) will be developed and shown to approximate the corresponding continuous fractional transform better than the simple definition.

It should be pointed out that matrix ***S*** *- which commutes with the DFT-IV matrix* ***G*** *- defined in [12] and studied in detail here is quite distinct than matrix* ***S*** *used in [6,7] which commutes with the DFT matrix* ***F***.

In section II some properties of matrix **G** will be surveyed. An elementary unitary matrix **P** will be introduced in section III and employed in defining a similarity transformation for reducing matrix **S** to a block diagonal form in section[2] IV. In section V a fully-fledged version of the fractional DFT-IV will be developed and compared with the simple version. Some simulation results will be presented in section VI.

II. The Kernel Matrix G of the DFT-IV

[2] A part of this work was presented as a conference paper [13].

One starts by the definition of the kernel matrix **G** of the DFT-IV of order N [12]:

$$G_{k,n} = \frac{1}{\sqrt{N}} W_N^{(k-0.5)(n-0.5)} \qquad , k, n = 1, \cdots, N \tag{1}$$

where $W_N = \exp\left(-j\frac{2\pi}{N}\right)$. It follows that matrix $\mathbf{G}$ is unitary and symmetric but *not Hermitian*.

Lemma 1: Matrix $\mathbf{G}$ is *centrosymmetric*, i.e.,

$$\mathbf{J}_N \mathbf{G} \mathbf{J}_N = \mathbf{G} \tag{2}$$

where $\mathbf{J}_N$ is the contra-identity matrix of order N defined by:

$$\mathbf{J}_N = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}. \tag{3}$$

Proof: By defining matrix **T** as $\mathbf{T} = \mathbf{J}_N \mathbf{G} \mathbf{J}_N$ and using (1), one gets:

$$\begin{aligned} T_{k,n} &= G_{N+1-k,N+1-n} \\ &= \frac{1}{\sqrt{N}} W_N^{(N+0.5-k)(N+0.5-n)} \\ &= \frac{1}{\sqrt{N}} W_N^{(k-0.5)(n-0.5)} \\ &= G_{k,n} \end{aligned} \qquad , \quad k, n = 1, \cdots, N. \tag{4}$$

(Q.E.D.)

Expressing $\mathbf{G}$ as:

$$\mathbf{G} = \mathbf{G_r} - j\mathbf{G_i} \tag{5}$$

it follows from the above lemma that both $\mathbf{G_r}$ and $\mathbf{G_i}$ are centrosymmetric.

Definition 1: Vector $\mathbf{u}$ is even symmetric if

$$\mathbf{Ju} = \mathbf{u}. \tag{6}$$

Definition 2: Vector $\mathbf{v}$ is odd symmetric if

$$\mathbf{Jv} = -\mathbf{v}. \tag{7}$$

It should be noticed that if N is odd, then the middle element of $\mathbf{v}$ will vanish.

Lemma 2: If vectors $\mathbf{u}$ and $\mathbf{v}$ are respectively even and odd symmetric and $\mathbf{A}$ is a centrosymmetric matrix, then:

a) $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

b) $\mathbf{Au}$ and $\mathbf{Av}$ are respectively even and odd symmetric.

Proof:

a) Equations (6), (7) and (3) imply that[3]: $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^H \mathbf{v} = \left(\mathbf{u}^H \mathbf{J}^H\right)\left(-\mathbf{Jv}\right) = -\mathbf{u}^H \mathbf{J}^2 \mathbf{v} = -\langle \mathbf{u}, \mathbf{v} \rangle$ where the fact that $\mathbf{J}^2 = \mathbf{I}$ has been used. Hence $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

b) The centrosymmetric condition (2) and Definition 1 above imply that: $\mathbf{J}(\mathbf{Au}) = (\mathbf{JAJ})(\mathbf{Ju}) = \mathbf{Au}$. Therefore $\mathbf{Au}$ is even symmetric. By the same token, one gets $\mathbf{J}(\mathbf{Av}) = -\mathbf{Av}$ implying that $\mathbf{Av}$ is odd symmetric.

(Q.E.D.)

Tseng [12] showed that:

$$\mathbf{G}^2 = -\mathbf{J}_N \tag{8}$$

and that the 4 distinct eigenvalues of **G** are:

$$\lambda_k = (-j)^{(k-1)} \quad , k = 1, \cdots, 4 . \tag{9}$$

Although the DFT-IV matrix **G** *and the DFT matrix* **F** *have the same set of 4 distinct eigenvalues, they have completely different eigenvectors. The eigenvectors of* **G** *are either even or odd symmetric [12] and those of* **F** *are either circularly even or odd [2]. Consequently the results reported here are distinct than those reported in [6,7].*

III. An Involutary Unitary Transformation Matrix P

One starts by defining an elementary square Matrix $\mathbf{P}$ as:

[3] The superscripts T, *, H respectively denote the transpose, the complex conjugate and the Hermitian transpose (i.e. the complex conjugate transpose).

$$\mathbf{P} = \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{I}_{\nu-1} & & \mathbf{J}_{\nu-1} \\ & \sqrt{2} & \\ \mathbf{J}_{\nu-1} & & -\mathbf{I}_{\nu-1} \end{bmatrix} \quad \text{for odd N} \tag{10-a}$$

and

$$\mathbf{P} = \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{I}_{\nu} & \mathbf{J}_{\nu} \\ \mathbf{J}_{\nu} & -\mathbf{I}_{\nu} \end{bmatrix} \quad \text{for even N} \tag{10-b}$$

where the integer ν is given by[4]:

$$\nu = \lceil 0.5N \rceil. \tag{11}$$

It follows that the first ν columns of **P** are even symmetric and the last $(N-\nu)$ columns are odd symmetric. Being real, symmetric and unitary, it follows that matrix **P** satisfies:

$$\mathbf{P} = \mathbf{P}^{T} = \mathbf{P}^{H} = \mathbf{P}^{-1}. \tag{12}$$

Consequently $\mathbf{P}$ is an involutary matrix because $\mathbf{P}^{2} = \mathbf{I}$. *A direct inspection reveals that matrix **P** defined above is distinctly different than its counterpart used in [6,7].*

In order to prepare for finding the effect of premultiplying a vector by $\mathbf{P}$, an arbitrary vector $\mathbf{x}$ will be expressed in partitioned form as:

$$\mathbf{x} = \begin{bmatrix} \mathbf{x_a} \\ x_{\nu} \\ \mathbf{x_b} \end{bmatrix} \quad \text{for odd N} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} \mathbf{x_a} \\ \mathbf{x_b} \end{bmatrix} \quad \text{for even N} \tag{13}$$

where the subvectors $\mathbf{x_a}$ and $\mathbf{x_b}$ are of dimension $(N-\nu)$. By defining vector $\mathbf{y}$ as:

$$\mathbf{y} = \mathbf{Px} \tag{14}$$

equations (10) and (13) result in:

$$\mathbf{y} = \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{x_a} + \mathbf{J}_{\nu-1}\mathbf{x_b} \\ \sqrt{\mathbf{2}}x_{\nu} \\ \mathbf{J}_{\nu-1}\mathbf{x_a} - \mathbf{x_b} \end{bmatrix} \quad \text{for odd N} \quad \text{and} \quad \mathbf{y} = \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{x_a} + \mathbf{J}_{\nu}\mathbf{x_b} \\ \mathbf{J}_{\nu}\mathbf{x_a} - \mathbf{x_b} \end{bmatrix} \quad \text{for even N} \tag{15}$$

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[4] The symbol $\lceil b \rceil$ denotes the smallest integer larger than or equal to b.

Lemma 3: Let vectors $\mathbf{x}$ and $\mathbf{y}$ be related by (14).

a) If $\mathbf{x}$ is even symmetric, then:

$$\mathbf{y} = \frac{1}{\sqrt{2}}\begin{bmatrix} 2\mathbf{x_a} \\ \sqrt{2}x_\nu \\ \mathbf{0} \end{bmatrix} \quad \text{for odd N} \quad \text{and} \quad \mathbf{y} = \frac{1}{\sqrt{2}}\begin{bmatrix} 2\mathbf{x_a} \\ \mathbf{0} \end{bmatrix} \quad \text{for even N}. \tag{16}$$

b) If $\mathbf{x}$ is odd symmetric, then:

$$\mathbf{y} = \frac{1}{\sqrt{2}}\begin{bmatrix} \mathbf{0} \\ 0 \\ -2\mathbf{x_b} \end{bmatrix} \quad \text{for odd N} \quad \text{and} \quad \mathbf{y} = \frac{1}{\sqrt{2}}\begin{bmatrix} \mathbf{0} \\ -2\mathbf{x_b} \end{bmatrix} \quad \text{for even N}. \tag{17}$$

c) If $\mathbf{x_b} = \mathbf{0}$, then $\mathbf{y}$ is even symmetric.

d) If $\mathbf{x_a} = \mathbf{0}$ and $x_\nu = 0$, then $\mathbf{y}$ is odd symmetric.

Proof:

a) Here $\mathbf{x_b} = \mathbf{Jx_a}$ and consequently $\mathbf{x_a} + \mathbf{Jx_b} = 2\mathbf{x_a}$. Equation (16) follows immediately.

b) Here $\mathbf{x_b} = -\mathbf{Jx_a}$ and only for odd N: $x_\nu = 0$. Consequently $\mathbf{x_a} + \mathbf{Jx_b} = \mathbf{0}$, $\mathbf{Jx_a} - \mathbf{x_b} = -2\mathbf{x_b}$ and (17) follows immediately.

c) and d) The proof follows from the definition of even and odd symmetric vectors.

(Q.E.D.)

Parts (a) and (b) of the above lemma imply that if $\mathbf{x}$ is even (odd) symmetric, then almost the second (first) half of $\mathbf{y}$ vanishes. Parts (c) and (d) are respectively the converse of parts (a) and (b).

IV. Tridiagonalization of a Nearly Tridiagonal Matrix S

A nearly tridiagonal matrix $\mathbf{S}$ which commutes with the DFT-IV matrix $\mathbf{G}$ will be reduced to an exactly tridiagonal matrix by a similarity transformation defined in terms of matrix $\mathbf{P}$ of last section. *The elements of the two diagonal blocks resulting from the similarity transformation will be derived*. Orthonormal even and odd symmetric eigenvectors of matrix $\mathbf{S}$ will be generated and proved to be eigenvectors of any centrosymmetric matrix which commutes with matrix $\mathbf{S}$. This holds *irrespective of the multiplicities of the eigenvalues of* $\mathbf{S}$.

