ON CONVERGENCE IN L-VALUED FUZZY TOPOLOGICAL SPACES

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Abstract. In this paper, we introduce the concept of L-fuzzy neighborhood systems using complete MV-algebras and present important links with the theory of L-fuzzy topological spaces. We investigate the relationships among the degrees of L-fuzzy r-adherent points (r-convergent, r-cluster and r-limit, respectively) in an L-fuzzy topological spaces. Also, we investigate the concept of LF- continuous functions and their properties.

Keywords: Complete MV- algebra, L -fuzzy topological spaces, L- neighborhood systems, r-convergent, r-cluster points, r-limit points.

0. Introduction

Šostak [25-29] introduced a new definition of L - fuzzy topology as the concept of the degree of the openness of fuzzy set. It is an extension of $I = [0, 1]$- fuzzy topology defined by Chang [1]. It has been developed in many directions [5,12-16,19]. The study of neighborhood systems and convergence of nets in Chang fuzzy topology was initiated by Pu Pao-Ming and Liu Yin Ming [19] and Liu Ying-Ming, Luo Mao-Kang [18]. In [33] M.S. Ying, introduced the degree to which a fuzzy point x_t belongs to a fuzzy subset λ by $m(x_t, \lambda) = min(1, 1 - t + \lambda(x))$ and gave the idea of graded neighborhood on fuzzy topological spaces. This plays an important role in the theory of convergence in Chang fuzzy topology see also [3,4,7,8,32]. Following M.S.Ying [33], Demirci [5] introduced the idea of graded neighborhood systems in smooth toplogical spaces [20] (a smooth topology is similar to fuzzy topology as defined by Šostak [25], Hazra and Samanta [12]) in a different approach but restricted himself to the I- valued fuzzy sets.

In this paper, we study the concept of L-fuzzy neighborhood systems and present important links with the theory of L-fuzzy topological spaces and investigate some of their

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properties. We investigate the relationships among the degrees of L-fuzzy r-adherent points (r-convergent, r-cluster and r-limit, respectively) nets in an L-fuzzy topological spaces. Also, we give some related examples to illustrate some of the introduced notions. In the end, we characterize LF- continuous functions in terms of some of the various notions introduced in this paper.

1.Preliminaries

Throughout the text we consider $(L, \leq, \wedge, \vee, 0, 1)$ as a completely distributive lattice with 0 and 1, respectively, being the universal upper and lower bound and $L_0 = L - \{0\}$. A lattice L is called order dense if for each $a, b \in L$ such that $a < b$, there exist $c \in L$ such that $a < c < b$. If L is a completely distributive lattice and $x \lhd \bigvee_{i\in\Gamma} y_i$, then there must be $i_0 \in \Gamma$ such that $x \lhd y_{i_0}$, where $x \lhd a$ means: $K \subset L$, $a \leq \bigvee K \Rightarrow \exists y \in K$ such that $x \leq y$. If $a \lhd b$ and $c \lhd d$, it is always has $a \wedge c \lhd b \wedge d$ [10] and some properties of $\lhd$ can be found in [18].

A completely distributive lattice $L = (L, \leq, \wedge, \vee, \odot, \rightarrow, 0, 1)$ (or L, in short) is called a residuated lattice [11,15,28,29] if it satisfies the following conditions: for each $x, y, z \in L$,

(R1) $(L, \odot, 1)$ is a commutative monoid,

(R2) if $x \leq y$, then $x \odot z \leq y \odot z$ ($\odot$ is isotone operation),

(R3) (Galois correspondence) $x \leq y \rightarrow z \Leftrightarrow x \odot y \leq z$.

In a residuated lattice L, $x' = x \rightarrow 0$ is called complement of $x \in L$.

A residuated lattice L is called a BL - algebra [11,15,29] if it satisfies the following conditions: for each $x, y, z \in L$,

(B1) $x \wedge y = x \odot (x \rightarrow y)$,

(B2) $x \vee y = [(x \rightarrow y) \rightarrow y] \wedge [(y \rightarrow x) \rightarrow x]$,

(B3) $(x \rightarrow y) \vee (y \rightarrow x) = 1$.

A BL - algebra is called an MV - algebra if $x = x''$, for each $x \in L$.

Lemma 1.1 [11,15,29]. Let L be a complete MV - algebra. For each $x, y, z \in L$, $\{y_i, x_i \mid i \in \Gamma\} \subset L$, we have the following properties.

(1) $x \odot y \leq x \wedge y \leq x \vee y$,

(2) $x \odot y \leq x, y$,

(3) If $y \leq z$, $(x \odot y) \leq (x \odot z)$, $x \rightarrow y \leq x \rightarrow z$ and $z \rightarrow x \leq y \rightarrow x$,

(4) $x \odot y = (x \rightarrow y')'$,

(5) $x \leq y$ iff $x' \geq y'$,

(6) $x \rightarrow y = y' \rightarrow x'$,

(7) $\bigwedge_{i\in\Gamma}(x \odot y_i) = x \odot (\bigwedge_{i\in\Gamma} y_i)$,

(8) $\bigvee_{i\in\Gamma}(x \odot y_i) = x \odot (\bigvee_{i\in\Gamma} y_i)$,

(9) $x \rightarrow 1 = 1, 0 \rightarrow x = 1, x \rightarrow x = 1$,

(10) $x \leq y \Leftrightarrow x \rightarrow y = 1$ and $1 \rightarrow x = x$,

(11) $x \rightarrow \bigwedge_{i\in\Gamma} y_i = \bigwedge_{i\in\Gamma}(x \rightarrow y_i)$,

(12) $(\bigvee_{i\in\Gamma} y_i) \rightarrow x = \bigwedge_{i\in\Gamma}(y_i \rightarrow x)$,

(13) $x \rightarrow \bigvee_{i\in\Gamma} y_i = \bigvee_{i\in\Gamma}(x \rightarrow y_i)$,

(14) $\bigwedge_{i\in\Gamma} y_i \rightarrow x = \bigvee_{i\in\Gamma}(y_i \rightarrow x)$,

(15) $\bigwedge_{i\in\Gamma} y_i' = (\bigvee_{i\in\Gamma} y_i)'$ and $\bigvee_{i\in\Gamma} y_i' = (\bigwedge_{i\in\Gamma} y_i)'$.

In this paper, we always assume that L is a complete MV - algebra. Let X be a nonempty set, the family L^X denotes the set of all L- fuzzy subsets of a given set X. For $\alpha \in L, \lambda \in L^X$, we denote $(\alpha \to \lambda), (\alpha \odot \lambda), \alpha_X \in L^X$ as $(\alpha \to \lambda)(x) = \alpha \to \lambda(x), (\alpha \odot \lambda)(x) = \alpha \odot \lambda(x), \alpha_X(x) = \alpha$.

A fuzzy point x_t for $t \in L_0$ is an element of L^X such that

$$x_t(y) = \begin{cases} t, & \text{if } y = x, \\ 0, & \text{if } y \neq x. \end{cases}$$

The set of all fuzzy points in X is denoted by $Pt(X)$. For $\lambda \in L^X$ and $x_t \in Pt(X), x_t \in \lambda$ iff $t \leq \lambda(x)$.

Given a mapping $\phi : X \to Y$, we write $\phi^{\leftarrow}$ for the mapping $L^Y \to L^X$ defined by $\phi^{\leftarrow}(\mu) = \mu \circ \phi$; and we write $\phi^{\rightarrow}$ for the mapping $L^X \to L^Y$ defined by $\phi^{\rightarrow}(\mu)(y) = \bigvee\{\mu(x) \mid \phi(x) = y\}$ for all $\mu \in L^X, y \in Y$.

For a given set X, define a binary mapping $S(-,-) : L^X \times L^X \to L$ as

$$S(\lambda, \mu) = \bigwedge_{x \in X} (\lambda(x) \to \mu(x)), \qquad \forall (\lambda, \mu) \in L^X \times L^X.$$

For each $\lambda, \mu \in L^X, S(\lambda, \mu)$ can be interpreted as the degree to which λ is fuzzy included in μ. It is called the L-fuzzy inclusion order [6].

