Hermite-Gaussian-Like Eigenvectors of the Discrete Fourier Transform Matrix Based on the Direct Utilization of the Orthogonal Projection Matrices on its Eigenspaces[1]

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Abstract

A new version is proposed for the Gram-Schmidt algorithm, the orthogonal procrustes algorithm and the sequential orthogonal procrustes algorithm for generating Hermite-Gaussian-like orthonormal eigenvectors for the discrete Fourier transform matrix F. This version is based on the direct utilization of the orthogonal projection matrices on the eigenspaces of matrix F rather than the singular value decomposition of those matrices for the purpose of generating initial orthonormal eigenvectors. The proposed version of the algorithms has the merit of achieving a significant reduction in the computation time.

Index Terms: Discrete fractional Fourier transform, Hermite-Gaussian-like orthonormal eigenvectors, orthogonal procrustes algorithm, sequential orthogonal procrustes algorithm, Gram-Schmidt algorithm, projection matrices.

I. INTRODUCTION

The development of the discrete fractional Fourier transform (DFRFT) necessitates the generation of orthonormal eigenvectors of the discrete Fourier transform (DFT) matrix F in order to satisfy the basic requirements of unitarity and index additivity that any legitimate definition of the DFRFT should possess. Since the multiplicities of the eigenvalues of the unitary matrix F – derived by McClellan and Parks [1] – are large, the dimensions of the corresponding eigenspaces are high and there is much freedom in the selection of the orthonormal eigenvectors of F. This freedom is exploited in achieving the goal of having the DFRFT approximate its continuous counterpart by demanding that the orthonormal eigenvectors of F be Hermite-Gaussian-like, i.e. be close to samples of the Hermite-Gaussian functions which are the eigenfunctions of the continuous fractional Fourier transform. Dickinson and Steiglitz proved that matrix F commutes with a real symmetric almost tridiagonal matrix S whose eigenvalues have a maximum multiplicity of two and that both matrices have a common set of eigenvectors [2]. Pei et al. showed that samples of the

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Hermite-Gaussian functions, taken in a specific way, form *approximate* eigenvectors of matrix F [3, 4]. They considered the eigenvectors of S as only *initial* eigenvectors of F and obtained *final* superior eigenvectors in the sense of better approximating samples of the Hermite-Gaussian functions [3]. They proposed two techniques; namely the Gram-Schmidt algorithm (GSA) and the orthogonal procrustes algorithm (OPA). In the first technique (GSA), vectors consisting of samples of the Hermite-Gaussian functions pertaining to one eigenvalue of F are projected on the corresponding eigenspace spanned by the *initial* orthonormal eigenvectors and are next orthonormalized by applying the Gram-Schmidt procedure. In the second technique (OPA), a unitary matrix Q is derived and premultiplied by a matrix whose columns are *initial* orthonormal eigenvectors pertaining to one eigenspace of F so that the resulting matrix will be as close as possible – in the sense of Frobenius norm – to the matrix whose columns are samples of the Hermite-Gaussian functions. In both the GSA and OPA, the procedure is applied to each eigenspace separately since those eigenspaces are mutually orthogonal due to the unitarity of matrix F. In a different development Candan et al. discretized the differential equation satisfied by the Hermite-Gaussian functions and showed that the solution of the resulting second order difference equation is given by the eigenvectors of an almost tridiagonal matrix[5] S [5, 6]. They regarded those eigenvectors as Hermite-Gaussian-like.

Recently Hanna, Seif and Ahmed [7] proposed a new technique for generating *initial* orthonormal eigenvectors of F by the singular value decomposition of the orthogonal projection matrices on the eigenspaces of F. They also proposed a new technique termed the sequential orthogonal procrustes algorithm (SOPA) for generating *final* eigenvectors of F given *initial* ones. This technique is based on the sequential derivation of the columns of the unitary matrix Q by solving a series of constrained minimization problems in contrast to the batch evaluation of that matrix by solving a single minimization problem as in the OPA. Hanna et al. proved that for each of the GSA, OPA and SOPA the *final* eigenvectors are invariant under the change of the *initial* eigenvectors.

The main objective of this paper is to show that the *final* eigenvectors of F can be directly generated given the orthogonal projection matrices without having to first find the singular value decomposition of those matrices for the sake of getting *initial* eigenvectors. This idea will be shown to be applicable to each of the GSA, OPA and SOPA.

[5] Strictly speaking, denoting matrix S in the work of Dickinson et al. [2] and Pei et al. [3] by S_1 and matrix S in the work of Candan et. al. [5] by S_2, the two matrices are related by $S_2 = S_1 - 4I$. Therefore S_1 and S_2 have the same eigenvectors.