Matrix $\mathbf{S}$ is defined as [12]:

$$\mathbf{S} = \begin{bmatrix} s_1 & 1 & & & & & -1 \\ 1 & s_2 & 1 & & & & \\ & 1 & \ddots & \ddots & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & 1 & \\ & & & & 1 & s_{N-1} & 1 \\ -1 & & & & & 1 & s_N \end{bmatrix} \tag{18}$$

where

$$s_n = 2\cos\left((2n-1)\frac{\pi}{N} \right) \qquad , n = 1, \cdots, N \, . \tag{19}$$

It follows that:

$$s_{N+1-n} = s_n \qquad , n = 1, \cdots, N \tag{20}$$

and consequently vector $\mathbf{s}$ defined by:

$$\mathbf{s} = \begin{bmatrix} s_1 & \cdots & s_N \end{bmatrix}^T \tag{21}$$

is even symmetric. *A direct inspection reveals that matrix* $\mathbf{S}$ *defined above in accordance to [12] is distinctly different than that defined in [6] (Eq (16) in [6]) and used in [7] (Eq (55) in [7]) in two aspects. First the diagonal elements are different. Second the two elements in the upper right and lower left corners are -1 here and +1 in [6, 7]. It follows that the two matrices have different sets of eigenvectors and the contribution of the present paper is distinct than that of [7].*

Let the columns of matrix $\mathbf{S}$ be denoted by $\mathbf{c_n}$, i.e.,

$$\begin{aligned} \mathbf{S} &= \begin{bmatrix} \mathbf{c_1} & \cdots & \mathbf{c_N} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{c_1} & \mathbf{c_2} & \mathbf{c_3} & \cdots & \mathbf{J}_N\mathbf{c_3} & \mathbf{J}_N\mathbf{c_2} & \mathbf{J}_N\mathbf{c_1} \end{bmatrix} \end{aligned} \tag{22}$$

where the second form is obtained by (18), (20) and (3). Consequently matrix $\mathbf{S}$ can be expressed in partitioned form as:

$$\mathbf{S} = \begin{cases} [\mathbf{C} \quad c_\nu \quad \mathbf{J}_N \mathbf{C} \mathbf{J}_{\nu-1}] & \text{for odd N} \\ [\mathbf{C} \quad \mathbf{J}_N \mathbf{C} \mathbf{J}_\nu] & \text{for even N} \end{cases} \tag{23}$$

where matrix $\mathbf{C}$ is defined by:

$$\mathbf{C} = \begin{cases} [\mathbf{c_1} \quad \mathbf{c_2} \quad \cdots \quad \mathbf{c}_{\nu-1}] & \text{for odd N} \\ [\mathbf{c_1} \quad \mathbf{c_2} \quad \cdots \quad \mathbf{c}_\nu] & \text{for even N} \end{cases} \tag{24}$$

Lemma 4: Let $\mathbf{S}$ be any matrix[5] of the form of (18) whose diagonal elements form an even symmetric vector and let $\mathbf{P}$ be the transformation matrix defined by (10), then

$$\mathbf{PSP}^{-1} = \begin{bmatrix} \mathbf{EV} & 0 \\ 0 & \mathbf{OD} \end{bmatrix} \tag{25}$$

where $\mathbf{EV}$ and $\mathbf{OD}$ are square matrices of order ν and $(N-\nu)$ respectively.

Proof:

Case a: N is odd

Using (12), (23) and (10), one gets:

$$\mathbf{SP^{-1}} = \mathbf{SP} = \frac{1}{\sqrt{2}}\left[(\mathbf{I}_N + \mathbf{J}_N)\mathbf{C} \quad \sqrt{2}\mathbf{c_v} \quad (\mathbf{I}_N - \mathbf{J}_N)\mathbf{C}\mathbf{J}_{\nu-1}\right]. \tag{26}$$

Premultiplying both sides of the above equation by the contra-identity matrix $\mathbf{J}$ and utilizing the fact that $\mathbf{J^2} = \mathbf{I}$ and the column $\mathbf{c_v}$ is even symmetric, one obtains:

$$\mathbf{J}_N\mathbf{SP^{-1}} = \frac{1}{\sqrt{2}}\left[(\mathbf{I}_N + \mathbf{J}_N)\mathbf{C} \quad \sqrt{2}\mathbf{c_v} \quad -(\mathbf{I}_N - \mathbf{J}_N)\mathbf{C}\mathbf{J}_{\nu-1}\right]. \tag{27}$$

By comparing (26) and (27) and calling to remembrance (6) and (7), one concludes that the first ν columns of $\mathbf{SP^{-1}}$ are even symmetric and the last $(N-\nu)$ columns are odd symmetric.

Case b: N is even

Using (12), (23) and (10), one gets:

[5] This lemma does not require that the diagonal elements of $\mathbf{S}$ be given by (19).

$$\mathbf{SP}^{-1} = \frac{1}{\sqrt{2}}\left[(\mathbf{I}_N + \mathbf{J}_N)\mathbf{C} \quad (\mathbf{I}_N - \mathbf{J}_N)\mathbf{C}\mathbf{J}_\nu\right]. \tag{28}$$

Premultiplying both sides by $\mathbf{J}_N$, one gets:

$$\mathbf{J}_N\mathbf{SP}^{-1} = \frac{1}{\sqrt{2}}\left[(\mathbf{I}_N + \mathbf{J}_N)\mathbf{C} \quad -(\mathbf{I}_N - \mathbf{J}_N)\mathbf{C}\mathbf{J}_\nu\right]. \tag{29}$$

By comparing (28) and (29), one concludes that the first ν columns of $\mathbf{SP}^{-1}$ are even symmetric and the last $(N-\nu)$ columns are odd symmetric.

The same conclusion has been reached whether N is odd or even. Therefore by virtue of parts (a) and (b) of Lemma 3, premultiplying $\mathbf{SP}^{-1}$ by $\mathbf{P}$ will zero the last $(N-\nu)$ elements of the first ν columns and the first ν elements of the last $(N-\nu)$ columns. Hence the validity of (25) is established. (Q.E.D.)

Theorem 1:

Let $\mathbf{S}$ be any matrix of the form of (18) whose diagonal elements form an even symmetric vector[6] and let $\mathbf{P}$ be the transformation matrix defined by (10), then the two diagonal blocks $\mathbf{EV}$ and $\mathbf{OD}$ appearing in (25) are symmetric tridiagonal matrices that are given explicitly by:

$$\mathbf{EV} = \begin{bmatrix} s_1 - 1 & 1 & & & & & & \\ 1 & s_2 & 1 & & & & & \\ & 1 & s_3 & 1 & & & & \\ & & 1 & \ddots & \ddots & & & \\ & & & \ddots & \ddots & 1 & & \\ & & & & 1 & s_{\nu-2} & 1 & \\ & & & & & 1 & s_{\nu-1} & \gamma \\ & & & & & & \gamma & s_\nu + \delta \end{bmatrix}, \tag{30}$$

[6] This Theorem does not require that the diagonal elements of $\mathbf{S}$ be given by (19).

$$\mathbf{OD}=\begin{bmatrix} s_{\varepsilon}-\delta & 1 & & & & \\ 1 & s_{\varepsilon-1} & 1 & & & \\ & 1 & \ddots & \ddots & & \\ & & \ddots & \ddots & 1 & \\ & & & 1 & s_2 & 1 \\ & & & & 1 & s_1+1 \end{bmatrix} \tag{31}$$

where

$$\gamma=\begin{cases}\sqrt{2} & \text{for odd N} \\ 1 & \text{for even N}\end{cases}, \quad \delta=\begin{cases}0 & \text{for odd N} \\ 1 & \text{for even N}\end{cases} \quad \text{and} \quad \varepsilon=\begin{cases}\nu-1 & \text{for odd N} \\ \nu & \text{for even N}\end{cases}. \tag{32}$$

The proof is given in Appendix A.

Generation of Orthonormal Eigenvectors for Matrix $\mathbf{S}$:

1. The real symmetric tridiagonal matrices $\mathbf{EV}$ and $\mathbf{OD}$ given by (30) and (31) are unreduced in the sense that all elements lying on the first upper (or lower) diagonal are nonzero. Consequently the eigenvalues of each matrix are distinct [14]. Therefore each matrix will have a complete set of orthonormal eigenvectors. More specifically

$$\mathbf{EV}=\mathbf{M_1\Lambda_1M_1^{-1}} \quad \text{and} \quad \mathbf{OD}=\mathbf{M_2\Lambda_2M_2^{-1}} \tag{33}$$

   where $\mathbf{\Lambda_1}$ and $\mathbf{\Lambda_2}$ are real diagonal matrices and $\mathbf{M_1}$ and $\mathbf{M_2}$ are real unitary modal matrices.

2. The eigendecomposition of the block diagonal matrix $\mathbf{PSP}^{-1}$ of (25) is given by:

$$\mathbf{PSP}^{-1}=\mathbf{M\Lambda M}^{-1} \tag{34}$$

   where

$$\mathbf{\Lambda}=\begin{pmatrix}\mathbf{\Lambda_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Lambda_2}\end{pmatrix} \quad \text{and} \quad \mathbf{M}=\begin{pmatrix}\mathbf{M_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{M_2}\end{pmatrix} \tag{35}$$

   and the modal matrix $\mathbf{M}$ is unitary.

3. The eigendecomposition of matrix $\mathbf{S}$ is:

$$\mathbf{S} = \mathbf{Q}\Lambda\mathbf{Q}^{-1} \qquad \text{where} \qquad \mathbf{Q} = \mathbf{P}^{-1}\mathbf{M} = \mathbf{PM}\,. \tag{36}$$

The modal matrix $\mathbf{Q}$ of $\mathbf{S}$ is unitary since (36) and (12) imply that $\mathbf{Q}^H\mathbf{Q} = \mathbf{M}^H\mathbf{P}^H\mathbf{PM} = \mathbf{I}$. Based on the block diagonal form of matrix $\mathbf{M}$ given by (35) and the relation $\mathbf{Q} = \mathbf{PM}$, one concludes – by virtue of parts (c) and (d) of Lemma 3 - that the first ν eigenvectors of $\mathbf{S}$ are even symmetric and the last $(N-\nu)$ eigenvectors are odd symmetric.

