On The Generalized Eigenvectors of a Class of Moment Matrices[1]

By

Magdy T. Hanna[2]

ABSTRACT

An investigation is made of the Eigenstructure of a class of lower triangular moment matrices which arose in the context of finding the forced response of IIR filters to typical excitations. It is found that the Jordan matrix can have at most two types of Jordan blocks. The modal matrix is shown to have a peculiar structure where the progenitors in the column partitions corresponding to the Jordan blocks have a certain pattern.

I. INTRODUCTION

Holtz and Campbell [1] presented an elegant technique for finding the forced response of an IIR filter to a typical set of excitation functions (i.e., polynomials, geometric progressions, sinusoids and products of these sequences). A class of moment matrices emerged and they proved that it has nice mathematical properties such as closure under matrix multiplication (which is commutative) and matrix inversion. The main objective of this paper is to investigate the Eigenstructure of this important class of matrices in search for a more complete characterization of its properties.

An IIR digital filter is described by the difference equation:

$$\sum_{j=0}^{N} d_j y_{n-j} = \sum_{j=-L}^{M} c_j x_{n-j} \tag{1}$$

where x_n and y_n are respectively the input and output sequences and d_j and c_j are constant coefficients. A typical excitation sequence having the form :

$$x_n = u^n \left(a_1 n^{r-1} + a_2 n^{r-2} + \cdots + a_{r-1} n + a_r\right) \tag{2}$$

can be compactly expressed as :

$$x_n = u^n TA \tag{3}$$

where T and A are the r-dimensional row and column vectors defined respectively by :

$$T = \begin{pmatrix} n^{r-1} & n^{r-2} & \cdots & 1 \end{pmatrix} \tag{4}$$

and

$$a = \begin{pmatrix} a_1 & a_2 & \cdots & a_r \end{pmatrix}^T \ . \tag{5}$$

---

[1] EDICS category : SP 2.1

[2]Mailing address : Magdy Tawfik Hanna : P.O. Box 568 , Alexandria, Egypt.

The corresponding forced response can be expressed as :

$$y_n = u^n TB \tag{6}$$

where

$$B = \begin{pmatrix} b_1 & b_2 & \cdots & b_r \end{pmatrix}^T \ . \tag{7}$$

It was shown [1] that the output vector B can be evaluated given the input vector A by the equation :

$$B = W_r^{-1} H_r A \tag{8}$$

where H_r and W_r are lower triangular moment matrices of order r defined respectively by :

$$H_r(i,j) = \begin{cases} \begin{pmatrix} r-j \\ r-i \end{pmatrix} G_{i-j} & 1 \le j \le i \le r \\ 0 & i < j \end{cases} \tag{9}$$

and

$$W_r(i,j) = \begin{cases} \begin{pmatrix} r-j \\ r-i \end{pmatrix} Q_{i-j} & 1 \le j \le i \le r \\ 0 & i < j \end{cases} \tag{10}$$

In the above equations $\begin{pmatrix} r \\ k \end{pmatrix}$ denotes the binomial coefficient and G_m and Q_m are the m th generalized moments - corresponding respectively to the input and output coefficients appearing in (1) - defined by :

$$G_m = \sum_{j=-L}^{M} c_j u^{-j} (-j)^m \tag{11}$$

and

$$Q_m = \sum_{j=0}^{N} d_j u^{-j} (-j)^m \ . \tag{12}$$

Define the r-dimensional vectors G and Q as[3] :

$$G = \begin{pmatrix} G_0 & G_1 & \cdots & G_{r-1} \end{pmatrix}^T \tag{13}$$

$$Q = \begin{pmatrix} Q_0 & Q_1 & \cdots & Q_{r-1} \end{pmatrix}^T \ . \tag{14}$$

One should notice that the matrices H_r and W_r of (8) are functions of the parameter u which is the base of the geometric progression term u^m appearing in (2), (3) and (6). By taking $u = e^{\gamma}$ where γ is a complex number, one can obtain the forced responses of the IIR

[3] The rows and columns of all matrices and vectors (apart from vectors G and Q defined by (13) and (14)) are indexed from 1 to r instead of from 0 to r as was done in [1].

filter to the products of exponentials, polynomials and sinusoids. One should also notice that vector T appearing in (3) and (6) and defined by (4) is time dependent.

