Weighted Least Squares Design of Two-dimensional Zero-Phase FIR Filters in the Continuous Frequency Domain

By

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ABSTRACT

Using matrix-vector formalism, a weighted least squares version of the design problem of zero-phase 2-D FIR filters is presented in the continuous frequency domain with no assumptions of a quadrantally symmetric or antisymmetric frequency response. By minimizing the weighted mean squared error between the desired and designed frequency responses, the resulting two matrix equations are found to decouple under the assumptions of the separability of the 2-D weighting function and the evenness of the 1-D weighting functions. Consequently closed-form expressions for the matrices of the filter coefficients are derived.

I. INTRODUCTION

Some work has been done in designing 2-D FIR filters using the least squares technique. Ahmad and Wang [1] applied the least squares criterion in the discrete frequency domain for designing filters with quadrantally symmetric or antisymmetric frequency response. Gu and Aravena [2] applied the weighted least squares criterion for designing the same kind of filters using a linear operator theory approach and ended up with a fixed-point problem in the form of an integral equation. By discretizing the continuous frequency variables, they obtained an affine equation for which they presented both explicit and iterative solutions. The weighting function they employed is exclusively 1 in the pass/stop band region and 0 in the transition band region. They employed some techniques [3] for updating the weighting function in order to equalize the peak ripples at the pass/stop band edges. Reweighted least squares strategies

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[3,4] which derive originally from Lawson's work [5] have been investigated. As an extension to their work, Aravena and Gu [6] treated the case of a general weighting function and filters having zero-phase frequency response but no quadrantal symmetry or antisymmetry. They ended up with a fixed-point problem expressed by two coupled matrix integral equations which can be discretized and solved only by an iterative technique. Gislason et. al. [7] briefly presented a weighted least squares solution for the 2-D FIR filter design problem based on a discrete frequency domain formulation.

In this paper an alternative weighted least squares treatment of the design problem of zero-phase 2-D FIR filters will be presented in the continuous frequency domain. Under the assumption of the separability of the weighting function, two algebraic matrix equations involving the filter coefficients will be derived; and under the reasonable assumption of the evenness of the individual weighting functions the two equations will be shown to decouple leading to closed-form expressions for the two matrices of the filter coefficients. This treatment is distinct from that presented in [2] and [6] since it leads to algebraic rather than integral equations, it does not require discretization for finding the solution, and it results in compact closed-form expressions for the matrices of filter coefficients.

II. Weighted Least Squares Optimization

The frequency response of a zero-phase 2-D FIR filter can be expressed as[2] :

$$H(\omega_1,\omega_2) = C_1^T(\omega_1) A C_2(\omega_2) + S_1^T(\omega_1) B S_2(\omega_2) \qquad (1)$$

where A and B are respectively (N_1+1) x (N_2+1) and N_1 x N_2 matrices of the filter coefficients and $C_1(\omega_1)$, $C_2(\omega_2)$, $S_1(\omega_1)$ and $S_2(\omega_2)$ are the vectors defined by :

$$C_i(\omega_i) = \begin{bmatrix} 1 & \cos(\omega_i) & \cos(2\omega_i) & . & . & . & \cos(N_i\omega_i) \end{bmatrix}^T \quad , \ i = 1,2 \qquad (2)$$

$$S_i(\omega_i) = \begin{bmatrix} \sin(\omega_i) & \sin(2\omega_i) & . & . & . & \sin(N_i\omega_i) \end{bmatrix}^T \quad , \ i = 1,2 \qquad (3)$$

[2]The superscript T denotes the transpose.

The optimization criterion will be defined as the integral of the weighted square of the difference between the frequency response $H(\omega_1,\omega_2)$ of the filter to be designed and the desired frequency response $D(\omega_1,\omega_2)$, i.e.,

$$E = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} W(\omega_1,\omega_2)\left[H(\omega_1,\omega_2) - D(\omega_1,\omega_2)\right]^2 d\omega_1 d\omega_2 \tag{4}$$

where $W(\omega_1,\omega_2)$ is a positive weighting function. The filter coefficient matrices will be evaluated by applying the minimization conditions :

$$\nabla_A E = 0 \quad , \quad \nabla_B E = 0 \ . \tag{5}$$

Here appeal will be made to the following lemma whose proof is given in the appendix.