4. The eigenvalues of $\mathbf{S}$ are those of the matrices $\mathbf{EV}$ and $\mathbf{OD}$. Since each of the latter two matrices has distinct eigenvalues, this proves that the maximum algebraic multiplicity of any eigenvalue of $\mathbf{S}$ is 2; *a fact that was mentioned without a proof in [12]*. Since the multiplicity will be 2 only when a diagonal element of $\mathbf{\Lambda_1}$ happens to equal a diagonal element of $\mathbf{\Lambda_2}$, one concludes that one of the eigenvectors of $\mathbf{S}$ corresponding to this repeated eigenvalue will be even symmetric and the other one will be odd symmetric. It should be emphasized that the orthonormality of the eigenvectors of $\mathbf{S}$ given by the columns of the unitary modal matrix $\mathbf{Q}$ holds *irrespective of the multiplicities of the eigenvalues*. The procedure presented here for generating $\mathbf{Q}$ is *computationally efficient* since matrix $\mathbf{M}$ is generated by computing the modal matrices $\mathbf{M_1}$ and $\mathbf{M_2}$ of the two tridiagonal matrices $\mathbf{EV}$ and $\mathbf{OD}$ having almost half the size of matrix $\mathbf{S}$.

Orthonormal Eigenvectors for Matrix $\mathbf{G}$:

Lemma 5: The modal matrix $\mathbf{Q}$ given by (36) – of any matrix $\mathbf{S}$ having the form of (18) and whose diagonal elements form an even symmetric vector[7] – is always a modal matrix of any centrosymmetric matrix $\mathbf{A}$ that commutes with $\mathbf{S}$ *irrespective of the multiplicities of the eigenvalues of* $\mathbf{S}$.

Proof:

If λ is a simple eigenvalue of $\mathbf{S}$, then the corresponding eigenvector will also be an eigenvector of $\mathbf{A}$ by the commutativity of $\mathbf{S}$ and $\mathbf{A}$ [15]. It remains to consider the case when λ is an eigenvalue of $\mathbf{S}$ with multiplicity 2. Let $\mathbf{u}$ and $\mathbf{v}$ be respectively the corresponding even and odd symmetric eigenvectors, i.e., $\mathbf{Su} = \lambda\,\mathbf{u}$ and $\mathbf{Sv} = \lambda\,\mathbf{v}$. Exploiting the commutativity of $\mathbf{S}$ and $\mathbf{A}$, one gets $\mathbf{S}(\mathbf{Au}) = \lambda\,(\mathbf{Au})$ and $\mathbf{S}(\mathbf{Av}) = \lambda\,(\mathbf{Av})$. This implies that $\mathbf{Au}$ and $\mathbf{Av}$ are eigenvectors of $\mathbf{S}$

[7] This lemma does not require that the diagonal elements of $\mathbf{S}$ be given by (19).

corresponding to the same λ. Consequently they can be expressed as linear combinations of $\mathbf{u}$ and $\mathbf{v}$ as follows:

$$\mathbf{Au} = \alpha_1 \mathbf{u} + \alpha_2 \mathbf{v}, \tag{37}$$

$$\mathbf{Av} = \beta_1 \mathbf{u} + \beta_2 \mathbf{v}. \tag{38}$$

Premultiplying the above two equations by $\mathbf{v}^{\mathbf{H}}$ and $\mathbf{u}^{\mathbf{H}}$ respectively and applying Lemma 2, one gets $\alpha_2 = 0$ and $\beta_1 = 0$. Consequently $\mathbf{Au} = \alpha_1 \mathbf{u}$ and $\mathbf{Av} = \beta_2 \mathbf{v}$. Therefore $\mathbf{u}$ and $\mathbf{v}$ are also eigenvectors of $\mathbf{A}$. (Q.E.D.)

Corollary: The modal matrix $\mathbf{Q}$ given by (36) of matrix $\mathbf{S}$ defined by (18) and (19) is always a modal matrix of the kernel matrices **G** and **H** of the discrete Fourier transform of type IV and the discrete Hartley transform of type IV respectively.

Proof:

The kernel matrix $\mathbf{H}$ of the DHT-IV is defined by:

$$\mathbf{H} = \mathbf{G_r} + \mathbf{G_i} \tag{39}$$

where the matrices $\mathbf{G_r}$ and $\mathbf{G_i}$ are defined by (5). According to Lemma 1 matrix $\mathbf{G}$ is centrosymmetric and the matrices $\mathbf{G_r}$ and $\mathbf{G_i}$ have the same property by virtue of (5). It follows from (39) that matrix $\mathbf{H}$ will also be centrosymmetric. Since each of the centrosymmetric matrices $\mathbf{G}$ and $\mathbf{H}$ commutes with $\mathbf{S}$ [12], the proof is immediately established[8].

V. Fractional Transforms

By shifting the Hermite-Gaussian functions by half a sample, sampling them and rearranging the samples after reversing the signs of those lying on the negative time axis, Tseng [12] showed that the resulting samples form vectors that are *approximate* eigenvectors of the DFT-IV matrix $\mathbf{G}$. On the other hand, starting by the second order differential equation satisfied by the Hermite-Gaussian functions and discretizing it after shifting the independent variable by half a sample, Tseng proved that the solution of the resulting second order difference equation is given by the eigenvectors of matrix $\mathbf{S}$ - defined by (18) and (19) – that have been proved to also be eigenvectors of matrix $\mathbf{G}$.

[8] Although Lemma 5 does not require that the diagonal elements of matrix $\mathbf{S}$ be given by (19), the corollary does require that because the commutativity of **G** and **S** and of **H** and **S** was proved in [12] under the assumption that the diagonal elements of $\mathbf{S}$ are given by (19).

Consequently the *exact* orthonormal eigenvectors obtained in the last section approximate the Hermite-Gaussian functions to some extent. One objective of the present section is to get *final exact* orthonormal eigenvectors of $\mathbf{G}$ that better approximate the Hermite-Gaussian functions than the eigenvectors of matrix $\mathbf{S}$ to be referred to as the *initial exact* eigenvectors. The superior *final* eigenvectors will be taken as a basis for developing a fully-fledged definition of the fractional DFT-IV (FDFT-IV) that will be next compared with the simple definition given in [12]. The results will be extended to define a fully-fledged Fractional DHT-IV (FDHT-IV).

(A) A Fully-Fledged Fractional DFT-IV

Tseng derived the following approximate key formula [12]:

$$\mathbf{G}\mathbf{u}_n \approx (-j)^n \mathbf{u}_n \tag{40}$$

where vector $\mathbf{u}_n$ is obtained by shifting the Hermite-Gaussian function of order n by half a sample, sampling it and rearranging the samples after reversing the signs of those lying on the negative time axis. Formula (40) indicates that $\mathbf{u}_n$ is an *approximate* eigenvector of matrix $\mathbf{G}$ corresponding to the *exact* eigenvalue $(-j)^n$ recalling that $\mathbf{G}$ has only the 4 distinct eigenvalues given by (9). Since the unitary matrix $\mathbf{G}$ of order N has only N linearly independent eigenvectors, one should select a set of indices $\Psi = \{n_1, n_2, \cdots, n_N\}$ such that the corresponding vectors $\mathbf{u}_{\mathbf{n_k}}$, $k = 1, \cdots, N$ will be adopted as the *approximate* Hermite-Gaussian-like eigenvectors of $\mathbf{G}$. The selection is guided by the following two factors:

- The eigenvalues $(-j)^{n_k}$, $k = 1, \cdots, N$ should satisfy the multiplicities requirement (given by Eq. (10) in [12]).
- The indices should generally be small in order to reduce the approximation error.

Therefore the set Ψ should be as given in Table 1 which upon examination reduces to the concise format of Table 2.

The Hermite-Gaussian-like *approximate* eigenvectors of $\mathbf{G}$ will be grouped as the columns of 4 matrices $\mathbf{U}_k$ corresponding to the 4 distinct eigenvalues λ_k of $\mathbf{G}$ given by (9). The procedure for obtaining orthonormal Hermite-Gaussian-like *exact* eigenvectors of $\mathbf{G}$ that approximate the Hermite-Gaussian functions better than the eigenvectors of matrix $\mathbf{S}$ is as follows:

a) Generate the real modal matrix Q of matrix $\mathbf{S}$ (defined by (18) and (19)) according to (36) and group its columns into 4 matrices $\mathbf{V}_k$, $k = 1, \cdots, 4$ according to the corresponding eigenvalues λ_k of matrix $\mathbf{G}$ that commutes with $\mathbf{S}$. These *exact* eigenvectors will be taken as only *initial* ones.

b) Generate real *approximate* Hermite-Gaussian-like eigenvectors of $\mathbf{G}$ and group them into 4 matrices $\mathbf{U}_k$.

c) For each eigenspace compute *final* real *exact* eigenvectors to form the columns of 4 matrices $\hat{\mathbf{U}}_k$ by applying either the Gram-Schmidt Algorithm (GSA) in the manner explained in [8] or the Orthogonal Procrustes Algorithm (OPA) delineated in [16] and applied in [8].

One should mention that the eigenspaces E_k, $k = 1, \cdots, 4$ corresponding to the 4 distinct eigenvalues of $\mathbf{G}$ have been dealt with separately since eigenspaces corresponding to distinct eigenvalues of a unitary matrix are orthogonal [17].

The modal decomposition of matrix $\mathbf{G}$ is:

$$\mathbf{G} = \hat{\mathbf{U}}\mathbf{D}\hat{\mathbf{U}}^H \tag{41}$$

where $\hat{\mathbf{U}}$ is the unitary modal matrix:

$$\hat{\mathbf{U}} = \begin{bmatrix} \hat{\mathbf{u}}_{\mathbf{n}_1} & \hat{\mathbf{u}}_{\mathbf{n}_2} & \cdots & \hat{\mathbf{u}}_{\mathbf{n}_N} \end{bmatrix} \tag{42}$$

and

$$\mathbf{D} = diag\left\{(-j)^{n_1}, \cdots, (-j)^{n_N}\right\}. \tag{43}$$

The kernel of a fully-fledged definition of the fractional discrete Fourier transform of type IV (FDFT-IV) of order a (corresponding to an angle of rotation α) is defined by:

$$\mathbf{G}^a = \mathbf{G}^{\frac{2}{\pi}\alpha} \equiv \hat{\mathbf{U}}\mathbf{D}^{\frac{2}{\pi}\alpha}\hat{\mathbf{U}}^H \tag{44}$$

where

$$\mathbf{D}^{\mathbf{a}} = \mathbf{D}^{\frac{2}{\pi}\boldsymbol{\alpha}} \equiv diag\left\{\exp(-j\alpha\, n_1), \cdots, \exp(-j\alpha\, n_N)\right\}. \tag{45}$$

The order of the transform a and the angle of rotation α are related by $a = \dfrac{2}{\pi}\alpha$. It should be mentioned that although matrix $\mathbf{D}$ of (43) has only 4 distinct diagonal elements given by (9), matrix $\mathbf{D}^a$ as defined by (45) can have up to N distinct diagonal elements. Consequently the definition of the kernel of the FDFT-IV given by (44) and (45) is a fully-fledged one.