Lemma 1.2 [6]. For each $\lambda, \mu, \rho, \mu_i \in L^X$, $i \in \Gamma$ and $e, x_t \in Pt(X)$, the following properties hold:
(1) $\lambda \leq \mu \Leftrightarrow S(\lambda, \mu) = 1$,
(2) $\lambda \leq \mu \Rightarrow S(\rho, \lambda) \leq S(\rho, \mu)$ and $S(\lambda, \rho) \geq S(\mu, \rho)$, for any $\rho \in L^X$,
(3) $S(x, \lambda) = \lambda(x)$; for any $\lambda \in L^X$,
(4) $S(x_t, \lambda) = 0$ iff $t = 1$ and $\lambda(x) = 0$,
(5) $S(e, \lambda) \wedge S(e, \mu) = S(e, \lambda \wedge \mu)$,
(6) $S(x_t, \bigwedge_{i \in \Gamma} \mu_i) = \bigwedge_{i \in \Gamma} S(x_t, \mu_i)$, for any $\{\mu_i\}_{i \in \Gamma} \subset L^X$,
(7) $S(x_t, \bigvee_{i \in \Gamma} \mu_i) = \bigvee_{i \in \Gamma} S(x_t, \mu_i)$, for any $\{\mu_i\}_{i \in \Gamma} \subset L^X$.

Lemma 1.3 [7]. Let $f : X \to Y$ be a mapping. Then the following statement hold:
(1) $S(\lambda, \mu) \leq S(f^{\rightarrow}(\lambda), f^{\rightarrow}(\mu))$, for each $\lambda, \mu \in L^X$
(2) $S(\rho, \nu) \leq S(f^{\leftarrow}(\rho), f^{\leftarrow}(\nu))$for each $\rho, \nu \in L^Y$.
In particular, if the mapping $f : X \to Y$ is bijective, then the equalities hold.

Definition 1.4 [15,25]. A map $\mathcal{T} : L^X \to L$ is called an *L-fuzzy topology* on X if it satisfies the following conditions:
(LO1) $\mathcal{T}(1_X) = \mathcal{T}(0_X) = 1$,
(LO2) $\mathcal{T}(\mu_1 \wedge \mu_2) \geq \mathcal{T}(\mu_1) \wedge \mathcal{T}(\mu_2)$, for all $\mu_1, \mu_2 \in L^X$,
(LO3) $\mathcal{T}(\bigvee_{i \in \Lambda} \mu_i) \geq \bigwedge_{i \in \Lambda} \mathcal{T}(\mu_i)$, for any $\{\mu_i\}_{i \in \Lambda} \subset L^X$.
The pair $(X, \mathcal{T})$ is called an *L-fuzzy topological space.*

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be L-fuzzy topologies on X. We say that $\mathcal{T}_1$ is *finer* than $\mathcal{T}_2$ ($\mathcal{T}_2$ is *coarser* than $\mathcal{T}_1$), denoted by $\mathcal{T}_2 \leq \mathcal{T}_1$, if $\mathcal{T}_2(\lambda) \leq \mathcal{T}_1(\lambda)$ for all $\lambda \in L^X$. Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be L-fuzzy topological space spaces. A map $f : (X, \mathcal{T}_1) \longrightarrow (Y, \mathcal{T}_2)$ is L-fuzzy continuous (LF- continuous, for short) if $\mathcal{T}_2(\lambda) \leq \mathcal{T}_1(f^{\leftarrow}(\lambda)), \forall \lambda \in L^Y$.

Theorem 1.5 [13,15]. Let $(X, \mathcal{T})$ be an L -fuzzy topological space. For each $r \in L_0$ and $\lambda \in L^X$, we define operators $I_\mathcal{T}, C_\mathcal{T} : L^X \times L_0 \to L^X$ as follows:

$$I_\mathcal{T}(\lambda, r) = \bigvee \{\rho \in L^X \mid \rho \leq \lambda, \mathcal{T}(\rho) \geq r\},$$

$$C_\mathcal{T}(\lambda, r) = \bigwedge \{\nu \in L^X \mid \lambda \leq \nu, \mathcal{T}(\nu') \geq r\}.$$

For each $\lambda, \mu \in L^X$ and $r, s \in L_0$, we have the following properties:
(I1) $\mathcal{I}_\mathcal{T}(1_X, r) = 1_X$,
(I2) $\mathcal{I}_\mathcal{T}(\lambda, r) \leq \lambda$,
(I3) If $\lambda \leq \mu$ and $r \leq s$, then $\mathcal{I}_\mathcal{T}(\lambda, s) \leq \mathcal{I}_\mathcal{T}(\mu, r)$,
(I4) $\mathcal{I}_\mathcal{T}(\lambda \wedge \mu, r \wedge s) \geq \mathcal{I}_\mathcal{T}(\lambda, r) \wedge \mathcal{I}_\mathcal{T}(\mu, s)$,
(I5) $\mathcal{I}_\mathcal{T}(\mathcal{I}_\mathcal{T}(\lambda, r), r) = \mathcal{I}_\mathcal{T}(\lambda, r)$,
(I6) $\mathcal{I}_\mathcal{T}(\lambda', r) = (\mathcal{C}_\mathcal{T}(\lambda, r))'$.

Definition 1.6[18]. Let D be a directed set. A function $T : D \to Pt(X)$ is called a fuzzy net in X. Let $\lambda \in L^X$, we say T is a fuzzy net in λ if $T(n) \in \lambda$ for every $n \in D$.

Definition 1.7[17,18]. Let T be a fuzzy net and $\lambda \in L^X$.
(1) T is often in λ if for each $n \in D$, there exists $n_0 \in D$ such that $n_0 \geq n$ and $T(n_0) \in \lambda$.
(2) T is finally in λ if there exists $n_0 \in D$ such that for each $n \in D$ with $n \geq n_0$, we have $T(n) \in \lambda$.

Definition 1.8[17,18]. Let $T : D \to Pt(X)$ and $U : E \to Pt(X)$ be two fuzzy nets. A fuzzy net U is called a subnet of T if there exists a function $N : E \to D$, called by a cofinal selection on T, such that:
(1) $U = T \circ N$.
(2) For every $n_0 \in D$, there exists $m_0 \in E$ such that $N(m) \geq n_0$, for $m \geq m_0$.

2. L - fuzzy neighborhood systems.

Definition 2.1. Let $\lambda \in L^X$ and $x_t \in Pt(X)$. Then the degree to which x_t belongs to λ is

$$S(x_t, \lambda) = \bigwedge_{x \in X} (t \longrightarrow \lambda(x)).$$

Definition 2.2. Let $(X, \mathcal{T})$ be an L - fuzzy topological space , $\lambda \in L^X$, $e \in Pt(X)$ and $r \in L_0$. The degree to which λ is a r-neighborhood of e is defined by

$$(\mathcal{N}^\mathcal{T})_e(\lambda, r) = \bigvee \{S(e, \mu) \mid \mu \leq \lambda,\ r \lhd \mathcal{T}(\mu)\}.$$

A mapping $(\mathcal{N}^\mathcal{T})_e : L^X \times L_0 \to L$ is called *the L-fuzzy neighborhood system* of e.

Theorem 2.3. Let $(X,\mathcal{T})$ be an L - fuzzy topological space and $(\mathcal{N}^{\mathcal{T}})_e$ the fuzzy neighborhood system of e. For all $\lambda,\mu\in L^X$ and $r,s\in L_0$, the following properties hold:

(1) $(\mathcal{N}^{\mathcal{T}})_e(0_X,r)=S(e,0_X)$and $(\mathcal{N}^{\mathcal{T}})_e(1_X,r)=1$,
(2) $(\mathcal{N}^{\mathcal{T}})_e(\lambda,r)\le S(e,\lambda)$,
(3) $(\mathcal{N}^{\mathcal{T}})_e(\lambda,r)\ge(\mathcal{N}^{\mathcal{T}})_e(\lambda,s)$, if $r\le s$,
(4) $(\mathcal{N}^{\mathcal{T}})_e(\lambda,r)\le(\mathcal{N}^{\mathcal{T}})_e(\mu,r)$, if $\lambda\le\mu$,
(5) $(\mathcal{N}^{\mathcal{T}})_e(\lambda_1,r)\wedge(\mathcal{N}^{\mathcal{T}})_e(\lambda_2,s)\le(\mathcal{N}^{\mathcal{T}})_e(\lambda_1\wedge\lambda_2,r\wedge s)$,
(6) $(\mathcal{N}^{\mathcal{T}})_e(\lambda,r)\le\bigvee\{(\mathcal{N}^{\mathcal{T}})_e(\mu,r)\mid\mu\le\lambda,S(d,\mu)\le(\mathcal{N}^{\mathcal{T}})_d(\mu,r)\ \ \forall d\in Pt(X)\}$,
(7) $(\mathcal{N}^{\mathcal{T}})_{x_t}(\lambda,r)=\bigwedge_{x\in X}\Big(t\to(\mathcal{N}^{\mathcal{T}})_{x_1}(\lambda,r)\Big)$.

Proof. (1),(3) and (4) are easily proved.