The matrices H_r of (9) (or W_r of (10)) - irrespective of the definition of G_m (or Q_m) - are the moment matrices whose mathematical properties were investigated by Holtz and Campbell and whose Eigenstructure will be investigated below.

II. The Eigenstructure

Since H_r is a triangular matrix with equal diagonal elements, all of its Eigenvalues are equal to the diagonal element G_0; in other words H_r has one Eigenvalue $\lambda = G_0$ with algebraic multiplicity r. The geometric multiplicity of this Eigenvalue, i.e. the number of linearly independent Eigenvectors associated with it, depends on the location of the first nonzero element (not counting G_0) of the vector G defined by (13). Let G_k be this element, i.e. ,

$$G_1 = G_2 = \cdots = G_{k-1} = 0 \quad \text{and} \quad G_k \neq 0 \ .$$

Here the smallest submatrix containing the nonzero elements of the matrix $(H_r - \lambda I)$ is a nonsingular lower triangular matrix of order (r - k). Consequently, the null space of $(H_r - \lambda I)$ has the dimension k. Therefore the matrix H_r has k linearly independent Eigenvectors and (r - k) generalized Eigenvectors [2-4]. Let m be the index of the Eigenvalue $\lambda = G_0$,i.e. let the largest Jordan block of H_r be m x m. Since it can be shown that for any r x r matrix C, m is the smallest integer such that [5, p. 132] :

$$\text{rank } (C - \lambda I)^m = r - \text{algebraic multiplicity of } \lambda \tag{15}$$

we find that : rank $(H_r - \lambda I)^m = 0$ and consequently :

$$(H_r - \lambda I)^m = 0 \ . \tag{16}$$

In view of the particular structure of the matrix $(H_r - \lambda I)$, the smallest submatrix containing the nonzero elements of the matrix $(H_r - \lambda I)^{(m-1)}$ is a nonsingular lower triangular submatrix of order :

$$q = r - (m-1)k \ . \tag{17}$$

Since multiplying $(H_r - \lambda I)^{(m-1)}$ by $(H_r - \lambda I)$ increases the number of zero rows by k, Eq.(16) can be satisfied only if $q \leq k$ and hence $m \geq r/k$, i.e.

$$m = \lceil r/k \rceil \tag{18}$$

where $\lceil x \rceil$ is the smallest integer greater than or equal to x.

Since the last k columns of the matrix $\left(H_r - \lambda I\right)$ are zero, the k linearly independent Eigenvectors of matrix H_r can be taken as the last k columns of the r x r identity matrix, i.e.,

$$x_i = e_i \qquad i = r-k+1,\cdots,r \tag{19}$$

where e_i is the unit vector whose i th element is unity and whose other elements are zeros. Let M and J be respectively the modal and Jordan matrices associated with the matrix H_r . The matrix M has k column partitions corresponding to the k Eigenvectors. Let f_i , i = 1 , ... , k be the progenitors, i.e. the first (numbered from the right for the sake of the clarity of the presentation) generalized Eigenvectors in these column partitions. Therefore the matrix M has the form :

$$M = \left(e_{r+1-k} \cdots f_k \mid \cdots \mid e_{r-1} \cdots f_2 \mid e_r \cdots f_1\right) \tag{20}$$

where in each partition the dots between the leading generalized Eigenvector f_i and the Eigenvector e_{r+1-i} represent other generalized Eigenvectors (if any) which can be obtained from f_i by successive multiplication by $\left(H_r - \lambda I\right)$.