In general the FDFT-IV of a time-domain signal represented by vector $\mathbf{x}$ is given by:

$$\mathbf{X}_\alpha = \mathbf{G}^{\frac{2}{\pi}\alpha}\mathbf{x}. \tag{46}$$

(B) A Simple Fractional DFT-IV

Instead of exploiting the possible variability of the diagonal elements of $\mathbf{D}^a$**,** Tseng [12] proposed a simple definition for the fractional DFT-IV based philosophically on confining those diagonal elements to the following 4 specifically selected values corresponding to the 4 distinct diagonal elements of $\mathbf{D}$:

$$\lambda_1^a = (1)^a = 1, \tag{47-a}$$

$$\lambda_2^a = (-j)^a = (\exp(-j0.5\pi))^{\frac{2}{\pi}\alpha} = \exp(-j\alpha), \tag{47-b}$$

$$\lambda_3^a = (-1)^a = (\exp(j\pi))^{\frac{2}{\pi}\alpha} = \exp(j2\alpha), \tag{47-c}$$

$$\lambda_4^a = (j)^a = (\exp(j0.5\pi))^{\frac{2}{\pi}\alpha} = \exp(j\alpha). \tag{47-d}$$

This restricted simple choice of the values of the diagonal elements of $\mathbf{D}^a$ allows $\mathbf{G}^a$ to be expressed as a power series in **G** given by:

$$\mathbf{G}^{\frac{2}{\pi}\alpha} = \sum_{k=0}^{\infty} c_k(\alpha)\mathbf{G}^k = \sum_{k=0}^{N-1} b_k(\alpha)\mathbf{G}^k = \sum_{k=0}^{3} a_k(\alpha)\mathbf{G}^k \ . \tag{48}$$

The second equality in the above formula is a consequence of Cayley-Hamilton theorem [17] and the third equality is due to the fact that $\mathbf{G}^4 = \mathbf{I}$ which follows from (8). Substituting (44) and (41) respectively in the left and right hand sides of (48) and exploiting the unitarity of the modal matrix $\hat{\mathbf{U}}$, one gets:

$$\mathbf{D}^{\frac{2}{\pi}\alpha} = \sum_{k=0}^{3} a_k(\alpha)\mathbf{D}^k \ . \tag{49}$$

In Appendix B it will be shown that the 4 coefficients in the above formula are given by:

$$a_0(\alpha) = 0.5(1+\exp(j\alpha))\cos\alpha\,, \tag{50-a}$$

$$a_1(\alpha) = 0.5(1-j\exp(j\alpha))\sin\alpha\,, \tag{50-b}$$

$$a_2(\alpha) = 0.5(\exp(j\alpha)-1)\cos\alpha\,, \tag{50-c}$$

$$a_3(\alpha) = -0.5(1+j\exp(j\alpha))\sin\alpha\,. \tag{50-d}$$

Substituting (48) in (46) and utilizing (8), one gets:

$$\mathbf{X}_\alpha = a_0(\alpha)\mathbf{x} + a_1(\alpha)\mathbf{Gx} - a_2(\alpha)\mathbf{Jx} - a_3(\alpha)\mathbf{JGx}\,. \tag{51}$$

The above formula implies that the simple definition of the FDFT-IV reduces to just a linear combination of the signal, its DFT-IV and their flipped versions.

It should be emphasized that the kernel matrix $\mathbf{G}^{\frac{2}{\pi}\alpha}$ of the fully-fledged definition of the FDFT-IV given by both (44) and (45) cannot be expressed as a power series in **G**. The reason is that such an expression would lead to the second and third equalities in (48) and consequently to (49). Since the diagonal matrix on the R.H.S. of (49) has only 4 distinct diagonal values while that appearing on the L.H.S. can have up to N distinct diagonal values in the case of the fully-fledged definition, one faces a contradiction. In fact the fully-fledged definition of the FDFT-IV is by far richer than the simple one. In the simulation section, it will be shown that the former is able to approximate the corresponding fractional continuous transform better than the latter.

(C) A Fully-Fledged Fractional DHT-IV

The kernel matrix **H** of the DHT-IV defined by (39) was shown to have only the two distinct eigenvalues ± 1 whose multiplicities are given by Fact 11 in [12].

In order to find out how vectors $\mathbf{u_n}$ (appearing in (40)) can serve as approximate eigenvectors of matrix **H**, one starts by substituting (5) in (40) to get:

$$\left(\mathbf{G_r} - j\mathbf{G_i}\right)\mathbf{u_n} \approx (-j)^n\,\mathbf{u_n}\,. \tag{52}$$

In order to equate the real and imaginary parts, one should consider the cases of even and odd n separately.

For even n, (52) results in:

$$\mathbf{G_r}\mathbf{u_n} \approx (-j)^n\,\mathbf{u_n}\,, \tag{53-a}$$

$$\mathbf{G_i}\mathbf{u_n} \approx 0\,. \tag{53-b}$$

Adding the above two equations and using definition (39) for matrix **H**, one gets:

$$\mathbf{H}\mathbf{u_n} \approx (-j)^n\,\mathbf{u_n}\,. \tag{54}$$

For odd n, (52) leads to:

$$\mathbf{G_r}\mathbf{u_n} \approx 0\,, \tag{55-a}$$

$$\mathbf{G_i}\mathbf{u_n} \approx (-j)^{n-1}\,\mathbf{u_n}\,. \tag{55-b}$$

By addition, one gets:

$$\mathbf{Hu_n} \approx (-j)^{n-1}\mathbf{u_n}\,. \tag{56}$$

Equations (54) and (56) for even and odd n respectively can be compactly expressed as:

$$\left.\begin{aligned} \mathbf{Hu_n} &\approx (-j)^n \mathbf{u_n} \\ \mathbf{Hu_{n+1}} &\approx (-j)^n \mathbf{u_{n+1}} \end{aligned}\right\} \qquad , n = 0, 2, 4, \cdots . \tag{57}$$

The above formula implies that for even n, the vectors $\mathbf{u_n}$ and $\mathbf{u_{n+1}}$ are *approximate* eigenvectors of matrix $\mathbf{H}$ of order N corresponding to the same *exact* eigenvalue $(-j)^n$ which has only two possible values ± 1. The selection of the set of indices $\Psi = \{n_1, \cdots, n_N\}$ of the approximate eigenvectors $\mathbf{u}_{\mathbf{n}_k}$ should satisfy the multiplicity requirement of the eigenvalues (as given by Fact 11 in [12]). This leads to Table 3 for the values of the index n to be used in (57) and the set Ψ of indices. A careful examination of Table 3 shows that:

$$\Psi = \{0, 1, 2, \cdots, N-1\}. \tag{58}$$

Consequently the corresponding diagonal matrix of exact eigenvalues is given by:

$$\mathbf{D} = Diag\left\{ \exp\left(-j\frac{\pi}{2}(0)\right), \exp\left(-j\frac{\pi}{2}(0)\right), \exp\left(-j\frac{\pi}{2}(2)\right), \exp\left(-j\frac{\pi}{2}(2)\right), \exp\left(-j\frac{\pi}{2}(4)\right), \exp\left(-j\frac{\pi}{2}(4)\right), \cdots \right\}. \tag{59}$$

The Hermite-Gaussian-like *approximate* eigenvectors of **H** will be grouped as the columns of 2 matrices $\mathbf{U_k}$ corresponding to the 2 distinct exact eigenvalues of **H**. The procedure for generating orthonormal Hermite-Gaussian-like *exact* eigenvectors of $\mathbf{H}$ that serve as a good approximation to the Hermite-Gaussian functions is given below:

a) Generate the real modal matrix of matrix $\mathbf{S}$ (defined by (18) and (19)) according to (36) and group its columns into 2 matrices $\mathbf{V}_k$, $k = 1, 2$ according to the two distinct eigenvalues ± 1 of matrix $\mathbf{H}$ that commutes with $\mathbf{S}$. These *exact* eigenvectors will be taken as only *initial* ones.

b) Generate *approximate* Hermite-Gaussian-like eigenvectors of $\mathbf{H}$ and group them into 2 matrices $\mathbf{U}_k$, $k = 1, 2$ according to the corresponding exact eigenvalues ± 1.

c) For each eigenspace compute *final exact* eigenvectors to form the columns of 2 matrices $\hat{\mathbf{U}}_k$, $k = 1, 2$ by applying either the Gram-Schmidt Algorithm or the Orthogonal Procrustes Algorithm.

The modal decomposition of $\mathbf{H}$ is:

$$\mathbf{H} = \hat{\mathbf{U}}\mathbf{D}\hat{\mathbf{U}}^H \tag{60}$$

where

$$\hat{\mathbf{U}} = \begin{bmatrix}\hat{\mathbf{u}}_{\mathbf{0}} & \hat{\mathbf{u}}_{\mathbf{1}} & \cdots & \hat{\mathbf{u}}_{\mathbf{N-1}}\end{bmatrix} \tag{61}$$

and $\mathbf{D}$ is given by (59).

The kernel of a fully-fledged definition of the fractional discrete Hartley transform of type IV (FDHT-IV) of order a is:

$$\mathbf{H}^a = \mathbf{H}^{\frac{2}{\pi}\alpha} \equiv \hat{\mathbf{U}}\mathbf{D}^{\frac{2}{\pi}\alpha}\hat{\mathbf{U}}^H \tag{62}$$

where

$$\mathbf{D}^{\frac{2}{\pi}\alpha} \equiv Diag\{\exp(-j0\alpha), \exp(-j0\alpha), \exp(-j2\alpha), \exp(-j2\alpha), \exp(-j4\alpha), \exp(-j4\alpha), \cdots\}. \tag{63}$$

VI. Simulation Results

(A) The Eigenvectors of the DFT-IV Matrix

Orthonormal eigenvectors of the kernel matrix of the DFT-IV are generated by each of the following three methods:

1) The S method where the eigenvectors of matrix **S** are computed in the manner delineated in section IV.
2) The Orthogonal Procrustes Algorithm (OPA).
3) The Gram-Schmidt Algorithm (GSA).