(2) It is proved from the following:

$$\begin{aligned}(\mathcal{N}^{\mathcal{T}})_e(\lambda,r)&=\bigvee\{S(e,\mu_i)\mid\mu_i\le\lambda,\ r\lhd\tau(\mu)\}\\&\le\bigvee\{S(e,\bigvee\mu_i)\mid\mu_i\le\lambda,\ r\lhd\tau(\mu)\}\\&\qquad\text{(by Lemma 1.2(2))}\\&\le\{S(e,\bigvee\mu_i)\mid\bigvee\mu_i\le\lambda,\ r\le\tau(\bigvee\mu_i)\}\\&\le S(e,\lambda).\end{aligned}$$

(5) If $a\lhd(\mathcal{N}^{\mathcal{T}})_e(\lambda_1,r)\wedge(\mathcal{N}^{\mathcal{T}})_e(\lambda_2,s)$, then $a\lhd(\mathcal{N}^{\mathcal{T}})_e(\lambda_1,r)$ and $a\lhd(\mathcal{N}^{\mathcal{T}})_e(\lambda_2,s)$, there exists $\rho_1\in L^X$ with $\rho_1\le\lambda_1$ and $r\lhd\mathcal{T}(\rho_1)$ such that $a\lhd S(e,\rho_1)$. Again, there exists $\rho_1\in L^X$ with $\rho_2\le\lambda_2$ and $r\lhd\mathcal{T}(\rho_2)$ such that $a\lhd S(e,\rho_2)$. So, $\rho_1\wedge\rho_2\le\lambda_1\wedge\lambda_2$,
$r\wedge s\lhd\mathcal{T}(\rho_1)\wedge\mathcal{T}(\rho_2)$ and $a\le S(e,\rho_1)\wedge S(e,\rho_2)=S(e,\rho_1\wedge\rho_2)\le(\mathcal{N}^{\mathcal{T}})_e(\lambda_1\wedge\lambda_2,r\wedge s)$. Hence,

$$(\mathcal{N}^{\mathcal{T}})_e(\lambda_1\wedge\lambda_2,r\wedge s)\ge(\mathcal{N}^{\mathcal{T}})_e(\lambda_1,r)\wedge(\mathcal{N}^{\mathcal{T}})_e(\lambda_2,s).$$

(6) If $r\lhd\mathcal{T}(\mu)$, then $S(d,\mu)=(\mathcal{N}^{\mathcal{T}})_d(\mu,r)$, for each $d\in Pt(X)$. It implies:

$$\begin{aligned}(\mathcal{N}^{\mathcal{T}})(\lambda,r)&=\bigvee\{S(e,\mu)\mid\mu\le\lambda,\ r\lhd\mathcal{T}(\mu)\}\\&=\bigvee\{(\mathcal{N}^{\mathcal{T}})_e(\mu,r)\mid\mu\le\lambda,S(d,\mu)=(\mathcal{N}^{\mathcal{T}})_d(\mu,r),\ \forall d\in Pt(X)\}\\&\le\bigvee\{(\mathcal{N}^{\mathcal{T}})_e(\mu,r)\mid\mu\le\lambda,\ S(d,\mu)\le(\mathcal{N}^{\mathcal{T}})_d(\mu,r),\ \forall d\in Pt(X)\}.\end{aligned}$$

(7) It proved from:

$$\begin{aligned}(\mathcal{N}^{\mathcal{T}})_{x_t}(\lambda,r)&=\bigvee\{S(x_t,\mu)\mid\mu\le\lambda,\ \mathcal{T}(\mu)\ge r\}\\&=\bigvee\{\bigwedge_{x\in X}(t\to\mu(x))\mid\mu\le\lambda,\ \mathcal{T}(\mu)\ge r\}\\&=\bigwedge_{x\in X}\{t\to\bigvee\{\mu(x)\mid\mu\le\lambda,\ \mathcal{T}(\mu)\ge r\}\}\\&\qquad\text{(by Lemma 1.2(7))}\\&=\bigwedge_{x\in X}(t\to(\mathcal{N}^{\mathcal{T}})_{x_1}(\lambda,r)).\end{aligned}$$

Theorem 2.4. Let X be a nonempty set. Let for each $e \in Pt(X)$, there is be given a mapping $\mathcal{N}_e : L^X \times L_0 \to L$ satisfying the above conditions (1)-(5). Define $\mathcal{T}_\mathcal{N} : L^X \to L$ by

$$\mathcal{T}_\mathcal{N}(\lambda) = \bigvee\{r \in L_0 \mid S(e, \lambda) = \mathcal{N}_e(\lambda, r),\ \forall e \in Pt(X)\}.$$

Then

(a) $\mathcal{T}_\mathcal{N}$ is an L- fuzzy topology on X.

(b) If $(\mathcal{N}^\mathcal{T})_e$ is the L-fuzzy neighborhood system of e induced by $(X, \mathcal{T})$, then $\mathcal{T}_{\mathcal{N}^\mathcal{T}} = \mathcal{T}$.

(c) If $\mathcal{N}_e$'s satisfy the conditions (6) and (7), then

$$\mathcal{T}_\mathcal{N}(\lambda) = \bigvee\{r \in L_0 \mid S(x, \lambda) = \mathcal{N}_x(\lambda, r),\ \forall x \in X\},$$

(d) $\mathcal{N}_{\mathcal{T}_\mathcal{N}} = \mathcal{N}$.

Proof. (a) (LO1) It is easily proved from Theorem 2.3(1).

(LO2) It is proved from the following:

$$\begin{aligned}
&\mathcal{T}_\mathcal{N}(\lambda_1) \wedge \mathcal{T}_\mathcal{N}(\lambda_2)\\
&\qquad = \Big(\bigvee\{r \in L_0 \mid S(e, \lambda_1) = \mathcal{N}_e(\lambda_1, r)\}\Big)\\
&\qquad \wedge \Big(\bigvee\{s \in L_0 \mid S(e, \lambda_2) = \mathcal{N}_e(\lambda_2, s)\}\Big)\\
&\qquad = \bigvee\{r \wedge s \in L_0 \mid S(e, \lambda_1) \wedge S(e, \lambda_2) = \mathcal{N}_e(\lambda_1, r) \wedge \mathcal{N}_e(\lambda_2, s)\}\\
&\qquad \le \bigvee\{r \wedge s \in L_0 \mid S(e, \lambda_1) \wedge S(e, \lambda_2) \le \mathcal{N}_e(\lambda_1 \wedge \lambda_2, r \wedge s)\}\\
&\qquad \le \bigvee\{r \wedge s \in L_0 \mid S(e, \lambda_1 \wedge \lambda_2) \le \mathcal{N}_e(\lambda_1 \wedge \lambda_2, r \wedge s)\}\\
&\qquad\qquad\qquad (\text{ by Lemma 1.2(5)})\\
&\qquad \le \mathcal{T}_\mathcal{N}(\lambda_1 \wedge \lambda_2).
\end{aligned}$$

(LO3)

If $a \triangleleft \bigwedge_{i\in\Gamma} \mathcal{T}_\mathcal{N}(\lambda_i)$, then $a \triangleleft \mathcal{T}_\mathcal{N}(\lambda_i)$ for each $i \in \Gamma$, Note that,

$$\mathcal{T}_\mathcal{N}(\lambda_i) = \bigvee\{r_i \in L_0 \mid S(e, \lambda_i) = \mathcal{N}_e(\lambda_i, r_i),\ \forall e \in Pt(X)\},$$

so, there exists $r_i \in L_0$, with $S(e, \lambda_i) = \mathcal{N}_e(\lambda_i, r_i)$ such that $a \triangleleft r_i$. Put $r = \bigwedge_{i\in\Gamma} r_i$, then $a \le r$. By Theorem 2.3, we have

$$S(e, \lambda_i) \le \mathcal{N}_e(\lambda_i, r_i) \le \mathcal{N}_e(\lambda_i, r) \le S(e, \lambda_i).$$

It implies $S(e, \lambda_i) = \mathcal{N}_e(\lambda_i, r)$. Furthermore, by Lemma 1.2(7), we have

$$\begin{aligned}
S(e, \bigvee_{i\in\Gamma} \lambda_i) &= \bigvee_{i\in\Gamma} S(e, \lambda_i) = \bigvee_{i\in\Gamma} \mathcal{N}_e(\lambda_i, r_i)\\
&\le \bigvee_{i\in\Gamma} \mathcal{N}_e(\lambda_i, r) \le \mathcal{N}_e(\bigvee_{i\in\Gamma} \lambda_i, r) \le S(e, \bigvee_{i\in\Gamma} \lambda_i).
\end{aligned}$$

So, $\mathcal{N}_e(\bigvee_{i\in\Gamma}\lambda_i, r) = S(e, \bigvee_{i\in\Gamma}\lambda_i)$. Hence, $\mathcal{T}_{\mathcal{N}}(\bigvee_{i\in\Gamma}\lambda_i) \geq r \geq a$. Therefore, $\mathcal{T}_{\mathcal{N}}(\bigvee_{i\in\Gamma}\lambda_i) \geq \bigwedge_{i\in\Gamma}\lambda_i(\lambda_i)$.