First the progenitors f_i corresponding to the largest m x m Jordan blocks will be found. Those generalized Eigenvectors should satisfy :

$$\left(H_r - \lambda I\right)^m f_i = 0$$

and

$$\left(H_r - \lambda I\right)^{(m-1)} f_i = e_{r+1-i} \ . \tag{21}$$

Since only the elements in the lower left q x q triangular part of the matrix $\left(H_r - \lambda I\right)^{(m-1)}$ are nonzero where q is given by (17), Equation (21) can be consistent only for r ≥ r+1-i ≥ (r - q) + 1 ,i.e. only for 1 ≤ i ≤ q . Moreover by taking the last (r - q) elements of each of the vectors f_i to be zero, the first q elements of each of these vectors can be uniquely determined from (21). Exploiting the lower triangular nature of $\left(H_r - \lambda I\right)^{(m-1)}$, we find that the first q progenitors f_i of the matrix H_r corresponding to the m x m Jordan blocks are given by :

$$f_i = \sum_{j=q+1-i}^{q} \alpha_{ij} e_j \qquad 1 \le i \le q \tag{22}$$

where for each i at least $\alpha_{i,q+1-i} \neq 0$.

Second if k > q , the progenitors f_i (i > q) corresponding to the (m-1) x (m-1) Jordan blocks will be found. Those progenitors should satisfy :

$$\left(H_r - \lambda I\right)^{(m-1)} f_i = 0$$

and

$$\left(H_r - \lambda I\right)^{(m-2)} f_i = e_{r+1-i} \quad . \tag{23}$$

The nonzero elements of the matrix $\left(H_r - \lambda I\right)^{(m-2)}$ lying in its lower left part form a nonsingular lower triangular matrix of order (q + k) where q is given by (17); consequently Eq.(23) is consistent only for r ≥ r+1-i ≥ r-(q+k)+1 ,i.e. for 1 ≤ i ≤ q+k . Since there are only a total of k progenitors, the remaining (k-q) progenitors f_i (q+1 ≤ i ≤ k) can be evaluated from Eq.(23) . By taking the last (r-(q+k)) elements of each of those vectors f_i to be zero, the first (q+k) elements of each of them can be uniquely determined from Eq. (23) . Exploiting the lower triangular nature of $\left(H_r - \lambda I\right)^{(m-2)}$ we find that the last (k-q) progenitors f_i corresponding to the (m-1) x (m-1) Jordan blocks are given by :

$$f_i = \sum_{j=q+k+1-i}^{q+k} \alpha_{ij} e_j \qquad q+1 \le i \le k \tag{24}$$

where for each i at least $\alpha_{i,q+k+1-i} \neq 0$.

Therefore the Jordan matrix of the matrix H_r corresponding to the modal matrix (20) is given by :

$$J = Diag\{J_k, \cdots, J_1\} \tag{25}$$

where the Jordan blocks J_i are m x m for 1 ≤ i ≤ q and (m-1) x (m-1) for (q+1) ≤ i ≤ k (if k > q), i.e. all Jordan blocks are of two types at most.

In the special case of m = 2 , q Jordan blocks will be 2 x 2 and the remaining (k-q) blocks (if any) will be 1 x 1 , i.e. the corresponding column partitions of the modal matrix have no generalized Eigenvectors. In the particular case of k = 1 there will be only one Jordan block of dimension r x r ; this is actually the case of simple degeneracy where all generalized Eigenvectors are associated with the single Eigenvector e_r .

The particular form of the progenitors given by Eqs. (22) and (24) and of the remaining generalized Eigenvectors (if any) obtained by successive multiplication by $\left(H_r - \lambda I\right)$ results in a certain structure of the modal matrix M of Eq (20), which will be illustrated in the examples given in the following section.

III. Illustrative Examples

Example 1 :

r = 12 and k = 4 .

From (18) and (17), we get m = 3 and q = 4 . Since q = k , all Jordan blocks are 3 x 3 . The Jordan matrix is :

$$J = Diag\{J_a, J_a, J_a, J_a\}$$

where each Jordan block is given by :

$$J_a = \begin{bmatrix} P_0 & 1 & 0 \\ 0 & P_0 & 1 \\ 0 & 0 & P_0 \end{bmatrix}.$$

The corresponding modal matrix is shown in Fig. 1 where an x represents a nonzero element (the rest of the elements are zero) and where the columns are numbered from the right to the left in accordance with the presentation of the previous section.