One should mention that even for the second and third methods, one should start by applying the **S** method in order to generate *initial* eigenvectors as stated in section V-A. Since the target is to approximate the Hermite-Gaussian functions, one computes the approximation error vectors defined as the difference between the exact eigenvectors (the columns of matrix **Q** of (36) for the **S** method and the columns of matrix $\hat{\mathbf{U}}$ appearing in (41) for the OPA and GSA) and the approximate eigenvectors (vectors $\mathbf{u}_n$, $n = 1, \cdots, N$ appearing in (40)). The norms of the approximation error vectors are plotted in Fig. 1 for $N = 18$ for each of the above three methods. The eigenvectors are numbered as $n = 1, \cdots, N$. Since eigenvectors of **G** corresponding to the eigenvalues $\lambda = \mp j$ are even symmetric and those corresponding to $\lambda = \pm 1$ are odd symmetric [12], the norms of the approximation error vectors corresponding to each set are separately plotted and are shown in Figs. 2 and 3. An inspection of Figs. 1-3 reveals that the OPA and GSA outperform the S method as expected.

The computation time of the eigenvectors for each of the three methods is recorded and given in Table 4 for several values of the order N of matrix **G**. It is obvious that either the GSA or the OPA has a longer computation time than the S method[9]. This is the minor cost that one should incur in return for getting a smaller approximation error.

In order to assess the numerical stability of the three methods, one should compute the orthonormality error matrix defined by:

$$\mathbf{E} = \hat{\mathbf{U}}^H \hat{\mathbf{U}} - \mathbf{I} \tag{64}$$

where $\hat{\mathbf{U}}$ is any unitary matrix, e.g. a modal matrix of the DFT-IV matrix **G**. In the absence of round off error - which is unavoidable - matrix **E** is identically zero. The maximum element in absolute value as well as the Frobenius norm of matrix **E** - corresponding to the modal matrix of **G** as evaluated by the three methods under consideration - are computed and given respectively in Tables 5 and 6 for several values of the order N. A quick examination of these two tables shows that for the S method and the OPA the orthonormality error[10] is negligible for all values of the order N. However for the GSA, the orthonormality error begins to be significant for values of N starting from 256. The plot of Fig. 1 for N = 18 is repeated in Fig. 4 for N = 512. Comparing the performance of the GSA and OPA, one notices that the OPA has the merit that the maximum error is smaller while the GSA has the merit that the threshold value of m (the serial number of the eigenvector) - where the approximation error becomes sensible - is larger. Moreover the spurious fluctuations near the end of the GSA plot are due to the numerical inaccuracy of the computation

[9] The computation is performed on a PC where some system related tasks are unavoidably concurrently taking place. This interprets the unexpected decrease in the computation time of the GSA for N = 32 compared to its value for N = 18 and that of the OPA for N = 64 compared to its value for N = 32. The same remark holds for the S method for N = 32 compared to N = 18. In general the estimated computation time is only reliable for large values of the order N.

[10] The orthonormality error and the approximation error are two quite distinct concepts.

manifested by the larger orthonormality error occurring for N = 512 as can be seen in Tables 5 and 6.

(B) The Transform of the Unit Impulse

The continuous FRactional Fourier Transform (FRFT) with an angle of rotation α [18] is defined by:

$$Y_\alpha(u) = \int_{-\infty}^{\infty} y(t) K_\alpha(t,u) dt \tag{65}$$

where the transform kernel is given by:

$$K_\alpha(t,u) = \begin{cases} \sqrt{\dfrac{1-j\cot\alpha}{2\pi}} \exp\left[j\left(\dfrac{t^2+u^2}{2}\cot\alpha - ut\operatorname{cosec}\alpha \right)\right] & \text{if } \alpha \text{ is not a multiple of } \pi \\ \delta(t-u) & \text{if } \alpha \text{ is a multiple of } 2\pi \\ \delta(t+u) & \text{if } (\alpha+\pi) \text{ is a multiple of } 2\pi \end{cases} \tag{66}$$

A Generalized FRactional Fourier Transform (GFRFT) can be defined by (65) where the transform kernel is given by:

$$K_\alpha(t,u) = \begin{cases} \sqrt{\dfrac{1-j\cot\alpha}{2\pi}} \exp\left[j\left(\dfrac{(t+t_0)^2+(u+u_0)^2}{2}\cot\alpha - (t+t_0)(u+u_0)\operatorname{cosec}\alpha \right)\right] \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } \alpha \text{ is not a multiple of } \pi \\ \delta(t-u) \qquad\qquad\qquad\qquad\qquad \text{if } \alpha \text{ is a multiple of } 2\pi \\ \delta(t+u) \qquad\qquad\qquad\qquad\qquad \text{if } (\alpha+\pi) \text{ is a multiple of } 2\pi \end{cases} \tag{67}$$

In order for the above continuous transform to parallel the DFT-IV, the two parameters t_0 and u_0 appearing in (67) will be taken as $t_0 = 0.5T$ and $u_0 = 0.5\Delta$ where T and Δ are respectively the sampling intervals in the time and frequency domains. Since the scope of the frequency spectrum that the DFT can cover is $2\pi / T$, the frequency spacing of the DFT is $\Delta = \dfrac{2\pi}{TN}$. From (65) and (67) one concludes that the GFRFT of an impulse is given by:

$$\begin{aligned} Y_\alpha(u) &= GFRFT[\delta(t)] = K_\alpha(0,u) \\ &= \sqrt{\frac{1-j\cot\alpha}{2\pi}} \exp\left[j\left(\frac{t_0^2+(u+u_0)^2}{2}\cot\alpha - t_0(u+u_0)\operatorname{cosec}\alpha \right)\right] \text{if } \alpha \text{ is not a multiple of } \pi \end{aligned} \tag{68}$$

In the special case of no fractionalization, i.e. $\alpha = 0.5\pi$, the above equation reduces to:

$$Y_{0.5\pi}(u) = \sqrt{\frac{1}{2\pi}} \exp\left[-jt_0\left(u+u_0\right)\right] . \tag{69}$$

Discretizing the frequency variable u according to:

$$u_k = (k-1)\Delta \quad , k = 1,\cdots,N \tag{70}$$

one obtains:

$$Y_{0.5\pi}(u_k) = \frac{1}{\sqrt{2\pi}} \exp\left[-j\left(k-\frac{1}{2}\right)\frac{\pi}{N}\right] \quad , k = 1,\cdots,N . \tag{71}$$

On the other hand the DFT-IV of a discrete-time impulse is obtained using (1) by assuming a time-domain vector $\begin{bmatrix}1 & 0 & \cdots & 0\end{bmatrix}^T$ and finding the corresponding frequency-domain vector as:

$$y_k = G_{k,1} = \frac{1}{\sqrt{N}} \exp\left[-j\frac{\pi}{N}\left(k-\frac{1}{2}\right)\right] \quad , k = 1,\cdots,N . \tag{72}$$

Equations (71) and (72) indicate that the continuous transform should be scaled by $\sqrt{\frac{2\pi}{N}}$ in order for the transforms of the continuous and discrete unit impulses to be identical in the absence of fractionalization. Figures 5 and 6 respectively show the fully-fledged and simple FDFT-IV of a discrete-time unit impulse for angles of rotation $\alpha = \frac{\pi}{8}, \frac{\pi}{4}, \frac{3}{8}\pi$ and $\frac{\pi}{2}$ using a transform order N = 18. Figure 7 shows the GFRFT of a continuous-time unit impulse as given by (68) and scaled by the factor $\sqrt{\frac{2\pi}{N}}$ for the same set of angles. In Figs 5-7, the solid line represents the real part and the dashed line represents the imaginary part. By examining the three figures, one notices that in the absence of fractionalization, i.e. $\alpha = 0.5\pi$, parts (d) are identical as expected. In the presence of fractionalization, i.e. $\alpha \neq 0.5\pi$, it is obvious that the plots in parts a, b, c of Fig. 6 - pertaining to the simple fractional transform - are quite lacking in variation and consequently they serve as a poor approximation of their counterparts in Fig. 7 pertaining to the continuous transform. On the contrary, parts a, b, c of Fig. 5 - pertaining to the fully-fledged fractional transform - have enough variations and consequently serve as a better approximation of their counterparts in Fig. 7 than those pertaining to the simple fractional transform.

VII. Conclusion

An eigenanalysis of an almost tridiagonal matrix **S** which commutes with the DFT-IV matrix **G** has been presented. By means of a similarity transformation defined in terms of a unitary matrix $\mathbf{P}$, matrix **S** has been reduced to a block diagonal form where the two diagonal blocks are symmetric exactly tridiagonal matrices and *explicit expressions* have been derived for the elements of those matrices in order to circumvent the need for performing the two matrix multiplications required for obtaining the transformed matrix $\mathbf{PSP}^{-1}$. Even and odd symmetric eigenvectors of **S** are systematically generated – in terms of the eigenvectors of the two diagonal blocks – and proved to always be eigenvectors of matrix **G** *irrespective of the multiplicities of the eigenvalues of* ***S***. The importance of the method of the present paper stems from the fact that a brute force application of a general eigenanalysis routine available in any software package will *not* generate one even and one odd symmetric eigenvectors to correspond to an eigenvalue of **S** of multiplicity 2. Moreover since the two diagonal blocks are of almost half the order of matrix **S**, the technique of the present paper has the extra merit of *computational efficiency*.

Eigenvectors of matrix **G** that approximate the Hermite-Gaussian functions better than those of matrix **S** have been generated by applying either the orthogonal procrustes algorithm (OPA) or the Gram-Schmidt algorithm (GSA). Those superior eigenvectors have been taken as basis for defining a fully-fledged version of the fractional DFT-IV.