(b)If $a \lhd \mathcal{T}_{\mathcal{N}}(\lambda)$, then there exists $r_0 \in L_0$ with $S(e,\lambda) = \mathcal{N}_e(\lambda, r_0)$ such that $r_0 \lhd \mathcal{T}(\lambda)$. Since

$$S(e,\lambda) = \mathcal{N}_e(\lambda, r_0) = \bigvee\{S(e,\mu_i) \mid \mu_i \leq \lambda,\ r_0 \lhd \mathcal{T}(\mu_i)\},$$

then, for each $x_1 \in Pt(X)$,

$$\lambda(x) = S(x_1,\lambda) = \bigvee\{S(x_1,\mu_i) \mid \mu_i \leq \lambda, r_0 \lhd \mathcal{T}(\mu_i)\}$$

$$= S(x_1, \bigvee_{i\in\Gamma}\mu_i) = \bigvee_{i\in\Gamma}\mu_i(x).$$

Thus, $\lambda = \bigvee \mu_i$. So, $\mathcal{T}(\lambda) \geq r_0 \geq a$. Hence, $\mathcal{T}_{\mathcal{N}}(\lambda) \leq \mathcal{T}(\lambda)$. We can easily obtained $\mathcal{T}_{\mathcal{N}}(\lambda) \geq \mathcal{T}(\lambda)$.

(c) We only show that $S(x_t,\lambda) = \mathcal{N}_{x_t}(\lambda, r),\ \forall x_t \in Pt(X)$
iff $S(x,\lambda) = \lambda(x) = \mathcal{N}_x(\lambda, r),\ \forall x \in X$.

($\Rightarrow$) It is trivial.

($\Leftarrow$) From the condition (7),

$$\begin{aligned}\mathcal{N}_{x_t}(\lambda, r) &= \bigwedge_{x\in X}\Big(t \to \mathcal{N}_{x_1}(\lambda, r)\Big)\\ &= \bigwedge_{x\in X}\Big(t \to S(x_1,\lambda)\Big)\\ &= \bigwedge_{x\in X}\Big(t \to \lambda(x)\Big)\\ &= S(x_t,\lambda).\end{aligned}$$

(d) From the proof of Theorem 2.3(6), we easily obtain $\mathcal{N}_{\mathcal{T}_{\mathcal{N}}} \geq \mathcal{N}$.

If $a \lhd (\mathcal{N}_{\mathcal{T}_{\mathcal{N}}})_e(\lambda, r) = \bigvee\{S(e,\mu) \mid \mu \leq \lambda,\ r \lhd \mathcal{T}_{\mathcal{N}}(\mu)\}$, there exists μ_0 with $\mu_0 \leq \lambda$, $r \lhd \mathcal{T}_{\mathcal{N}}(\mu_0)$ such that $a \lhd S(e,\mu_0)$. Note that

$$\mathcal{T}_{\mathcal{N}}(\mu_0) = \bigvee\{t \in L_0 \mid S(e,\mu_0) = \mathcal{N}_e(\mu_0, t),\ \forall e \in Pt(X)\},$$

there exists $t_0 \in L_0$ with $S(e,\mu_0) = \mathcal{N}_e(\mu_0, t_0)$ such that $r \lhd t_0$ (thus $r \leq t_0$). So, $a \lhd \mathcal{N}_e(\mu_0, t_0) \leq \mathcal{N}_e(\mu_0, r) \leq \mathcal{N}_e(\lambda, r)$. Therefore, $\mathcal{N}_{\mathcal{T}_{\mathcal{N}}} \leq \mathcal{N}$.

By Theorem 2.4, we have the following Corollary:

Corollary 2.5. The set of all L - fuzzy topologies on X and the set of all L - fuzzy neighborhood systems on X are in one to one correspondence.

Example 2.6. Let $L = [0, 1]$, $X = \{a, b\}$ be a set, $x \to y = min(1 - x + y, 1)$ and $\mu \in L^X$ be defined as follows:

$$\mu(a) = 0.3,\ \mu(b) = 0.4.$$

We define an L- fuzzy topology on X as:

$$\mathcal{T}(\lambda) = \begin{cases} 1, & \text{if } \lambda = 0_X \text{ or } 1_X, \\ \frac{1}{2}, & \text{if } \lambda = \mu, \\ 0, & \text{otherwise.} \end{cases}$$

From Definition 2.2, $\mathcal{N}_{a_1}, \mathcal{N}_{b_2} : L^X \times L_0 \to L$ as follows:

$$\mathcal{N}_{a_1}(\lambda, r) = \begin{cases} 1, & \text{if } \lambda = 1_X,\ r \in L_0 \\ 0.3, & \text{if } 1_X \neq \lambda \geq \mu,\ 0 < r \leq \frac{1}{2} \\ 0, & \text{otherwise.} \end{cases}$$

$$\mathcal{N}_{b_1}(\lambda, r) = \begin{cases} 1, & \text{if } \lambda = 1_X,\ r \in L_0 \\ 0.4, & \text{if } 1_X \neq \lambda \geq \mu,\ 0 < r \leq \frac{1}{2} \\ 0, & \text{otherwise.} \end{cases}$$

From Theorem 2.4 (c), we have

$$\mathcal{T}_{\mathcal{N}}(\lambda) = \begin{cases} 1, & \text{if } \lambda = 0_X \text{ or } 1_X, \\ \frac{1}{2}, & \text{if } \lambda = \mu, \\ 0, & \text{otherwise.} \end{cases}$$

3. R- Convergence

Definition 3.1. Let $(X, \mathcal{T})$ be an L -fuzzy topological space, $\lambda \in L^X, e \in Pt(X)$ and $r \in L_0$. Then the degree to which a fuzzy net T in X r-convergent to e and T r-cluster to e are defined, respectively, as follows:

$$Con_e(T, r) = \bigwedge\{\mathcal{N}'_e(\lambda, r) \mid T \text{ is often in } \lambda'\},$$

$$Cl_e(T, r) = \bigwedge\{\mathcal{N}'_e(\lambda, r) \mid T \text{ is finally in } \lambda'\}.$$

Definition 3.2. Let $(X, \mathcal{T})$ be be an L - fuzzy topological space, $\lambda \in L^X, e \in Pt(X)$ and $r \in L_0$. Then the degree to which e r-adherent point of λ is defined by

$$Ad_e(\lambda, r) = \mathcal{N}'_e(\lambda', r).$$

Proposition 3.3. Let $(X,\mathcal{T})$ be an L-fuzzy topological space. For each $\lambda \in L^X$, $e, x_t \in Pt(X)$ and $r \in L_0$, we have
(1) $S(e, I_{\mathcal{T}}(\lambda, r)) = \mathcal{N}_e(\lambda, r)$,
(2) $S(e, C'_{\mathcal{T}}(\lambda, r)) = Ad'_e(\lambda, r)$,
(3) $Ad_{x_t}(\lambda, r) = \bigvee_{x\in X}(t \odot Ad_x(\lambda, r))$.

Proof.
(1) From Lemma 1.2(7), we have

$$\begin{aligned} S(e, I_{\mathcal{T}}(\lambda, r)) &= S(e, \bigvee\{\mu_i \mid \mu_i \le \lambda, \mathcal{T}(\mu_i) \ge r\}) \\ &= \bigvee\{S(e, \mu_i) \mid \mu_i \le \lambda, \mathcal{T}(\mu_i) \ge r\}) \\ &= \mathcal{N}_e(\lambda, r). \end{aligned}$$

(2) From Theorem 1.5, we have

$$\begin{aligned} S(e, C'_{\mathcal{T}}(\lambda, r)) &= S(e, I_{\mathcal{T}}(\lambda', r)) \\ &= \mathcal{N}_e(\lambda', r) \qquad (\text{ by (1) }) \\ &= Ad'_e(\lambda, r). \end{aligned}$$

(3) From Theorem 2.3(7), we have

$$\begin{aligned} Ad_{x_t}(\lambda, r) &= \mathcal{N}'_{x_t}(\lambda', r) \\ &= \Big(\bigwedge_{x\in X}(t \to \mathcal{N}_{x_t}(\lambda', r)\Big)' \\ &= \bigvee_{x\in X}\Big(t \to \mathcal{N}_{x_t}(\lambda', r)\Big)' \\ &= \bigvee_{x\in X}\Big(t \odot \mathcal{N}'_{x_1}(\lambda', r)\Big) \\ &\qquad (\text{ by Lemma 1.2(4) }) \\ &= \bigvee_{x\in X}(t \odot Ad_x(\lambda, r)). \end{aligned}$$

Theorem 3.4. Let $(X,\mathcal{T})$ be an L-fuzzy topological space. Let $T : D \to Pt(X)$ be fuzzy net and $U : E \to Pt(X)$ a subnet of S. For $r, s \in L_0$, the following properties hold:
(1) If $r_1 \le r_2$, $Con_e(T, r_1) \le Con_e(T, r_2)$, and $Cl_e(T, r_1) \le Cl_e(T, r_2)$,
(2) $Con_e(T, r) \le Cl_e(T, r)$,
(3) $Cl_e(U, r) \le Cl_e(T, r)$,
(4) $Con_e(T, r) \le Con_e(U, r)$,
(5) $Con_{x_t}(T, r) = \bigvee_{x\in X}(t \odot Con_x(T, r))$, and
$Cl_{x_t}(T, r) = \bigvee_{x\in X}(t \odot Cl_x(T, r)$.