Some background material will be given in section II. The modified forms of the GSA, OPA and SOPA based on the direct utilization of the orthogonal projection matrices rather than the generation of *initial* eigenvectors will be developed in sections III - V. Some simulation results will be presented in section VI demonstrating the reduction in the computation time achieved by the proposed version of the algorithms.

II. BACKGROUND MATERIAL AND NOTATION

The DFT matrix $F = \left(f_{m,n}\right)$ of order N is defined by:

$$f_{m,n} = \frac{1}{\sqrt{N}} W^{(m-1)(n-1)} \qquad , m, n = 1, \cdots, N \tag{1}$$

where $W = \exp\left(-j\frac{2\pi}{N}\right)$. Matrix F has the four distinct eigenvalues [1]:

$$\lambda_k = (-j)^{k-1} \qquad , k = 1, \cdots 4 . \tag{2}$$

Because matrix F is unitary, it is diagonalizable and its eigenspaces E_k , $k = 1, \cdots, 4$ are orthogonal to one another [8]. Explicit expressions have been derived for the four orthogonal projection matrices P_k on E_k in [7]. By means of the singular value decomposition of P_k, it has been possible to obtain orthonormal basis for E_k given by the columns of an N x r_k matrix V_k, i.e.

$$P_k = V_k V_k^H \tag{3}$$

where[6] r_k is the dimension of E_k (which is the multiplicity of λ_k) derived in [1]. (The reader is referred to [7] for a proof of the above formula and to Appendix A for some clarification). Those eigenvectors are regarded as only *initial* ones and are used for obtaining *final* Hermite-Gaussian-like eigenvectors that will form the columns of an N x r_k matrix $\hat{U}_k$. A preliminary step toward computing $\hat{U}_k$ is to generate an N x r_k matrix U_k whose columns are *approximate* rather than *exact* eigenvectors corresponding to the *exact* eigenvalue λ_k by taking samples of the Hermite-Gaussian functions [3], [4]. Because of the orthogonality of the eigenspaces of F due to its unitarity, each eigenspace will be dealt with separately. In order to simplify the notation, the subscript k will be dropped in the remainder of this paper. The

[6] The superscripts H, * and T respectively denote the Hermitian transpose, the complex conjugate and the transpose operations.

$N \times r_k$ matrices $V_k, U_k, \hat{U}_k$ will be written as the $N \times r$ matrices $V, U, \hat{U}$ respectively. The space E_k will be denoted by E and the corresponding orthogonal projection matrix P_k will be denoted by P.

***Lemma 1*:**

The orthogonal projection matrix P on any space is invariant under the change of the orthonormal basis of that space.

Proof: See Appendix A.

III. THE GRAM-SCHMIDT ALGORITHM

Let u_n be a vector of N elements obtained by sampling the Hermite-Gaussian function of order n in the manner delineated in [3]. Pei et al. proved that u_n is an *approximate* eigenvector of matrix F of order N [3], [4]. Given an orthonormal basis $v_m, m = 1, \cdots, r$ of the eigenspace E (namely the columns of the matrix V of (A-1) in Appendix A), one projects u_n on E to get the *exact* eigenvector $\tilde{u}_n$ as:

$$\tilde{u}_n = \sum_{m=1}^{r} \langle v_m, u_n \rangle v_m \quad , \quad n = 1, \cdots, r. \tag{4}$$

Here it is suggested to exploit the availability of the projection matrix P on E – as derived in [7] – to directly get $\tilde{u}_n$ from u_n according to:

$$\tilde{u}_n = Pu_n \quad , n = 1, \cdots, r. \tag{5}$$

The above r vector equations can be compactly expressed as one matrix equation:

$$\tilde{U} = PU \tag{6}$$

where

$$U = \begin{pmatrix} u_1 & \cdots & u_r \end{pmatrix}, \tag{7}$$

$$\tilde{U} = \begin{pmatrix} \tilde{u}_1 & \cdots & \tilde{u}_r \end{pmatrix}. \tag{8}$$

Since the exact eigenvectors $\tilde{u}_n, n = 1, \cdots, r$ are not orthogonal, one can apply the Gram-Schmidt orthonormalization procedure to get a set of orthonormal eigenvectors $\hat{u}_n, n = 1, \cdots, r$ that can be arranged as the columns of the target matrix $\hat{U}$ defined by:

$$\hat{U} = \begin{pmatrix} \hat{u}_1 & \cdots & \hat{u}_r \end{pmatrix}. \tag{9}$$

One should notice that the modification suggested here is only in obtaining the projected vectors $\tilde{u}_n, n = 1, \cdots, r$ using (5) instead of (4). No modification is made in the orthonormalization procedure contributed by Gram and Schmidt. Given the projection matrix P, it is faster and more straightforward to get $\tilde{u}_n$ using (5) than to first perform a singular value decomposition to get the matrix V according to (3) in preparation for using (4).