Appendix A

(Proof of Theorem 1)

Case a: N is odd

From (26), one gets:

$$\mathbf{PSP}^{-1} = [\mathbf{A_1} \quad \mathbf{a_2} \quad \mathbf{A_3}] \tag{A-1}$$

where the $N \times (\nu - 1)$ submatrices $\mathbf{A_1}$ and $\mathbf{A_3}$ and the column vector $\mathbf{a_2}$ are given by:

$$\mathbf{A_1} = \mathbf{P}\frac{1}{\sqrt{2}}(\mathbf{I_N} + \mathbf{J_N})\mathbf{C}\,, \tag{A-2}$$

$$\mathbf{a_2} = \mathbf{Pc}_\nu\,, \tag{A-3}$$

$$\mathbf{A_3} = \mathbf{P}\frac{1}{\sqrt{2}}(\mathbf{I_N} - \mathbf{J_N})\mathbf{CJ}_{\nu-1}\,. \tag{A-4}$$

It is straightforward to show that:

$$\mathbf{I_N} + \mathbf{J_N} = \begin{bmatrix} \mathbf{I}_{\nu-1} & & \mathbf{J}_{\nu-1} \\ & 2 & \\ \mathbf{J}_{\nu-1} & & \mathbf{I}_{\nu-1} \end{bmatrix}, \tag{A-5}$$

$$\mathbf{I_N} - \mathbf{J_N} = \begin{bmatrix} \mathbf{I}_{\nu-1} & & -\mathbf{J}_{\nu-1} \\ & 0 & \\ -\mathbf{J}_{\nu-1} & & \mathbf{I}_{\nu-1} \end{bmatrix}. \tag{A-6}$$

The above two equations together with (10-a) lead to:

$$\mathbf{P}\frac{1}{\sqrt{2}}(\mathbf{I_N} + \mathbf{J_N}) = \begin{bmatrix} \mathbf{I}_{\nu-1} & & \mathbf{J}_{\nu-1} \\ & \sqrt{2} & \\ \mathbf{0} & & \mathbf{0} \end{bmatrix}, \tag{A-7}$$

$$\mathbf{P}\frac{1}{\sqrt{2}}(\mathbf{I_N} - \mathbf{J_N}) = \begin{bmatrix} \mathbf{0} & & \mathbf{0} \\ & 0 & \\ \mathbf{J}_{\nu-1} & & -\mathbf{I}_{\nu-1} \end{bmatrix}. \tag{A-8}$$

Substituting (A-7) and (24) in (A-2), one gets:

$$
\mathbf{A_1}=\begin{bmatrix}\mathbf{I}_{\nu-1} & & \mathbf{J}_{\nu-1}\\ & \sqrt{2} & \\ \mathbf{0} & & \mathbf{0}\end{bmatrix}\begin{bmatrix}s_1 & 1 & & & \\ 1 & s_2 & 1 & & \\ & 1 & s_3 & \ddots & \\ & & \ddots & \ddots & 1\\ & & & 1 & s_{\nu-1}\\ & & & & 1\\ & & & & \\ & & & & \\ -1 & & & & \end{bmatrix}=\begin{bmatrix}s_1-1 & 1 & & & \\ 1 & s_2 & 1 & & \\ & 1 & s_3 & \ddots & \\ & & \ddots & \ddots & 1\\ & & & 1 & s_{\nu-1}\\ & & & & \sqrt{2}\\ - & - & - & - & -\\ & & \mathbf{0} & & \end{bmatrix}. \tag{A-9}
$$

Substituting (A-8) and (24) in (A-4), one gets:

$$
\mathbf{A_3}=\begin{bmatrix}\mathbf{0} & & \mathbf{0}\\ & 0 & \\ \mathbf{J}_{\nu-1} & & -\mathbf{I}_{\nu-1}\end{bmatrix}\begin{bmatrix} & & & 1 & s_1\\ & & 1 & s_2 & 1\\ & \iddots & s_3 & 1 & \\ 1 & \iddots & \iddots & & \\ s_{\nu-1} & 1 & & & \\ 1 & & & & \\ & & & & \\ & & & & \\ & & & & -1\end{bmatrix}=\begin{bmatrix} & & \mathbf{0} & & \\ - & - & - & - & -\\ s_{\nu-1} & 1 & & & \\ 1 & \ddots & \ddots & & \\ & \ddots & s_3 & 1 & \\ & & 1 & s_2 & 1\\ & & & 1 & s_1+1\end{bmatrix}. \tag{A-10}
$$

Substituting (10-a) in (A-3) and calling to remembrance the definition of $\mathbf{c}_\nu$ as the ν^{th} column of matrix **S** as given by (22) and (18), one gets:

$$
\mathbf{a_2}=\mathbf{P}\begin{pmatrix}\mathbf{0}\\ 1\\ s_\nu\\ 1\\ \mathbf{0}\end{pmatrix}=\begin{pmatrix}\mathbf{0}\\ \sqrt{2}\\ s_\nu\\ 0\\ \mathbf{0}\end{pmatrix}. \tag{A-11}
$$

Substituting (A-9), (A-10) and (A-11) in (A-1), one gets:

$$
\mathbf{PSP^{-1}} = \left[\begin{array}{cccccc|cccccc}
s_1-1 & 1 & & & & & & & & & & \\
1 & s_2 & 1 & & & & & & & & & \\
 & 1 & s_3 & \ddots & & & & & & & & \\
 & & \ddots & \ddots & 1 & & & & & & & \\
 & & & 1 & s_{\nu-1} & \sqrt{2} & & & & & & \\
 & & & & \sqrt{2} & s_\nu & & & & & & \\
- & - & - & - & - & - & - & - & - & - & - & \\
 & & & & & & s_{\nu-1} & 1 & & & & \\
 & & & & & & 1 & \ddots & \ddots & & & \\
 & & & & & & & \ddots & s_3 & 1 & & \\
 & & & & & & & & 1 & s_2 & 1 & \\
 & & & & & & & & & 1 & s_1+1 &
\end{array}\right]. \tag{A-12}
$$

By comparing (A-12) and (25), one concludes that (30) and (31) hold with $\gamma = \sqrt{2}$, $\delta = 0$ and $\varepsilon = \nu - 1$.

Case b: N is even

From (28), one gets:

$$\mathbf{PSP}^{-1} = [\mathbf{B_1} \quad \mathbf{B_2}] \tag{A-13}$$

where the $N \times \nu$ submatrices $\mathbf{B_1}$ and $\mathbf{B}_2$ are given by:

$$\mathbf{B_1} = \mathbf{P}\frac{1}{\sqrt{2}}(\mathbf{I_N} + \mathbf{J_N})\mathbf{C}, \tag{A-14}$$

$$\mathbf{B_2} = \mathbf{P}\frac{1}{\sqrt{2}}(\mathbf{I_N} - \mathbf{J_N})\mathbf{CJ}_\nu. \tag{A-15}$$

It is straightforward to show that:

$$\mathbf{I_N} + \mathbf{J_N} = \begin{bmatrix} \mathbf{I}_\nu & \mathbf{J}_\nu \\ \mathbf{J}_\nu & \mathbf{I}_\nu \end{bmatrix}, \tag{A-16}$$

$$\mathbf{I_N} - \mathbf{J_N} = \begin{bmatrix} \mathbf{I}_\nu & -\mathbf{J}_\nu \\ -\mathbf{J}_\nu & \mathbf{I}_\nu \end{bmatrix}. \tag{A-17}$$

The above two equations together with (10-b) lead to:

$$\mathbf{P}\frac{1}{\sqrt{2}}(\mathbf{I_N} + \mathbf{J_N}) = \begin{bmatrix} \mathbf{I}_\nu & \mathbf{J}_\nu \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \tag{A-18}$$

$$\mathbf{P}\frac{1}{\sqrt{2}}\left(\mathbf{I_N}-\mathbf{J_N}\right)=\begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{J}_\nu & -\mathbf{I}_\nu\end{bmatrix}. \tag{A-19}$$

Substituting (A-18) and (24) in (A-14), one gets:

$$\mathbf{B_1}=\begin{bmatrix}\mathbf{I}_\nu & \mathbf{J}_\nu\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix} s_1 & 1 & & & \\ 1 & s_2 & 1 & & \\ & 1 & s_3 & \ddots & \\ & & \ddots & \ddots & 1\\ & & & 1 & s_\nu\\ & & & & 1\\ & & & & \\ & & & & \\ -1 & & & & \end{bmatrix}=\begin{bmatrix} s_1-1 & 1 & & & \\ 1 & s_2 & 1 & & \\ & 1 & s_3 & \ddots & \\ & & \ddots & \ddots & 1\\ & & & 1 & s_\nu+1\\ - & - & - & - & -\\ & & \mathbf{0} & & \\ & & & & \\ & & & & \end{bmatrix}. \tag{A-20}$$

Substituting (A-19) and (24) in (A-15), one gets:

$$\mathbf{B_2}=\begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{J}_\nu & -\mathbf{I}_\nu\end{bmatrix}\begin{bmatrix} & & & 1 & s_1\\ & & 1 & s_2 & 1\\ & \iddots & s_3 & 1 & \\ 1 & \iddots & \iddots & & \\ s_\nu & 1 & & & \\ 1 & & & & \\ & & & & \\ & & & & -1\end{bmatrix}=\begin{bmatrix} & & & & \\ & & \mathbf{0} & & \\ & & & & \\ - & - & - & - & -\\ s_\nu-1 & 1 & & & \\ 1 & \ddots & \ddots & & \\ & \ddots & s_3 & 1 & \\ & & 1 & s_2 & 1\\ & & & 1 & s_1+1\end{bmatrix}. \tag{A-21}$$

Substituting the above two equations in (A-13), one gets:

$$\mathbf{PSP^{-1}}=\left[\begin{array}{ccccc|ccccc} s_1-1 & 1 & & & & & & & & \\ 1 & s_2 & 1 & & & & & & & \\ & 1 & s_3 & \ddots & & & & & & \\ & & \ddots & \ddots & 1 & & & & & \\ & & & 1 & s_\nu+1 & & & & & \\ \hline & & & & & s_\nu-1 & 1 & & & \\ & & & & & 1 & \ddots & \ddots & & \\ & & & & & & \ddots & s_3 & 1 & \\ & & & & & & & 1 & s_2 & 1\\ & & & & & & & & 1 & s_1+1\end{array}\right]. \tag{A-22}$$

By comparing (A-22) and (25), one concludes that (30) and (31) hold with $\gamma=1$, $\delta=1$ and $\varepsilon=\nu$.

(Q.E.D.)