Proof. (1) It easily proved.

(2) If T is finally in λ', T is often in λ'. Hence

$$\begin{aligned} Con_e(T,r) &= \bigwedge\{\mathcal{N}'_e(\lambda,r) \mid T \text{ is often in } \lambda'\} \\ &\le \bigwedge\{\mathcal{N}'_e(\lambda,r) \mid T \text{ is finally in } \lambda'\} \\ &= Cl_e(T,r). \end{aligned}$$

(3) If T is finally in λ', U is finally in λ'. Hence

$$\begin{aligned} Cl_e(U,r) &= \bigwedge\{\mathcal{N}'_e(\lambda,r) \mid U \text{ is finally in } \lambda'\} \\ &\le \bigwedge\{\mathcal{N}'_e(\lambda,r) \mid T \text{ is finally in } \lambda'\} \\ &= Cl_e(T,r). \end{aligned}$$

(4) Let U be often in λ'. We will show that T is often in λ'. Let $n \in D$. Since $U : E \to Pt(X)$ is a subnet of T, there exists a cofinal selection $N : E \to D$. For each $n \in D$, there exists $m \in E$ such that $N(k) \ge n$ for $k \ge m$. Since U is often in λ', for $m \in E$, there exists $m_0 \in E$ such that $m_0 \ge m$ for $U(m_0) \in \lambda'$. Put $n_0 = N(m_0)$. Then $n_0 \ge n$ and $T(n_0) = T(N(m_0)) = T(n_0) \in \lambda'$. Thus, U is often in λ'. Hence

$$\begin{aligned} Con_e(T,r) &= \bigwedge\{\mathcal{N}'_e(\lambda,r) \mid T \text{ is often in } \lambda'\} \\ &\le \bigwedge\{\mathcal{N}'_e(\lambda,r) \mid U \text{ is often in } \lambda'\} \\ &= Con_e(U,r). \end{aligned}$$

(5)

$$\begin{aligned} Con_{x_t}(T,r) &= \bigwedge\{\mathcal{N}'_{x_t}(\lambda,r) \mid T \text{ is often in } \lambda'\} \\ &= \bigwedge\{\Big(\bigwedge_{x\in X} (t \to \mathcal{N}_{x_1}(\lambda,r)\Big)' \mid T \text{ is finally in } \lambda'\} \\ &\qquad (\text{ by Theorem 2.3(7) }) \\ &= \bigvee_{x\in X} \bigwedge\{(t \to \mathcal{N}_{x_1}(\lambda,r)\Big)' \mid T \text{ is finally in } \lambda'\} \\ &= \bigvee_{x\in X} \bigwedge\{t \odot \mathcal{N}'_{x_1}(\lambda,r) \mid T \text{ is finally in } \lambda'\} \\ &\qquad (\text{ by Lemma 1.1(4) }) \\ &= \bigvee_{x\in X} \Big(t \odot \bigwedge\{\mathcal{N}'_{x_1}(\lambda,r) \mid T \text{ is finally in } \lambda'\}\Big) \\ &= \bigvee_{x\in X} (t \odot Con_x(T,r)). \end{aligned}$$

The other case is similar.

Proposition 3.5. Let $(X, \mathcal{T})$ be an L-fuzzy topological space , T be a fuzzy net , $e \in Pt(X)$ and $r \in L_0$. Then we have:

$$\begin{aligned} Ad_e(\lambda, r) &= \bigvee \{Con_e(T, r) \mid T \text{ is a fuzzy net in } \lambda\} \\ &= \bigvee \{Cl_e(T, r) \mid T \text{ is a fuzzy net in } \lambda\} \end{aligned}$$

Proof. Since T is finally in λ, T is often in λ. We easily show

$$\begin{aligned} Ad_e(\lambda, r) &= \mathcal{N}'_e(\lambda', r) \\ &\geq \bigvee \{Cl_e(T, r) \mid T \text{ is a fuzzy net in } \lambda\} \\ &\geq \bigvee \{Con_e(T, r) \mid T \text{ is a fuzzy net in } \lambda\} \end{aligned}$$

We only show that

$$Ad_e(\lambda, r) \leq \bigvee \{Con_e(T, r) \mid T \text{ is a fuzzy net in } \lambda\}.$$

Let $Ad_e(\lambda, r) = t$. If $t > 0$, then $\mathcal{N}'_e(\lambda', r) = t$. Put $D = \{\mu \in L^X \mid \mathcal{N}_e(\mu, r) > t'\}$. Define a relation on D by

$$\mu_1 \preceq \mu_2 \text{ iff } \mu_1 \geq \mu_2, \ \ \forall \mu_1, \mu_2 \in D.$$

For each $\mu_1, \mu_2 \in D$, since by Theorem 2.3(5),

$$\mathcal{N}_e(\mu_1 \wedge \mu_2, r) \geq \mathcal{N}_e(\mu_1, r) \wedge \mathcal{N}_e(\mu_2, r) > t'.$$

Hence, $\mu_1 \wedge \mu_2 \in D$ and $\mu_1, \mu_2 \preceq \mu_1 \wedge \mu_2$. Thus, $(D, \preceq)$ is a directed set. For each $\mu \in D$, that is, $\mathcal{N}_e(\mu, r) > t'$, we have $\mu \not\leq \lambda'$, that is, there exists $x \in X$ such that $\lambda(x) > \mu'(x)$. Thus, we can define a fuzzy net $T_0 : D \to Pt(X)$ by $T_0(\mu) = x_{\lambda(x)}$ where $T_0(\mu) \in \lambda$ and $\lambda(x) = T_0(\mu)(x) > \mu'(x)$.

We will show that if $\mu \in D$, then T_0 is not often in μ'. Suppose T_0 is often in μ'. For $\mu \in D$, there exists $\rho \in D$ such that $\mu \preceq \rho$ such that

$$T_0(\rho) = y_{\lambda(y)} \in \mu'$$

and $\lambda(y) = T_0(\rho)(y) > \rho'(y)$. Since $\mu \preceq \rho$ implies $\mu \geq \rho$. It implies

$$\lambda(y) \leq \mu'(y) \leq \rho'(y).$$

It is contradiction for the definition of T_0. Thus, if T_0 is often in μ', then $\mu \notin D$, that is, $\mathcal{N}_e(\mu, r) \leq t'$. Therefore,

$$\begin{aligned} &\bigvee \{Con_e(T, r) \mid T \text{ is a fuzzy net in } \lambda\} \\ &\geq Con_e(T, r) \\ &= \bigwedge \{\mathcal{N}'_e(\mu, r) \mid T_0 \ \text{ is often in } \ \mu'\} \\ &\geq t = Ad_e(\lambda, r). \end{aligned}$$

Theorem 3.6. Let $(X, \mathcal{T})$ be L-fuzzy topological space and $T, U : D \to Pt(X)$ fuzzy nets such that $T(n) \vee U(n), T(n) \wedge U(n) \in Pt(X)$ for each $n \in D$. Define fuzzy nets $T \vee U, T \wedge U : D \to Pt(X)$ by, for each $n \in D$,

$$(T \vee U)(n) = T(n) \vee U(n),\ (T \wedge U)(n) = T(n) \wedge U(n).$$

For each $r \in L_0$, the following properties hold:
(1) If $T(n) \leq U(n)$ for all $n \in D$, then

$$Cl_e(T, r) \leq Cl_e(U, r),\ \ Con_e(T, r) \leq Con_e(U, r).$$

(2) $Cl_e(T \wedge U, r) \leq Cl_e(T, r) \wedge Cl_e(U, r)$.
(3) $Con_e(T \vee U, r) \geq Con_e(T, r) \vee Con_e(U, r)$.
(4) $Con_e(T \wedge U, r) \leq Con_e(T, r) \wedge Con_e(U, r)$.
(5) If L is order dense , then $Cl_e(T \vee U, r) = Cl_e(T, r) \vee Cl_e(U, r)$.