IV. THE MODIFIED ORTHOGONAL PROCRUSTES ALGORITHM

Given the matrix U of *approximate* eigenvectors defined by (7) and an N x r matrix V whose columns are *initial exact* orthonormal eigenvectors of F, one seeks a matrix $\hat{U}$ of the form:

$$\hat{U} = VQ \tag{10}$$

where Q is a unitary matrix of order r derived according to the orthogonal procrustes algorithm (OPA) such that the squared Frobenius norm $\left\| U - \hat{U} \right\|_F^2$ is minimized. The evaluation of matrix Q according to the OPA was given in [9] for the real case and will be expounded upon in Appendix B for the case of a complex vector space. One should mention that although in general the OPA does not require the orthonormality of the columns of the matrix V, that matrix should have orthonormal columns in order to guarantee the orthonormality of the columns of the target matrix $\hat{U}$ according to (10). The OPA can be summarized in the following three steps:

1. Form the square matrix C of order r:

   $$C = V^H U \,. \tag{11}$$

2. Find the full size[7] singular value decomposition of C:

   $$C = ADB^H \tag{12}$$

   where A and B are unitary matrices and D is a real diagonal matrix of the singular values of C, all of order r.

3. Compute the matrix Q:

   $$Q = AB^H \,. \tag{13}$$

[7] Explaining the need for using the full size version of the singular value decomposition technique has been one of the reasons for the inclusion of Appendix B. The primary reason has been the presentation of a detailed derivation of the OPA when the vector space is over the field of complex scalars.

Our objective here is to develop a modified version of the above algorithm that does not require the explicit use of the matrix V . From (11) and (12), one obtains:

$$V^H U = ADB^H \text{ .} \tag{14}$$

Premultiplying both sides by V , one obtains:

$$VV^H U = VADB^H \text{ .} \tag{15}$$

Upon defining the N x r matrix H:

$$H = VA \tag{16}$$

and utilizing (3), one can express (15) as:

$$PU = HDB^H \text{ .} \tag{17}$$

From (10), (13) and (16), one gets:

$$\hat{U} = VAB^H = HB^H \text{ .} \tag{18}$$

It should be mentioned that the columns of the matrix H of (16) are orthonormal due to the unitarity of the matrix A and the orthonormality of the columns of the matrix V. Based on (17) and (18), the modified OPA can be described by the following steps:

1. Form the N x r matrix G:

   $$G = PU \text{ .} \tag{19}$$

2. Find the economy size singular value decomposition of G:

   $$G = HDB^H \tag{20}$$

   where B is a unitary matrix of order r, H is an N x r matrix with orthonormal columns and D is a diagonal matrix of order r.

3. Compute the matrix $\hat{U}$:

   $$\hat{U} = HB^H \text{ .} \tag{21}$$

V. THE MODIFIED SEQUENTIAL ORTHOGONAL PROCRUSTES ALGORITHM

In the sequential OPA (SOPA), the columns of the unitary matrix Q appearing in (10) are sequentially generated by solving a series of constrained minimization problems. In the s th stage the column q_s of Q is derived by minimizing the functional:

$$J_s = \|u_s - \hat{u}_s\|_2^2 \tag{22}$$

subject to the constraints that q_s is orthogonal to q_k , $k = 1, \cdots, (s-1)$ and is of unit norm. The solution of this problem is given by [7]:

$$q_s = \frac{1}{\|x_s\|} x_s \qquad , \ s = 1, \cdots, r \tag{23}$$

where $x_s \in \mathbf{C}^r$ and is given by:

$$x_s = \begin{cases} V^H u_1 & , s = 1 \\ \left(I - C_{s-1}^H C_{s-1}\right) V^H u_s & , s = 2, \cdots, r \end{cases} \tag{24}$$

and the $(s-1) \times r$ matrix C_{s-1} is defined by:

$$C_{s-1} = \begin{pmatrix} q_1^H \\ \vdots \\ q_{s-1}^H \end{pmatrix}. \tag{25}$$