Appendix B
(Derivation of (50))

By equating the diagonal elements of both sides of (49) using the 4 distinct eigenvalues of D (given by (9)) on the R.H.S. and the corresponding specifically selected fractional powers (given by (47)) on the L.H.S., one gets:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}\begin{bmatrix} a_0(\alpha) \\ a_1(\alpha) \\ a_2(\alpha) \\ a_3(\alpha) \end{bmatrix} = \begin{bmatrix} 1 \\ \exp(-j\alpha) \\ \exp(j2\alpha) \\ \exp(j\alpha) \end{bmatrix}. \tag{B-1}$$

By interchanging the second and third rows, the above equation can be compactly expressed as:

$$\mathbf{A}\begin{bmatrix} a_0(\alpha) \\ a_1(\alpha) \\ a_2(\alpha) \\ a_3(\alpha) \end{bmatrix} = \begin{bmatrix} 1 \\ \exp(j2\alpha) \\ \exp(-j\alpha) \\ \exp(j\alpha) \end{bmatrix} \tag{B-2}$$

where

$$\mathbf{A} = \begin{pmatrix} \mathbf{B} & \mathbf{B} \\ \mathbf{C} & -\mathbf{C} \end{pmatrix}, \tag{B-3}$$

$$\mathbf{B} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \tag{B-4}$$

$$\mathbf{C} = \begin{pmatrix} 1 & -j \\ 1 & j \end{pmatrix}. \tag{B-5}$$

Matrix A of (B-3) can be expressed as:

$$\mathbf{A} = \begin{pmatrix} \mathbf{B} & \mathbf{O} \\ \mathbf{O} & \mathbf{C} \end{pmatrix}\begin{pmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{I} & -\mathbf{I} \end{pmatrix}. \tag{B-6}$$

and it is straightforward to show that:

$$\begin{pmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{I} & -\mathbf{I} \end{pmatrix}\begin{pmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{I} & -\mathbf{I} \end{pmatrix} = \mathbf{2I}. \tag{B-7}$$

From the above two equations, one concludes that the inverse of A is given by:

$$\mathbf{A}^{-1} = \frac{\mathbf{1}}{\mathbf{2}}\begin{pmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{I} & -\mathbf{I} \end{pmatrix}\begin{pmatrix} \mathbf{B}^{-1} & \mathbf{O} \\ \mathbf{O} & \mathbf{C}^{-1} \end{pmatrix} = \frac{\mathbf{1}}{\mathbf{2}}\begin{pmatrix} \mathbf{B}^{-1} & \mathbf{C}^{-1} \\ \mathbf{B}^{-1} & -\mathbf{C}^{-1} \end{pmatrix}. \tag{B-8}$$

Using (B-4), (B-5) and (B-8), one obtains the following unique solution of (B-2):

$$\begin{bmatrix} a_0(\alpha) \\ a_1(\alpha) \\ a_2(\alpha) \\ a_3(\alpha) \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & j & -j \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -j & j \end{bmatrix} \begin{bmatrix} 1 \\ \exp(j2\alpha) \\ \exp(-j\alpha) \\ \exp(j\alpha) \end{bmatrix}. \tag{B-9}$$

By algebraic manipulation, one directly gets (50).

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Table 1: The set of indices $\Psi = \{n_1, \cdots, n_N\}$ of the eigenvectors of the DFT-IV matrix G

N	$n_1, n_2, \cdots, n_N$
4m	$0,1,2,\cdots,(4m-2),(4m-1)$
4m+1	$0,1,2,\cdots,(4m-1),(4m+1)$
4m+2	$0,1,2,\cdots,4m,(4m+1)$
4m+3	$0,1,2,\cdots,(4m+1),(4m+3)$

Table 2: A concise form for the set of indices $\Psi = \{n_1, \cdots, n_N\}$ of the eigenvectors of matrix G

N	$n_1, \cdots, n_N$
Odd	$0,\ 1,\ 2,\ \cdots,\ N-2,\ N$
Even	$0,\ 1,\ 2,\ \cdots,\ N-2,\ N\text{-}1$

Table 3: The set of indices $\Psi = \{n_1, \cdots, n_N\}$ of the eigenvectors of the DHT-IV matrix H

N	The value of n in (57)	$n_1, n_2, \cdots, n_N$
4m	$0,2,\cdots,(4m-4),(4m-2)$	$0,1,2,3,\cdots,(4m-2),(4m-1)$
4m+1	$0,2,\cdots,(4m-2),4m$	$0,1,2,3,\cdots,(4m-2),(4m-1),4m$
4m+2	$0,2,\cdots,(4m-2),4m$	$0,1,2,3,\cdots,4m,(4m+1)$
4m+3	$0,2,\cdots,4m,(4m+2)$	$0,1,2,3,\cdots,4m,(4m+1),(4m+2)$

Table 4: The computation time (in seconds) of the eigenvectors.

N	S method	GSA	OPA
18	0.015625	0.046875	0.015625
32	0.000000	0.015625	0.031250
64	0.015625	0.062500	0.015625
128	0.062500	0.093750	0.062500
256	0.343750	0.437500	0.406250
512	3.359375	5.109375	4.796875
1024	29.171875	47.359375	42.484375
2048	230.093750	383.937500	365.156250

Table 5: The maximum orthonormality error.

N	S method	GSA	OPA
18	1.33227E-15	6.66E-16	1.9984E-15
32	1.88738E-15	8.88E-16	2.22045E-15
64	1.9984E-15	4.37E-15	2.88658E-15
128	5.10703E-15	2.33E-11	3.55271E-15
256	6.66134E-15	0.526586	3.77476E-15
512	8.88178E-15	0.999515	4.77396E-15
1024	9.32587E-15	0.994361	7.32747E-15
2048	1.53211E-14	0.998968	1.19904E-14

Table 6: The Frobenius norm of the orthonormality error matrix.

N	S method	GSA	OPA
18	2.71163E-15	2.24E-15	4.67783E-15
32	6.11936E-15	3.97E-15	9.034E-15
64	1.04443E-14	1.62E-14	1.55815E-14
128	2.2952E-14	3.38E-11	2.83594E-14
256	4.31009E-14	1.017684	5.20255E-14
512	8.36262E-14	18.56104	9.72064E-14
1024	1.67385E-13	50.52326	2.00532E-13
2048	3.29626E-13	148.6696	4.20158E-13

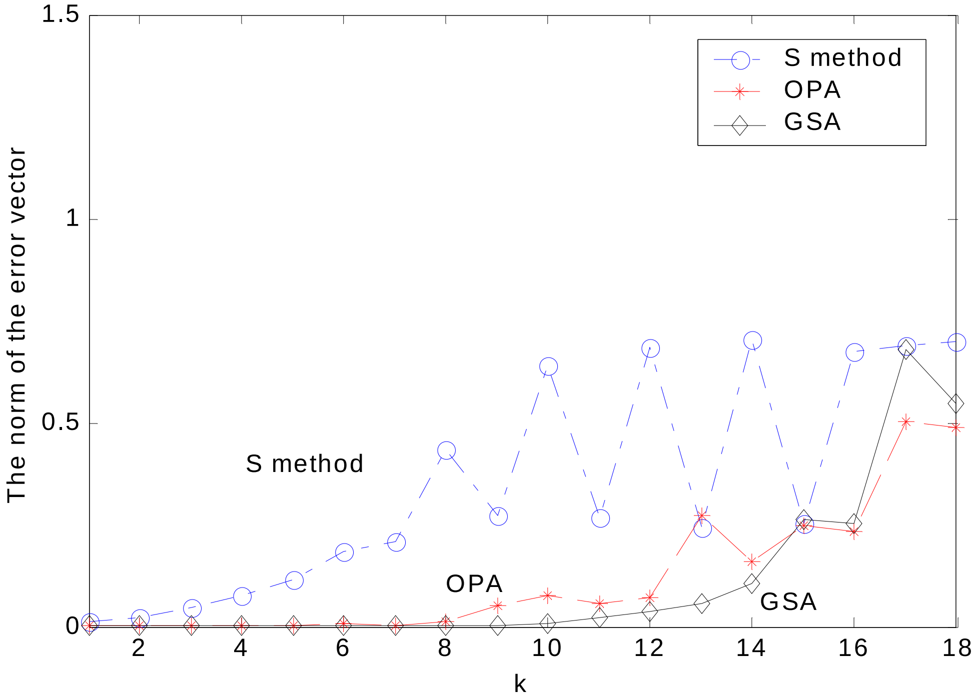

Fig. 1: The norm of the approximation error vectors between the exact and approximate eigenvectors for N = 18.
(all eigenvectors)

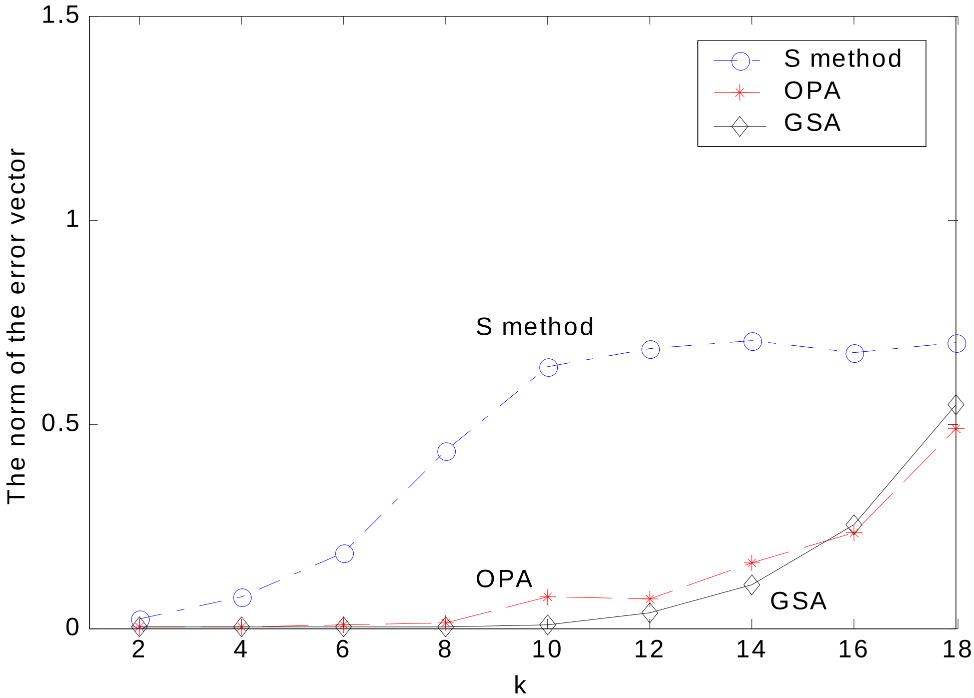

Fig. 2: The norm of the approximation error vectors between the exact and approximate eigenvectors for N = 18.
(for only the even symmetric eigenvectors)