Proof. (1) Let U be finally (often) in λ. Then T be finally (often) in λ, respectively. Thus it is trivial. (2),(3) and (4) are easily proved.

(5) Since $T \leq T \vee U$ and $U \leq T \vee U$, by (1), we have

$$Cl_e(T \vee U, r) \geq Cl_e(T, r) \vee Cl_e(U, r).$$

Suppose that $Cl_e(T \vee U, r) \not\geq Cl_e(T, r) \vee Cl_e(U, r)$. Since L is order dense, then there exist $t \in L_0$ and a fuzzy point $e \in Pt(X)$ such that

$$Cl_e(T \vee U, r) > t > Cl_e(T, r) \vee Cl_e(U, r).$$

Since $Cl_e(T, r) < t$ and $Cl_e(U, r) < t$, by the definition Cl_e, there exist $\lambda, \mu \in L^X$ such that T and U are finally in λ' and μ', respectively, with

$$Cl_e(T, r) \vee Cl_e(U, r) \leq \mathcal{N}'_e(\lambda, r) \vee \mathcal{N}'_e(\mu, r) < t.$$

Since T is finally in λ', there exists $n_1 \in D$ such that $T(n) \in \lambda'$ for every $n \in D$ with $n \geq n_1$. Since U is finally in μ', there exists $n_2 \in D$ such that $T(n) \in \mu'$ for every $n \in D$ with $n \geq n_2$. Let $n_3 \in D$ such that $n_3 \geq n_1$ and $n_3 \geq n_2$. For $n \geq n_3$, we have

$$(T \vee U)(n) \leq \lambda' \vee \mu' = (\lambda \wedge \mu)'.$$

Thus, $(T \vee U)$ is finally in $(\lambda \wedge \mu)'$. It implies

$$\begin{aligned} Cl_e(T \vee U, r) &\leq \mathcal{N}'_e(\lambda \wedge \mu, r) \\ &\leq \mathcal{N}'_e(\lambda, r) \vee \mathcal{N}'_e(\mu, r) < t. \end{aligned}$$

It is a contradiction. Hence, we have

$$Cl_e(T \vee U, r) \leq Cl_e(T, r) \vee Cl_e(U, r).$$

□

Example 3.7. Let $(L = [0,1], \rightarrow)$ be defined as Example 2.6. Let $X = \{a, b\}$ be a set and $\mu \in I^X$ as follows:

$$\mu(x) = 0.3,\ \mu(y) = 0.4.$$

We define L- fuzzy topology $\mathcal{T} : I^X \rightarrow I$ as follows:

$$\mathcal{T}(\lambda) = \begin{cases} 1, & \text{if } \lambda = 0_X \text{ or } 1_X, \\ \frac{1}{2}, & \text{if } \lambda = \mu, \\ 0, & \text{otherwise.} \end{cases}$$

(1) In general, $Cl_e(T \wedge U, r) \neq Cl_e(T, r) \wedge Cl_e(U, r)$.
Let N be a natural numbers. Define fuzzy nets $T, U : N \rightarrow Pt(X)$ by

$$T(n) = x_{a_n},\ a_n = 0.8 + (-1)^n 0.2.$$

$$U(n) = x_{b_n},\ b_n = 0.8 + (-1)^{n+1} 0.2.$$

From Theorem 3.6, $(T \wedge U)(n) = x_{0.6}$ is a fuzzy net. Let $e = x_{0.3}$. From Definition 3.1, we have for $0 < r \leq \frac{1}{2}$,

$$Cl_e(x_{0.6}, r) = 1 - \mathcal{N}_e(\mu, r) = 1 - m(x_{0.3}, \mu) = 0.$$

Since T or U is finally in 1_X,

$$Cl_e(T, r) = 1 - \mathcal{N}_e(0_X, r) = 1 - m(x_{0.3}, 0_X) = 0.3.$$

Similarly, $Cl_e(U, r) = 0.3$. For $0 < r \leq \frac{1}{2}$,

$$0 = Cl_e(T \wedge U, r) \neq Cl_e(T, r) \wedge Cl_e(U, r) = 0.3.$$

(2) In general, $Con_e(T \vee U, r) \neq Con_e(T, r) \vee Con_e(U, r)$.
Define fuzzy nets $T, U : N \rightarrow Pt(X)$ by

$$T(n) = x_{a_n},\ a_n = 0.6 + (-1)^n 0.2.$$

$$U(n) = x_{b_n},\ b_n = 0.6 + (-1)^{n+1} 0.2.$$

From Theorem 3.6, $(T \vee U)(n) = x_{0.8}$ is a fuzzy net. Let $e = x_{0.3}$. For all $r \in I_0$,

$$Ad_e(x_{0.8}, r) = 1 - \mathcal{N}_e(0_X, r) = 1 - m(x_{0.3}, 0_X) = 0.3.$$

Since T or U is often in μ', for $0 < r \leq \frac{1}{2}$,

$$Cl_e(T, r) = 1 - \mathcal{N}_e(\mu, r) = 1 - m(x_{0.3}, \mu) = 0.$$

Similarly, $Cl_e(U, r) = 0$. For $0 < r \leq \frac{1}{2}$

$$0.3 = Con_e(T \vee U, r) > (Con_e(T, r) \vee Con_e(U, r)) = 0.$$

4. Fuzzy r-limit nets and LF- continuous mappings

Definition 4.1. Let $(X, \mathcal{T})$ be an L-fuzzy toplogical space. Let $T : D \to Pt(X)$ be fuzzy net in X , $e \in Pt(X)$ and $r \in L_0$. Then the degree to which T r-limit to e is defined , denoted by $Lim_e(T, r) = t$, if $Cl_e(T, r) = Con_e(T, r) = t$.

Theorem 4.2. Let $(X, \mathcal{T})$ be L-fuzzy topological space and $T, U : D \to Pt(X)$ be fuzzy nets such that $T(n) \vee U(n) \in Pt(X)$ for each $n \in D$. If If L is order dense , $Cl_e(T, r) = Con_e(T, r)$ and $Cl_e(U, r) = Con_e(U, r)$, then

$$Lim_e(T \vee U, r) = Lim_e(T, r) \vee Lim_e(U, r).$$

Proof. From Theorem 3.6, $T \vee U$ is a fuzzy net. We easily proved it from the followings:

$$Cl_e(T \vee U, r) = Cl_e(T, r) \vee Cl_e(U, r) \qquad \text{(by Theorem 3.6(2))}$$

(since $Cl_e(T, r) = Con_e(T, r)$ and $Cl_e(U, r) = Con_e(U, r)$,)

$$\begin{aligned} &= Con_e(T, r) \vee Con_e(U, r) \\ &\leq Con_e(T \vee U, r) && \text{(by Theorem 3.6(4))} \\ &\leq Cl_e(T \vee U, r). && \text{(by Theorem 3.6(2))} \quad \square \end{aligned}$$

Theorem 4.3. Let $(X, \mathcal{T})$ be L-fuzzy topological space. Let T be a fuzzy net and $\mathcal{H} = \{U \mid U \text{ is a subnet of } T\}$. Then, if L is an order dense, the following statements hold:

(1) $Con_e(T, r) = \bigwedge_{T \in \mathcal{H}} Cl_e(U, r)$.
(2) $Cl_e(T, r) = \bigvee_{T \in \mathcal{H}} Con_e(U, r)$.

Proof. (1) For each $U \in \mathcal{H}$, by Theorem 3.4, we have

$$Con_e(T, r) \leq Con_e(U, r) \leq Cl_e(U, r) \leq Cl_e(T, r).$$

Hence

$$Con_e(T, r) \leq \bigwedge_{U \in \mathcal{H}} Cl_e(U, r).$$

Suppose

$$Con_e(T, r) \not\geq \bigwedge_{U \in \mathcal{H}} Cl_e(U, r).$$

Then there exist $x_p \in Pt(X)$ and $t \in L_0$ such that

$$Con_{x_p}(T, r) < t < \bigwedge_{U \in \mathcal{H}} Cl_{x_p}(U, r).$$

Since $Con_{x_p}(T, r) < t$, there exists $\mu \in L^X$ with T is often in μ' such that

$$Con_{x_p}(T, r) \leq \mathcal{N}'_{x_p}(\mu, r) < \bigwedge_{U \in \mathcal{H}} Cl_{x_p}(U, r).$$

Since T is often in μ', for each $n \in D$ there exists $N(n) \in D$ with $N(n) \geq n$ and $T(N(n)) \in \mu'$. Hence there exists a cofinal selection $N : E \to D$ such that $U = T \circ N$. Thus U is a subnet of T and U is finally in μ'. It is a contradiction.