The SOPA as presented above necessitates the knowledge of some *initial* orthonormal eigenvectors forming the columns of matrix V. Our objective here is to develop a modified version of the SOPA that directly utilizes the orthogonal projection matrices of the DFT matrix on its eigenspaces so that one can save the preliminary step of having to perform a singular value decomposition of those matrices for the sake of finding *initial* orthonormal eigenvectors. One starts by taking the s th column of both sides of (10) to get:

$$\hat{u}_s = V q_s \qquad , s = 1, \cdots, r. \tag{26}$$

Upon substituting (23) and exploiting the orthonormality of the columns of V, one gets:

$$\hat{u}_s = \frac{1}{\|x_s\|} V x_s = \frac{1}{\|z_s\|} z_s \tag{27}$$

where

$$z_s = V x_s \qquad , s = 1, \cdots, r. \tag{28}$$

From (28), (24) and (3), it follows that:

$$z_1 = V V^H u_1 = P u_1 \tag{29}$$

and

$$\begin{aligned} z_s &= V\left(I - C_{s-1}^H C_{s-1}\right) V^H u_s \\ &= \left[\mathrm{V}\mathrm{V}^{\mathrm{H}} - \left(V C_{s-1}^H\right)\left(V C_{s-1}^H\right)^H\right] u_s \qquad , s = 2, \cdots, r \\ &= \left(\mathrm{P} - \hat{\mathrm{U}}_{\mathrm{s-1}} \hat{U}_{s-1}^H\right) u_s \end{aligned} \tag{30}$$

where

$$\hat{U}_{s-1} \equiv V C_{s-1}^H \qquad , s = 2, \cdots, r. \tag{31}$$

By virtue of (25) and (26), the above equation reduces to:

$$\hat{U}_{s-1} = V\left(q_1 \quad \cdots \quad q_{s-1}\right) = \left(\hat{u}_1 \quad \cdots \quad \hat{u}_{s-1}\right). \tag{32}$$

From (9) and (32), it follows that the target matrix $\hat{U}$ is simply $\hat{U}_r$.

Based on (27), (29), (30) and (32), the modified SOPA can be summarized by the following steps:

1) For $s = 1$:

   a) $z_1 = Pu_1$

   b) $\hat{u}_1 = \frac{1}{\|z_1\|} z_1$

   c) $\hat{U} = \hat{u}_1$.

2) For $s = 2, \cdots, r$:

   a) $z_s = \left(\mathrm{P} - \hat{\mathrm{U}}\hat{\mathrm{U}}^{\mathrm{H}}\right) u_s$

   b) $\hat{u}_s = \frac{1}{\|z_s\|} z_s$

   c) Augment matrix $\hat{U}$ by the column vector $\hat{u}_s$.

VI. SIMULATION RESULTS

Our goal is the assessment of the computational performance of the proposed version of the techniques based on the direct utilization of the projection matrices as contrasted to the previous version [7] requiring the singular value decomposition of those matrices for the sake of finding *initial* orthonormal eigenvectors of the DFT matrix F. *Final* Hermite-Gaussian-like eigenvectors of the matrix F of orders $N = 64,\ 128,\ 192,\ 210$ have been computed using each of the three algorithms: OPA, GSA and SOPA. For each algorithm both the proposed and previous versions are used and the ratio between the computation times of the two versions is given in Table 1. Obviously the proposed version has the merit of having a reduced computation time.

VII. CONCLUSION

A new version of each of the GSA, OPA and SOPA – for generating *final* Hermite Gaussian like eigenvectors of the DFT matrix after computing the orthogonal projection matrices on its eigenspaces – has been proposed. It is based on the direct utilization of those matrices rather than finding their singular value decomposition for the sake of generating *initial* orthonormal eigenvectors. The proposed version achieves a definite significant reduction in the computation time.

APPENDIX A

(Proof of Lemma 1)

Let V and W be two N x r matrices each with orthonormal columns that can be regarded as orthonormal basis of an r-dimensional subspace of the N-dimensional complex space $\mathbb{C}^N$ $(r < N)$, i.e.

$$V = \begin{pmatrix} v_1 & \cdots & v_r \end{pmatrix}, \tag{A-1}$$

$$W = \begin{pmatrix} w_1 & \cdots & w_r \end{pmatrix}. \tag{A-2}$$

Representing each column of W in terms of the columns of V, one gets:

$$w_n = \sum_{m=1}^{r} v_m \alpha_{mn} \quad , \ n = 1, \cdots, r . \tag{A-3}$$

The above r vector equations can be compactly expressed as:

$$W = VG \tag{A-4}$$

where G is a square matrix of order r. It follows immediately that:

$$W^H W = G^H \left(V^H V\right) G . \tag{A-5}$$

Consequently the orthonormality of the columns of each of the two matrices V and W implies that:

$$G^H G = I . \tag{A-6}$$

Therefore G is unitary and (A-4) results in:

$$WW^H = VGG^H V^H = VV^H . \tag{A-7}$$

By virtue of (3), it follows that the orthogonal projection matrix P can also be expressed as:

$$P = WW^H . \tag{A-8}$$

(Q.E.D.)