Lemma :

The gradient of the scalar function :

$$f(A) = u^T A v \tag{6}$$

where A is an M_1 x M_2 matrix and u and v are respectively M_1- and M_2-dimensional vectors is given by :

$$\nabla_A f(A) = uv^T \ . \tag{7}$$

Substituting (1) into (4) and applying the above lemma, we get :

$$\begin{aligned} \nabla_A E &= 2 \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} W(\omega_1,\omega_2)\left[C_1^T A C_2 + S_1^T B S_2 - D\right] C_1 C_2^T d\omega_1 d\omega_2 \\ &= 2 \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} W(\omega_1,\omega_2)\left[C_1 C_1^T A C_2 C_2^T + C_1 S_1^T B S_2 C_2^T - D C_1 C_2^T\right] d\omega_1 d\omega_2 \end{aligned} \tag{8}$$

$$\begin{aligned} \nabla_B E &= 2 \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} W(\omega_1,\omega_2)\left[C_1^T A C_2 + S_1^T B S_2 - D\right] S_1 S_2^T d\omega_1 d\omega_2 \\ &= 2 \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} W(\omega_1,\omega_2)\left[S_1 C_1^T A C_2 S_2^T + S_1 S_1^T B S_2 S_2^T - D S_1 S_2^T\right] d\omega_1 d\omega_2 \end{aligned} \tag{9}$$

Assuming that the two-dimensional weighting function $W(\omega_1,\omega_2)$ is separable, i.e.,

$$W(\omega_1,\omega_2) = W_1(\omega_1) W_2(\omega_2) \tag{10}$$

and substituting (8) and (9) into (5) one gets the following coupled system of two matrix equations :

$$P_1 A P_2 + R_1 B R_2^T = U \ , \qquad (11)$$

$$R_1^T A R_2 + Q_1 B Q_2 = V \qquad (12)$$

where the square symmetric matrices P_1, P_2, Q_1 and Q_2 are of order (N_1+1), (N_2+1), N_1 and N_2 respectively and the rectangular matrices R_1, R_2, U and V are (N_1+1) x N_1, (N_2+1) x N_2, (N_1+1) x (N_2+1) and N_1 x N_2 respectively. These matrices are defined as follows :

$$P_i = \int_{-\pi}^{\pi} W_i(\omega) C_i(\omega) C_i^T(\omega) d\omega \quad , \text{i} = 1,2 \qquad (13)$$

$$Q_i = \int_{-\pi}^{\pi} W_i(\omega) S_i(\omega) S_i^T(\omega) d\omega \quad , \text{i} = 1,2 \qquad (14)$$

$$R_i = \int_{-\pi}^{\pi} W_i(\omega) C_i(\omega) S_i^T(\omega) d\omega \quad , \text{i} = 1,2 \qquad (15)$$

$$U = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} W(\omega_1, \omega_2) D(\omega_1, \omega_2) C_1(\omega_1) C_2^T(\omega_2) d\omega_1 d\omega_2 \qquad (16)$$

$$V = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} W(\omega_1, \omega_2) D(\omega_1, \omega_2) S_1(\omega_1) S_2^T(\omega_2) d\omega_1 d\omega_2 \qquad (17)$$

Actually the separability condition of (10) has been imposed in order to be able to define the matrices P_1, P_2, Q_1, Q_2, R_1 and R_2 .

The system of two coupled matrix equations given by (11) and (12) is equivalent to $(N_1+1)(N_2+1) + N_1 N_2$ simultaneous linear equations. Under the reasonable assumption that $W_1(\omega_1)$ and $W_2(\omega_2)$ are even functions of their respective arguments, it can be shown that :

$$R_1 = R_2 = 0 \ . \qquad (18)$$

Consequently (11) and (12) de-couple to :

$$P_1 A P_2 = U \ , \qquad (19)$$

$$Q_1 B Q_2 = V \ . \qquad (20)$$

Since the 4 matrices P_1, P_2, Q_1 and Q_2 defined by (13) and (14) are at least positive semidefinite and are actually positive definite and consequently nonsingular for a wide range of weighting functions $W_1(\omega_1)$ and $W_2(\omega_2)$, the unique solution of the above two matrix equations is given by :

$$A = P_1^{-1} U P_2^{-1} \ , \qquad (21)$$

$$B = Q_1^{-1} V Q_2^{-1} \ . \qquad (22)$$

In the special case of $W_1(\omega_1) = W_2(\omega_2) = 1$, it can be shown from (13) and (14) that the 4 matrices P_1, P_2, Q_1 and Q_2 become diagonal and reduce to :

$$P_i = 2\pi \, Diag\{1 \quad 0.5 \quad . \quad . \quad . \quad 0.5\} \quad , i = 1,2 \tag{23}$$

and

$$Q_i = \pi I \quad , i = 1,2 \tag{24}$$

where I is the identity matrix. Consequently the filter coefficient matrices of (21) and (22) reduce to :

$$A = \frac{1}{\pi^2} Diag\{0.5 \quad 1 \quad . \quad . \quad . \quad 1\} U \, Diag\{0.5 \quad 1 \quad . \quad . \quad . \quad 1\} , \tag{25}$$

$$B = \frac{1}{\pi^2} V \; . \tag{26}$$

III. SIMULATION EXAMPLE

The same fan filter presented as example 5.3 in [2] where the desired frequency response is given by :

$$D(\omega_1, \omega_2) = \begin{cases} 1 & \omega_1\omega_2 > 0 \;\; , \;\; \varepsilon \le |\omega_i| \le \pi - \varepsilon \;\; , \;\; i = 1,2 \\ 0 & \omega_1\omega_2 < 0 \;\; , \;\; \varepsilon \le |\omega_i| \le \pi - \varepsilon \;\; , \;\; i = 1,2 \\ \text{do not care} & \text{elsewhere} \end{cases} \tag{27}$$

will be designed using the technique of this paper. In order to be able to compare with the results of the cited reference, the individual weighting functions of (10) will be taken as :

$$W_i(\omega_i) = \begin{cases} 1 & \varepsilon \le |\omega_i| \le \pi - \varepsilon \\ 0 & \text{elsewhere} \end{cases} \tag{28}$$