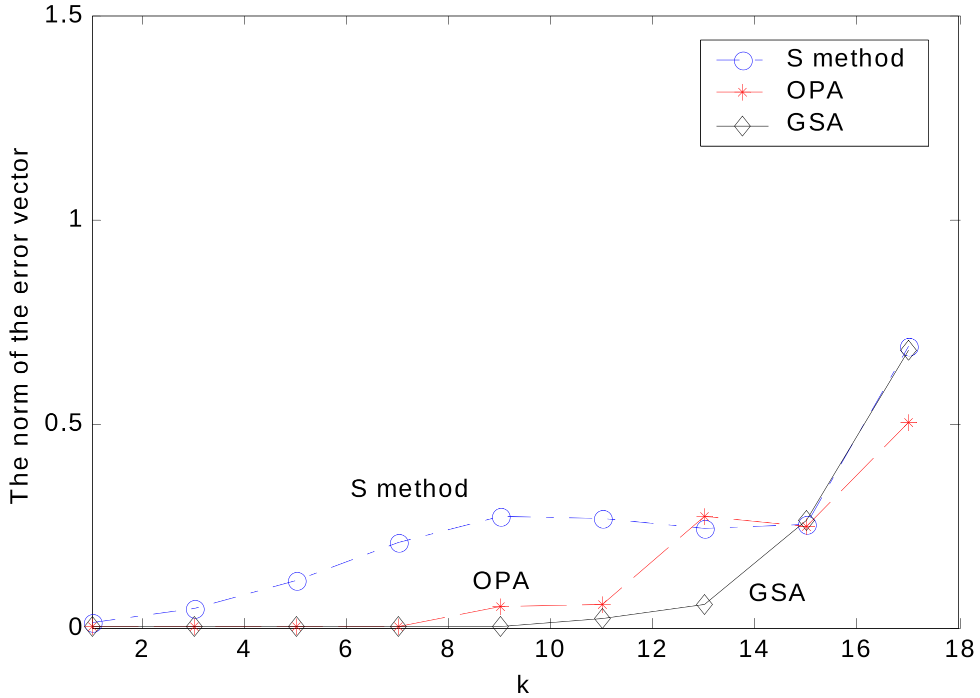

Fig. 3: The norm of the approximation error vectors between the exact and approximate eigenvectors for N = 18.
(for only the odd symmetric eigenvectors)

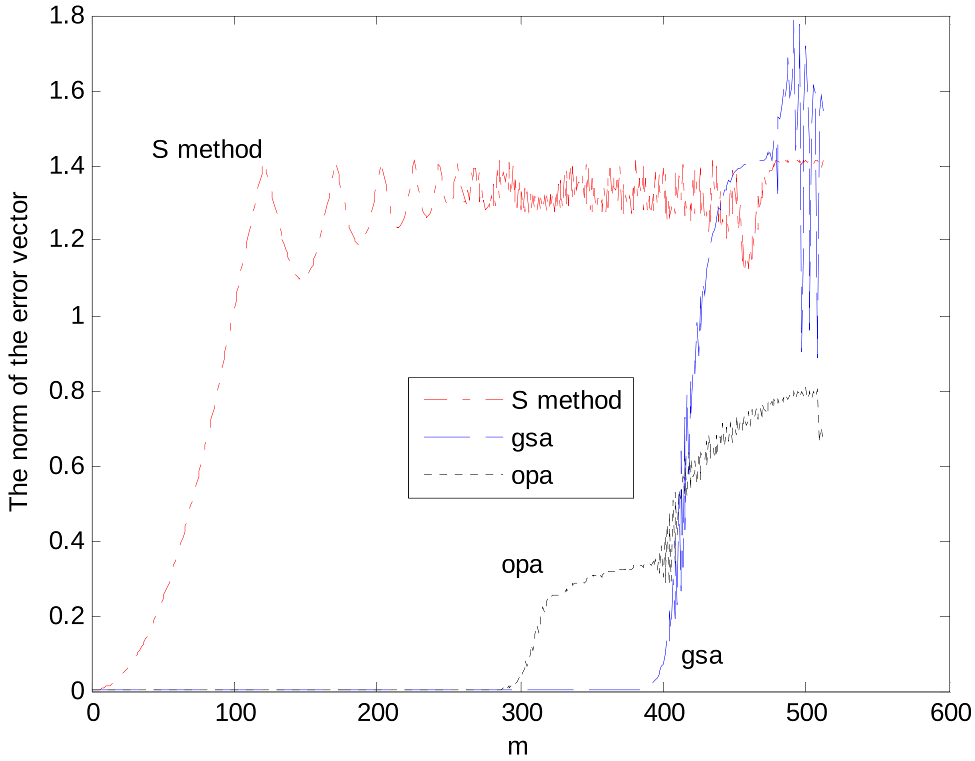

Fig. 4: The norm of the approximation error vectors between the exact and approximate eigenvectors for N = 512.
(all eigenvectors)

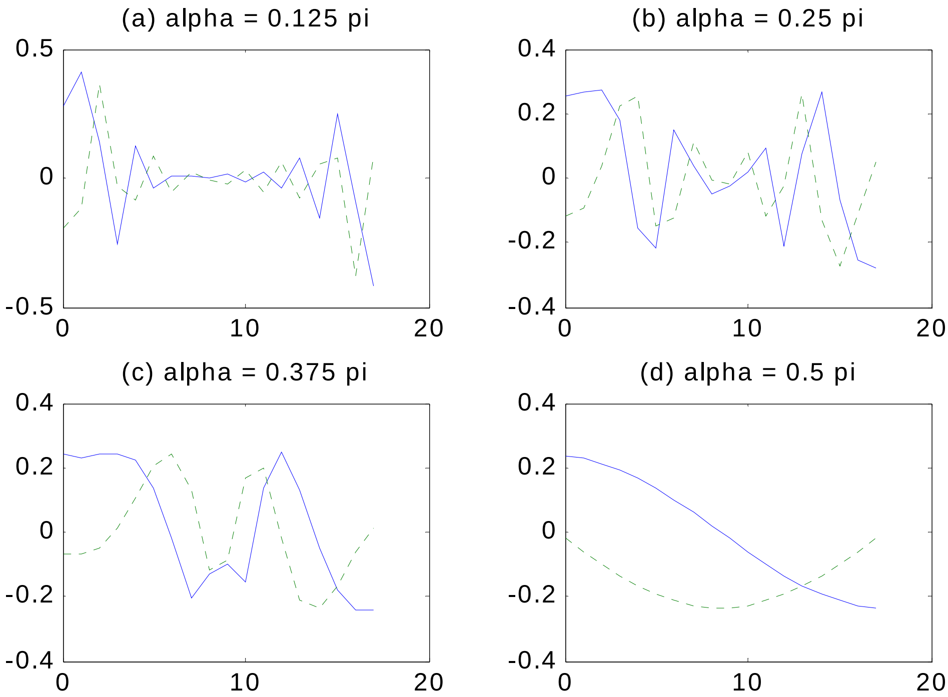

Fig. 5: The fully-fledged FDFT-IV of a discrete-time impulse for N = 18.

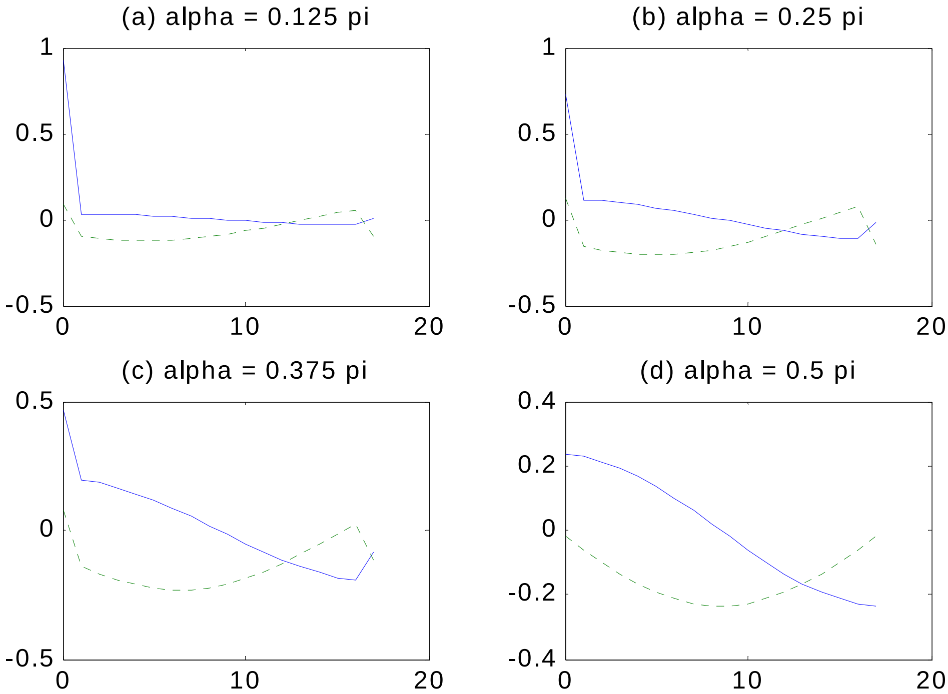

Fig. 6: The simple FDFT-IV of a discrete-time impulse for N = 18.

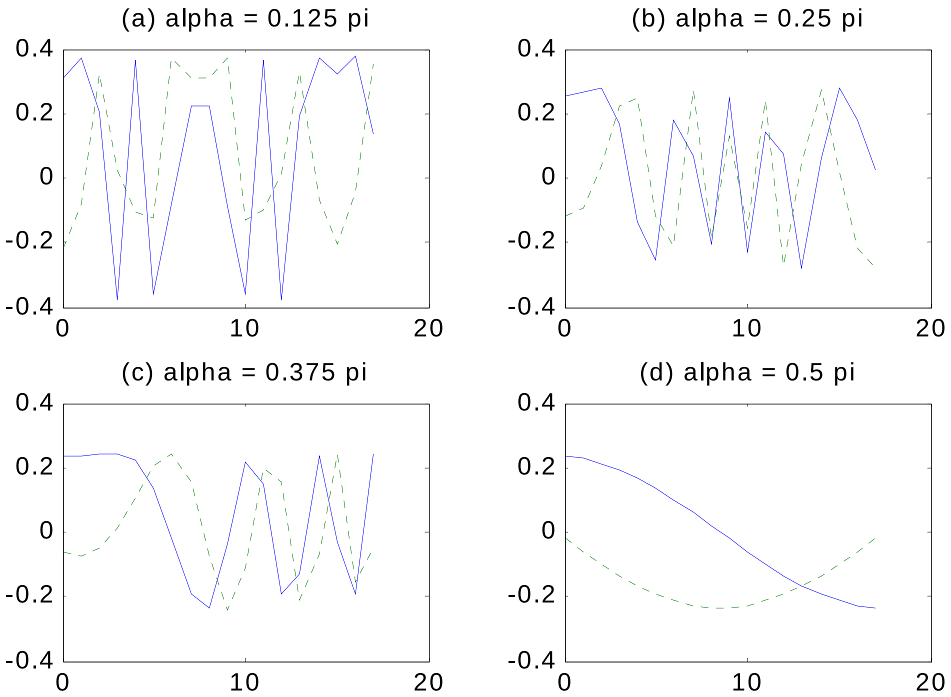

Fig. 7: The (scaled) GFRFT of a continuous-time impulse for N = 18.

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