APPENDIX B

(Derivation of the orthogonal procrustes algorithm in the complex domain)

Statement of the problem: Given arbitrary $N \, x \, r$ complex matrices U and V, find the square unitary matrix Q of order r that minimizes:

$$J_a = \|U - VQ\|_F^2 \tag{B-1}$$

where $\|\cdots\|_F$ is the Frobenius norm.

Solution: By virtue of the definition of the Frobenius norm , one gets:

$$\begin{aligned} \|U - VQ\|_F^2 &= tr\left[(U - VQ)^H (U - VQ)\right] \\ &= tr\left(U^H U\right) + tr\left(Q^H V^H VQ\right) - tr\left(Q^H V^H U\right) - tr\left(U^H VQ\right). \end{aligned} \tag{B-2}$$

Upon using the properties of the trace of a matrix $[tr(\cdots)]$ and the unitarity of Q, one obtains:

$$tr\left(Q^H V^H VQ\right) = tr\left(QQ^H V^H V\right) = tr\left(V^H V\right), \tag{B-3}$$

$$tr\left(U^H VQ\right) = tr\left[\left(Q^H V^H U\right)^*\right] = \left[tr\left(Q^H V^H U\right)\right]^*. \tag{B-4}$$

Substituting the above two equations in (B-2), one gets:

$$\|U - VQ\|_F^2 = tr\left(U^H U\right) + tr\left(V^H V\right) - 2Real\left[tr\left(Q^H V^H U\right)\right]. \tag{B-5}$$

Since matrix Q appears only in the last term of the above equation, it follows that minimizing J_a defined by (B-1) is equivalent to maximizing J_b defined by:

$$J_b = Real\left[tr\left(Q^H C\right)\right] \tag{B-6}$$

where C is the square matrix of order r defined by:

$$C = V^H U. \tag{B-7}$$

The full size singular value decomposition of C is:

$$C = ADB^H \tag{B-8}$$

where A and B are unitary matrices of order r and D is the real diagonal matrix of the singular values:

$$D = Diag\{\sigma_1, \cdots, \sigma_r\} \tag{B-9}$$

with $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r \geq 0$.

It follows from (B-8) that:

$$tr\left(Q^H C\right) = tr\left(Q^H ADB^H\right) = tr\left(B^H Q^H AD\right) = tr(ZD) \tag{B-10}$$

where Z is a square matrix of order r defined by:

$$Z = B^H Q^H A. \tag{B-11}$$

The unitarity of A, B and Q implies the unitarity of Z. From (B-6), (B-9) and (B-10) it follows that:

$$J_b = \sum_{k=1}^{r} Real\left(z_{kk}\right)\sigma_k. \tag{B-12}$$

The unitarity of matrix Z implies that its k th column has a unity Euclidean norm and consequently $|z_{kk}| \le 1$ with the equality holding iff the k th column is ce_k where e_k is the k th unit vector and c is a complex scalar that satisfies $|c| = 1$. Since $Real(z_{kk}) \le |z_{kk}|$ with the equality holding iff z_{kk} is real and positive, it follows that $Real(z_{kk}) \le 1$ with the equality holding iff the k th column is e_k. Consequently (B-12) results in

$$J_b \le \sum_{k=1}^{r} \sigma_k \qquad \text{(B-13)}$$

with the equality holding iff the columns of Z are e_k , $k = 1, \cdots, r$. Therefore the functional J_b is maximized when

$$Z = I . \qquad \text{(B-14)}$$

From (B-11) and (B-14), one obtains[8]:

$$Q^H = BA^H \qquad \text{(B-15)}$$

and consequently the desired matrix Q is given by:

$$Q = AB^H . \qquad \text{(B-16)}$$

(Q.E.D.)

Table 1: The ratio between the computation times of the proposed and previous versions of the algorithms

N	Algorithm		
	OPA	GSA	SOPA
64	0.2619	0.25	0.4818
128	0.1423	0.1609	0.6743
192	0.1316	0.1394	0.6209
210	0.1117	0.1449	0.5959

[8] This step necessitates that the singular value decomposition used in (B-8) be the full size version rather than an economy size version.

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