(2) From (1), we have

$$\bigvee_{U \in \mathcal{H}} Con_e(U, r) \leq Cl_e(T, r).$$

Conversely, let $Cl_e(T, r) = t > 0$. Then $\mathcal{N}_e(\lambda, r) \leq t'$, for T is finally in λ'. Let $F = \{\mu \mid \mathcal{N}_e(\mu, r) > t'\}$. Define a relation on $E = D \times F$ by

$$(m, \mu_1) \leq (n, \mu_2) \text{ iff } m \leq n,\ \mu_1 \geq \mu_2.$$

Then $(E, \leq)$ is a directed set. If $\mu \in F$, then T is not finally in μ'. For each $(n, \mu) \in E$, there exists $N(n, \mu) \in D$ with $N(n, \mu) \geq n$ such that $T(N(n, \mu)) \not\leq \lambda'$. So, we can define $N : E \to D$. For each $n_0 \in D$ and $\mu_0 \in F$, there exists $N(n_0, \mu_0) \in D$ with $N(n_0, \mu_0) \geq n_0$ such that $T(N(n_0, \mu_0)) \not\leq \mu_0'$. Hence for every $(n, \mu) \geq (n_0, \mu_0)$, since $n \geq n_0$, we have $N(n, \mu) \geq n \geq n_0$. Therefore N is a cofinal selection on T. So, $U = T \circ N$ is a fuzzy subnet of T and U is finally to every member of F. If U is often in λ', then U is not finally of λ, that is, $\lambda \notin F$. Thus

$$\bigvee_{U \in \mathcal{H}} Con_e(T, r) = \bigwedge \{\mathcal{N}_e'(\lambda, r) \mid U \text{ is often in } \lambda'\} \geq t.$$

Since t is arbitrary, we complete the proof.

□

Theorem 4.4. Let L be an order dense , $(X, \mathcal{T})$ be L-fuzzy topological space and T be a fuzzy net. If every subnet U of T has a subnet K of U such that $Lim_e(K, r) = t$, then $Lim_e(T, r) = t$,

Proof. Let $\mathcal{H} = \{U \mid\ U \text{ is a subnet of } T\}$. For each $U \in \mathcal{H}$, since U has a subnet K with $Lim_{\mathcal{T}}(K, r) = t$, by Theorem 3.4(4), we have

$$Con_e(U, r) \leq Con_e(K, r) = Cl_e(K, r) = t.$$

Hence, by Theorem 4.3(2),

$$Cl_e(T, r) = \bigvee_{U \in \mathcal{H}} Con_e(U, r) \leq t.$$

Conversely, by Theorem 3.4(2),

$$t = Con_e(K, r) = Cl_e(K, r) \leq Cl_e(U, r).$$

Hence, by Theorem 4.3(1),

$$t \leq \bigwedge_{U \in \mathcal{H}} Cl_e(U, r) = Con_e(T, r).$$

Hence, $Cl_e(T, r) \leq Con_e(T, r)$. Since $Con_e(T, r) \leq Cl_e(T, r)$ from Theorem 3.4(2), $Cl_e(T, r) = Con_e(T, r)$, that is, $Lim_e(T, r) = t$. □

Example 4.5. Let $(L = [0,1], \to)$ be defined as Example 3.7. Let N be a natural number set. Define a fuzzy net $T : N \to Pt(X)$ by

$$T(n) = x_{a_n},\ a_n = 0.6 + (-1)^n 0.2.$$

Let $e = x_{0.3}$. Since T is often in μ', for $0 < r \le \frac{1}{2}$,

$$Con_e(T, r) = 1 - \mathcal{N}_e(\mu, r) = 1 - m(x_{0.3}, \mu) = 0.$$

Since T is finally in 1_X, for each $r \in I_0$

$$Cl_e(T, r) = 1 - \mathcal{N}_e(0_X, r) = 1 - m(x_{0.3}, 0_X) = 0.3.$$

Thus, since $Con_e(T, r) \ne Cl_e(T, r)$ for $0 < r \le \frac{1}{2}$, $Lim_e(T, r)$ does not exists.

Since $Con_e(T, r) = Cl_e(T, r) = 0.3$ for $\frac{1}{2} < r \le 1$, $Lim_e(T, r) = 0.3$.

Theorem 4.6. Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be L-fuzzy topological spaces. For every fuzzy net T in X , $x_t \in Pt(X)$, $r \in L_0$ and $\lambda \in L^X$, the following statements are equivalent.

(1) $f : (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ is LF- continuous.

(2) $\mathcal{N}_{f^{\to}(e)}(\mu, r) \le \bigvee\{\mathcal{N}_e(\lambda, r) \mid f^{\to}(\lambda) \le \mu\}$.

(3) $Cl_e(T, r) \le Cl_{f^{\to}(e)}(f \circ T, r)$

(4) $Con_e(T, r) \le Con_{f^{\to}(e)}(f \circ T, r)$

(5) $f^{\to}(C_{\mathcal{T}_1}(\lambda, r)) \le C_{\mathcal{T}_2}(f^{\to}(\lambda), r)$.

(6) $C_{\mathcal{T}_1}(f^{\leftarrow}(\mu), r)) \le f^{\leftarrow}(C_{\mathcal{T}_2}(\mu), r)$.

(7) $f^{\leftarrow}(I_{\mathcal{T}_2}(\mu, r)) \le I_{\mathcal{T}_1}(f^{\leftarrow}(\mu), r)$.

Proof. (1) $\Rightarrow$ (2) For any $\rho \in L^Y$ such that $\mathcal{T}_2(\rho) \ge r$ and $\rho \le \mu$. Since f is LF-continuous, then $\mathcal{T}_1(f^{\leftarrow}(\rho)) \ge \mathcal{T}_2(\rho) \ge r$, we have by Lemma 1.3 (2)

$$\begin{aligned} S(f^{\to}(e), \rho) &\le S(e, f^{\leftarrow}(\rho)) \qquad (e = x_t, f^{\to}(e) = f(x)_t) \\ &= \mathcal{N}_e(f^{\leftarrow}(\rho), r)) \qquad (\mathcal{T}_1(f^{\to}(f^{\leftarrow})(\rho)) \ge r) \\ &\le \bigvee\{\mathcal{N}_e(\lambda, r) \mid f^{\to}(\lambda) \le \mu\}. \qquad (f^{\to}(f^{\leftarrow}(\rho)) \le \rho \le \mu). \end{aligned}$$

Thus, $\mathcal{N}_{f^{\to}(e)}(\mu, r) \le \bigvee\{\mathcal{N}_e(\lambda, r) \mid f^{\to}(\lambda) \le \mu\}$.

(2) $\Rightarrow$ (3) If $f^{\to}(\lambda) \le \mu$ and $f \circ T$ is finally in μ', there exists $n_0 \in D$ such that for all $n \ge n_0$, $f(T(n)) \in \mu'$. Let $T(n) = x_t$. Then

$$t \le \mu'(f(x)) \le (f(\lambda))'(f(x)) \le \lambda'(x).$$

It implies $T(n) \in \lambda'$. Therefore, T is finally in λ'.

$$\begin{aligned} & Cl_e(T, r) \\ &= \bigwedge\{\mathcal{N}'_e(\lambda, r) \mid T \text{ is finally in } \lambda'\} \\ &\le \bigwedge\{\mathcal{N}'_e(\lambda, r) \mid \exists\mu, f^{\to}(\lambda) \le \mu, f \circ T \text{ is finally in } \mu'\} \\ &= \bigwedge\{\{\bigvee\{\mathcal{N}_e(\lambda, r) \mid f^{\to}(\lambda) \le \mu\}', f \circ T \text{ is finally in } \mu'\} \\ &\le \bigwedge\{\mathcal{N}'_{f^{\to}(e)}(\mu, r), f \circ T \text{ is finally in } \mu'\} \\ &= Cl_e(f \circ T, r) \quad (\text{ by (2) }) \end{aligned}$$

(3)$\Rightarrow$(4) Every subnet $U : E \to Pt(Y)$ of $f(T)$, there exists a cofinal selection $N : E \to D$ such that $U = f(T) \circ N = f \circ (T \circ N)$. Put $K = T \circ N$. Then K is a subnet of T. We can prove it from the followings:

$$\begin{aligned}
Con_e(T,r) &\leq Con_e(K,r) && \text{(by Theorem 3.4(5))}\\
&\leq Cl_e(K,r) && \text{(by Theorem 3.4(2))}\\
&\leq Cl_{f^{\to}(e)}(f \circ K, r) && \text{(by (3))}\\
&= Cl_{f^{\to}(e)}(f \circ (T \circ N), r)\\
&= Cl_{f^{\to}(e)}(U, r).
\end{aligned}$$

From Theorem 3.4 (2), we have $Con_e(T,r) \leq Con_{f^{\to}(e)}(f \circ T, r)$.