Example 2 :

r = 14 and k = 6 .

From (18) and (17) we get m = 3 and q = 2 . The Jordan matrix is :

$$J = Diag\{J_b, J_b, J_b, J_b, J_a, J_a\}$$

where

$$J_a = \begin{bmatrix} P_0 & 1 & 0 \\ 0 & P_0 & 1 \\ 0 & 0 & P_0 \end{bmatrix} \quad \text{and} \quad J_b = \begin{bmatrix} P_0 & 1 \\ 0 & P_0 \end{bmatrix}$$

and the corresponding modal matrix is shown in Fig. 2 .

Example 3 :

r = 24 and k = 7 .

Here m = 4 and q = 3 . Therefore 3 Jordan blocks are 4 x 4 and 4 blocks are 3 x 3 . The modal matrix is shown in Fig. 3 .

Example 4 :

r = 6 and k = 1 .

Here m = 6 and q = 1 . This is the simple degeneracy case where there is only 1 Jordan block of order 6 . The modal matrix is shown in Fig. 4 .

IV. CONCLUSION

The Eigenvalue problem of a certain class of moment matrices is considered. It is shown that the Jordan matrix has no more than 2 types of Jordan blocks. the modal matrix has a corresponding certain pattern of zero and nonzero elements.

REFERENCES

[1] H. Holtz and B.J. Campbell, “An approach to solving linear constant coefficient difference and differential equation,” *IEEE Transactions on Acoustics, Speech and Signal Processing*, vol. 38, no. 4, pp. 654-662, April 1990.

[2] J.H. Wilkinson, *The Algebraic Eigenvalue Problem*, Oxford University Press, London, 1965.

[3] G. Strang, *Linear Algebra and its Applications*, 2 nd edition, Academic Press, New York, 1980.

[4] A.S. Deif, *Advanced Matrix Theory for Scientists and Engineers*, John Wiley & Sons, New York, 1982.

[5] W.L. Brogan, *Modern Control Theory*, Prentice Hall / Quantum Publishers, Englewood Cliffs, New Jersey, 1982.

$$
M = \begin{array}{cc}
\begin{bmatrix}
0 & 0 & x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & x & 0 & 0 & x & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & x & 0 & 0 & x & 0 & 0 & x & 0 & 0 & 0 \\
0 & 0 & x & 0 & 0 & x & 0 & 0 & x & 0 & 0 & x \\
0 & x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & x & 0 & 0 & x & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & x & 0 & 0 & x & 0 & 0 & x & 0 & 0 & 0 & 0 \\
0 & x & 0 & 0 & x & 0 & 0 & x & 0 & 0 & x & 0 \\
x & x & 0 & 0 & x & 0 & 0 & x & 0 & 0 & x & 0 \\
0 & x & 0 & x & x & 0 & 0 & x & 0 & 0 & x & 0 \\
0 & x & 0 & 0 & x & 0 & x & x & 0 & 0 & x & 0 \\
0 & x & 0 & 0 & x & 0 & 0 & x & 0 & x & x & 0
\end{bmatrix} &
\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \end{matrix} \\
\begin{matrix} 12 & & 10 & & 8 & & 6 & & 4 & & 2 & 1 \end{matrix} &
\end{array}
$$

Fig. 1 : The modal matrix of example 1

(An x represents a nonzero element)

$$
M = \begin{array}{cc}
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & 0 & 0 & x \\
0 & x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & x & 0 & x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & x & 0 & x & 0 & x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & x & 0 & x & 0 & x & 0 & x & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & x & 0 & x & 0 & x & 0 & x & 0 & x & 0 & 0 & 0 & 0 \\
0 & x & 0 & x & 0 & x & 0 & x & 0 & x & 0 & 0 & x & 0 \\
x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & 0 & 0 & x & 0 \\
0 & 0 & x & 0 & 0 & 0 & 0 & 0 & 0 & x & 0 & 0 & x & 0 \\
0 & 0 & 0 & 0 & x & 0 & 0 & 0 & 0 & x & 0 & 0 & x & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & x & 0 & 0 & x & 0 & 0 & x & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & x & 0 & 0 & x & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & 0 & x & x & 0
\end{bmatrix} &
\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \\ 13 \\ 14 \end{matrix} \\
\begin{matrix} 14 & & 12 & & 10 & & 8 & & 6 & & 4 & & 2 & 1 \end{matrix} &
\end{array}
$$