The symmetric matrices P_1, P_2, Q_1, Q_2 which appear in the closed-form expressions (21), (22) have been evaluated analytically and their elements are given below where i = 1,2 :

$$(P_i)_{1,1} = 2(\pi - 2\varepsilon) \tag{29}$$

$$(P_i)_{n+1,1} = \frac{2}{n}\left[(-1)^{n+1} - 1\right]\sin(n\varepsilon) \qquad 1 \le n \le N_i \tag{30}$$

$$(P_i)_{n+1,n+1} = \pi - 2\varepsilon - \frac{1}{n}\sin(2n\varepsilon) \qquad 1 \le n \le N_i \tag{31}$$

$$\begin{aligned}(P_i)_{n_1+1,n_2+1} = {} & \frac{1}{(n_1+n_2)}\left[(-1)^{n_1+n_2+1} - 1\right]\sin((n_1+n_2)\varepsilon) \\ & + \frac{1}{(n_1-n_2)}\left[(-1)^{n_1-n_2+1} - 1\right]\sin((n_1-n_2)\varepsilon) \qquad 1 \le n_1, n_2 \le N_i \quad , n_1 \ne n_2\end{aligned} \tag{32}$$

$$\left(Q_i\right)_{n,n} = \pi - 2\varepsilon + \frac{1}{n}\sin(2n\varepsilon) \quad 1 \le n \le N_i \tag{33}$$

$$\begin{aligned}\left(Q_i\right)_{n_1,n_2} &= \frac{1}{(n_1 - n_2)}\left[(-1)^{n_1-n_2+1} - 1\right]\sin((n_1 - n_2)\varepsilon) \\ &\quad - \frac{1}{(n_1 + n_2)}\left[(-1)^{n_1+n_2+1} - 1\right]\sin((n_1 + n_2)\varepsilon) \quad 1 \le n_1, n_2 \le N_i, n_1 \ne n_2\end{aligned} \tag{34}$$

The matrices U and V have been evaluated in terms of the vectors f_1, f_2, g_1, g_2 as :

$$U = 2 f_1 f_2^T \tag{35}$$

$$V = 2 g_1 g_2^T \tag{36}$$

where the elements of these vectors are given below with i=1,2 :

$$\left(f_i\right)_1 = \pi - 2\varepsilon \tag{37}$$

$$\left(f_i\right)_{n+1} = \frac{1}{n}\left[(-1)^{n+1} - 1\right]\sin(n\varepsilon) \ , \ 1 \le n \le N_i \tag{38}$$

$$\left(g_i\right)_n = \frac{1}{n}\left[1 - (-1)^n\right]\cos(n\varepsilon) \quad , \quad 1 \le n \le N_i \tag{39}$$

Fig. 1 shows the frequency response of a filter designed with $N_1 = N_2 = 15$ and $\varepsilon = 0.1\pi$. By comparing this figure with its counterparts, namely Figs 3a,b of [2] - which were obtained after 149 and 3577 iterations respectively - it is obvious that it is superior to the first figure and as good as the second one.

IV. CONCLUSION

A weighted least squares treatment of the design problem of zero-phase 2-D FIR filters has been presented in the continuous frequency domain resulting in closed-form expressions for the filter coefficients under the reasonable assumptions of the separability of the 2-D weighting function and the evenness of the 1-D weighting functions.

APPENDIX

Proof of the Lemma :

Let the columns of matrix A be denoted by a_i and the elements of vector v by v_i , i.e.,

$$A = \begin{pmatrix} a_1 & . & . & . & a_n \end{pmatrix} \quad , \tag{A1}$$

$$v = \begin{pmatrix} v_1 & . & . & . & v_n \end{pmatrix}^T \quad . \tag{A2}$$

The scalar function f(A) :

$$f(A) = u^T A v \tag{A3}$$

can be expressed as :

$$f(A) = \sum_{i=1}^{n} \left(u^T a_i \right) v_i \quad . \tag{A4}$$

Consequently the gradient of f(A) with respect to vector a_k is given by :

$$\nabla_{a_k} f(A) = u v_k \quad . \tag{A5}$$

Defining the gradient of f(A) with respect to matrix A as :

$$\nabla_A f(A) = \begin{pmatrix} \nabla_{a_1} f & . & . & . & \nabla_{a_n} f \end{pmatrix} \tag{A6}$$

and substituting (A5), one gets :

$$\nabla_A f(A) = u v^T \quad . \tag{A7}$$

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Figure Captions

Fig. 1 : A perspective plot of the frequency response of the designed 2-D FIR filter .