(4) $\Rightarrow$ (5) From Theorem 1.5 and Proposition 3.3 (2),

$$S(x_1, C'_{\mathcal{T}_1}(\lambda, r)) = C'_{\mathcal{T}_1}(\lambda, r)(x) = Ad'_x(\lambda, r).$$

It implies

$$C_{\mathcal{T}_1}(\lambda, r)(x) = Ad_x(\lambda, r). \tag{X}$$

Thus, we have

$$\begin{aligned}
&f^{\to}(C_{\mathcal{T}_1}(\lambda, r))(y)\\
&= \bigvee \{C_{\mathcal{T}_1}(\lambda, r)(x) \mid f(x) = y\}\\
&= \bigvee \{Ad_x(\lambda, r) \mid f(x) = y\} && \text{(by (X))}\\
&= \bigvee_{f(x)=y} \bigvee \{Con_x(T, r) \mid T \text{ is fuzzy net in } \lambda\} && \text{(by Proposition 3.5)}\\
&\leq \bigvee_{f(x)=y} \bigvee \{Con_y(f \circ T, r) \mid T \text{ is fuzzy net in } \lambda\} \text{(by (4))}\\
&= \bigvee \{Con_y(f \circ T, r) \mid T \text{ is fuzzy net in } \lambda\}\\
&\leq \bigvee \{Con_y(T, r) \mid T \text{ is fuzzy net in } f^{\to}(\lambda)\}\\
&= Ad_y(f^{\to}(\lambda), r) && \text{(by Proposition 3.5)}\\
&= C_{\mathcal{T}_2}(f^{\to}(\lambda), r)(y). && \text{(by (X))}
\end{aligned}$$

(5) $\Rightarrow$ (6) and (6) $\Rightarrow$ (7) are easily proved .

(7) $\Rightarrow$ (1) We will show that $\mathcal{T}_1(f^{\leftarrow}(\mu)) \geq \mathcal{T}_2(\mu)$, for all $\mu \in L^Y$.

Let $\mathcal{T}_2(\mu) = 0$. It is trivial.

Let $\mathcal{T}_2(\mu) = r > 0$. Since $\mathcal{T}_N = \mathcal{T}_2$ from Theorem 2.4(b), we have, for all $y \in Y$,

$$S(y, \mu) = \mathcal{N}_y(\mu, r).$$

It implies, for all $x \in X$,

$$S(f(x), \mu) = S(x, f^{\leftarrow}(\mu)) = \mathcal{N}_{f(x)}(\mu, r).$$

Since $f^{\leftarrow}(I_{\mathcal{T}_2}(\mu,r)) = f^{\leftarrow}(\mu)$,

$$S(x, f^{-1}(\mu)) = S(x, f^{\leftarrow}(I_{\mathcal{T}_2}(\mu, r))$$

(Since $f^{\leftarrow}(I_{\mathcal{T}_2}(\mu,r)) \leq I_{\mathcal{T}_1}(f^{\leftarrow}(\mu),r),$)

$$\begin{aligned} &\leq S(x, I_{\mathcal{T}_1}(f^{\leftarrow}(\mu), r)) \\ &= \mathcal{N}_x(f^{\leftarrow}(\mu), r). \text{ (by Proposition 3.3(1))} \end{aligned}$$

Thus, by Theorem 2.3(2), we have

$$S(x, f^{\leftarrow}(\mu)) = \mathcal{N}_x(f^{\leftarrow}(\mu), r).$$

Hence, $\mathcal{T}_1(f^{\leftarrow}(\mu)) \geq r$. □

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References

1. C.L.Chang, *Fuzzy topological spaces*, J. Math. Anal. Appl. **24** (1968), 182-190.
2. K.C.Chattopadhyay and S.K. Samanta, *Fuzzy topology*, Fuzzy Sets and Systems **54** (1993), 207-212.
3. S.L.Chen and J.S.Cheng, *On convergence of nets of L-fuzzy sets*, J. Fuzzy Math. **2** (1994), 517-524.
4. S.L.Chen and J.S.Cheng, *$\theta-$Convergence of nets of L-fuzzy sets and its applications*, Fuzzy Sets and Systems **86** (1997), 235-240.
5. M.Demirci, *Neighborhood structures in smooth topological spaces*, Fuzzy Sets and Systems **92** (1997), 123-128.
6. J. Fang, *Stratified L-ordered convergence structures*, Fuzzy Sets and Systems **161** (2010), 2130-2149.
7. J. Fang, *The relationship between L-ordered convergence structures and strong L-topologies*, Fuzzy Sets and Systems **161** (2010), 2923-2944.
8. D.N.Georgiou and B.K.Papadopoulos, *Convergences in fuzzy topological spaces*, Fuzzy Sets and Systems **101** (1999), 495-504.
9. J.A.Goguen, *The fuzzy Tychonoff theorem*, J. Math. Anal. Appl. **43** (1973), 734-742.
10. G.Gierz, K.H.Hofmann and etc.,Continuous Lattices and 305 Domains, Cambridge University Press,Cambridge (2003).
11. P. Hájek, *Metamathematices of Fuzzy Logic*, Kluwer Academic Publishers, Dordrecht (1998).
12. R.N.Hazara, S.K. Samanta and K.C.Chattopadhyay, *Fuzzy topology redefined*, Fuzzy Sets and Systems **45** (1992), 79-82.
13. R.N.Hazara, S.K. Samanta and K.C.Chattopadhyay, *Gradation of openness; Fuzzy topology*, Fuzzy Sets and Systems **49(2)** (1992), 237–242.
14. U. Höhle, *Many Valued Topology and its Applications*, Kluwer Academic Publisher, Boston, 2001.
15. U. Höhle, S. E. Rodabaugh, *Mathematics of Fuzzy Sets, Logic, Topology and Measure Theory*, The Handbooks of Fuzzy Sets Series, Volume 3, Kluwer Academic Publishers, Dordrecht (1999).
16. U. Höhle, A. Šostak, *Axiomatic foundations of fixed-basis fuzzy topology*, Chapter 3 in [15], 123-272.
17. H. Lai, D. Zhang, *Fuzzy preorder and fuzzy topology*, Fuzzy Sets and Systems **157** (2008), 1865-1885..
18. Liu Ying-Ming, Luo Mao-Kang, Fuzzy topology, World Scientific Publishing Co., Singapore , 1997.
19. Pu Pao-Ming and Liu Ying-Ming, Fuzzy topology I; Neighborhood structures of a fuzzy point and moore - Smith convergence, J.Math.Anall.Appl. **67** (1980), 571-599.
20. A.A. Ramadan, *Smooth topological spaces*, Fuzzy Sets and Systems **48(2)** (1992), 371-375.

21. A.A.Ramadan, S.N.El-Deeb, and M.A.Abdel-Sattar, *On Smooth topological spaces IV*, Fuzzy Sets and Systems **199** (2001), 473-482.
22. A.A. Ramadan, *L -fuzzy interior systems*, Computers and Math. with Appl. **62** (2011), 4301-4307.
23. S. E. Rodabaugh, *Point - set lattice theoretic topology*, Fuzzy Sets and Systems **40** (1991), 297-375.
24. S. E. Rodabaugh, E. P. Klement, *Toplogical and Algebraic Structures in Fuzzy Sets*, The Handbook of Recent Developments in the Mathematics of Fuzzy Sets, , Kluwer Academic Publishers, (Boston/Dordrecht/London) (2003).
25. A. P. Šostak, *On a fuzzy topological structure*, Suppl. Rend. Circ. Matem. Palermo2 Ser II **11** (1985), 89-103.
26. A.P.Šostak, *On the neighborhood structure of a fuzzy topological spaces*, Zb. Rodova Univ.Nis. Ser Math. **4** (1990), 7-14.
27. A. P. Šostak, *Basic structures of fuzzy topology*, J. of Math. Sciences **78, no 6** (1996), 662-701.
28. E. Turunen, *Algebraic structures in fuzzy logic*, Fuzzy Sets and Systems **52** (1992), 181-188.
29. E. Turunen, *Mathematics behind fuzzy logic*, A Springer-Verlag Co., (1999).
30. Wang Guo-Jun, *Pointwise topology on completely distributive lattices*, Fuzzy Sets and Systems **30** (1989), 53-62.
31. R.R.Yager, *On a general class of fuzzy conectivies*, Fuzzy Sets and Systems **4** (1980), 235-242.
32. W. Yao, *On many-valued L - fuzzy convergence spaces*, Fuzzy Sets and Systems **159** (2008), 2503-2519.
33. M.S. Ying, *On the method of neighborhood systems in fuzzy topology*, Fuzzy Sets and Systems **68** (1994), 227-238.

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