Fig. 2 : The modal matrix of example 2

(An x represents a nonzero element)

$$M = \begin{array}{cc}
\begin{bmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&x&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&x&0&0&0&x&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&x&0&0&0&x&0&0&0&x\\
0&0&x&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&x&0&0&x&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&x&0&0&x&0&0&x&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&x&0&0&x&0&0&x&0&0&x&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&x&0&0&x&0&0&x&0&0&x&0&0&x&0&0&0&0&0&0&0&0&0\\
0&0&x&0&0&x&0&0&x&0&0&x&0&0&x&0&0&0&x&0&0&0&0&0\\
0&0&x&0&0&x&0&0&x&0&0&x&0&0&x&0&0&0&x&0&0&0&x&0\\
0&x&0&0&0&0&0&0&0&0&0&0&0&0&x&0&0&0&x&0&0&0&x&0\\
0&x&0&0&x&0&0&0&0&0&0&0&0&0&x&0&0&0&x&0&0&0&x&0\\
0&x&0&0&x&0&0&x&0&0&0&0&0&0&x&0&0&0&x&0&0&0&x&0\\
0&x&0&0&x&0&0&x&0&0&x&0&0&0&x&0&0&0&x&0&0&0&x&0\\
0&x&0&0&x&0&0&x&0&0&x&0&0&x&x&0&0&0&x&0&0&0&x&0\\
0&x&0&0&x&0&0&x&0&0&x&0&0&x&x&0&0&x&x&0&0&0&x&0\\
0&x&0&0&x&0&0&x&0&0&x&0&0&x&x&0&0&x&x&0&0&x&x&0\\
x&x&0&0&x&0&0&x&0&0&x&0&0&x&x&0&0&x&x&0&0&x&x&0\\
0&x&0&x&x&0&0&x&0&0&x&0&0&x&x&0&0&x&x&0&0&x&x&0\\
0&x&0&0&x&0&x&x&0&0&x&0&0&x&x&0&0&x&x&0&0&x&x&0\\
0&x&0&0&x&0&0&x&0&x&x&0&0&x&x&0&0&x&x&0&0&x&x&0\\
0&x&0&0&x&0&0&x&0&0&x&0&x&x&x&0&0&x&x&0&0&x&x&0\\
0&x&0&0&x&0&0&x&0&0&x&0&0&x&x&0&x&x&x&0&0&x&x&0\\
0&x&0&0&x&0&0&x&0&0&x&0&0&x&x&0&0&x&x&0&x&x&x&0
\end{bmatrix} &
\begin{matrix} 1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12\\13\\14\\15\\16\\17\\18\\19\\20\\21\\22\\23\\24 \end{matrix} \\
\begin{matrix} & & 22 & & & 19 & & & 16 & & & 13 & & & & 9 & & & 5 & & & & 1 & \end{matrix} &
\end{array}$$

Fig. 3 : The modal matrix of example 3

(An x represents a nonzero element)

$$M = \begin{array}{cc}
\begin{bmatrix}
0&0&0&0&0&x\\
0&0&0&0&x&0\\
0&0&0&x&x&0\\
0&0&x&x&x&0\\
0&x&x&x&x&0\\
x&x&x&x&x&0
\end{bmatrix} &
\begin{matrix} 1\\2\\3\\4\\5\\6 \end{matrix} \\
\begin{matrix} 6 & 5 & 4 & 3 & 2 & 1 \end{matrix} &
\end{array}$$

Fig. 4 : The modal matrix of example 4

(An x represents a nonzero element)