Design of Circularly Symmetric Two-Dimensional Linear-Phase Lowpass FIR Filters using Closed-Form Expressions

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ABSTRACT

A design technique is presented for two-dimensional linear phase lowpass FIR filters based on a constrained minimization formulation. The minimization criterion is a convex combination of two criteria representing measures of the error between the actual and ideal frequency responses in the pass and stop bands. The constraints can for example represent flatness conditions at the origin. The result is applied to the case of a circularly symmetric lowpass filter and closed-form expressions are derived for the elements of the relevant matrices which appear in the minimization criterion so that numerical integration can be avoided.

I. INTRODUCTION

Recently 2-D FIR eigenfilters were designed by minimizing a quadratic error criterion defined by the integral of the square of the difference between the frequency response of the designed filter and a scaled version of the ideal frequency response [1]. The scaling factor was introduced in order to get a quadratic form as an essential requirement of the eigenfilter formulation. The technique proposed here is distinct from that of [1] in two respects. First, a more meaningful minimization criterion is used where no scaling of the ideal frequency response is employed. Second, instead of the single quadratic normalization constraint required by the eigenfilter formulation of [1], a set of linear constraints will be imposed which can for example represent flatness conditions of the frequency response at the origin of the 2D frequency plane. In the case of a circularly symmetric lowpass filter, closed-form expressions will be derived for the elements of the pertinent matrices so that numerical integration can be avoided.

The frequency response of a 2-D FIR filter having a first quadrant causality and a real impulse response which is symmetric about the center of its support can be expressed as :

$$H(e^{j\omega_1}, e^{j\omega_2}) = \exp\left\{-j\left[\omega_1\left(0.5\left(N_1-1\right)\right)+\omega_2\left(0.5\left(N_2-1\right)\right)\right]\right\} H_a(\omega_1,\omega_2) \tag{1}$$

where $H_a(\omega_1,\omega_2)$ is given below for the case of odd N_1 and N_2 being treated in this paper :

$$H_a(\omega_1,\omega_2) = \sum_{n_1=0}^{0.5(N_1-1)} \sum_{n_2=0}^{0.5(N_2-1)} a\left(n_1,n_2\right)\cos\left(\omega_1 n_1\right)\cos\left(\omega_2 n_2\right) \ . \tag{2}$$

The amplitude function $H_a(\omega_1,\omega_2)$ can be compactly expressed as[1] :

$$H_a\left(\omega_1,\omega_2\right) = u^T(\omega_1) B v(\omega_2) \tag{3}$$

where

$$u(\omega_1) = \left[1 \quad \cos(\omega_1) \quad \cos(2\omega_1) \quad . \ . \ . \quad \cos\left(\left(M_1-1\right)\omega_1\right)\right]^T \tag{4}$$

and $v(\omega_2)$ is analogously defined with ω_1 and M_1 respectively replaced by ω_2 and M_2 where :

[1]The superscript T denotes the transpose.

$$M_i = 0.5\left(N_i + 1\right) \qquad , i = 1 , 2 \ . \tag{5}$$

In Eq (3) the elements of the square matrix B are related to the coefficients $a(n_1,n_2)$ of Eq (2) which in turn are related to the impulse response of the filter.

In order to be able to express the amplitude function $H_a(\omega_1,\omega_2)$ as an inner product of two vectors; one of them consisting of the filter coefficients, a vector x will be defined as a concatenation of the rows of matrix B and a vector $s(\omega_1,\omega_2)$ will be defined as the Kronecker product of the vectors $u(\omega_1)$ and $v(\omega_2)$:

$$s\left(\omega_1, \omega_2\right) = u(\omega_1) \otimes v(\omega_2) \tag{6}$$

Therefore Eq (3) can be rewritten as :

$$H_a\left(\omega_1, \omega_2\right) = s^T (\omega_1, \omega_2) x \ . \tag{7}$$

II. The Constrained Minimization Problem

The total mean squared error criterion is defined as [1] :

$$E = \alpha E_s + (1-\alpha) E_p \qquad 0 \le \alpha \le 1 \tag{8}$$

where E_s and E_p are respectively mean squared error measures in the stop and pass domains Ω_s and Ω_p defined by :

$$E_s = \iint\limits_{\Omega_s} W(\omega_1, \omega_2) e^2(\omega_1, \omega_2) d\omega_1 d\omega_2 \tag{9}$$

and an analogous expression holds for E_p with the subscript s replaced by p. In the above equation $W(\omega_1, \omega_2)$ is a positive weighting function and $e(\omega_1,\omega_2)$ is the error between the actual frequency response $H_a(\omega_1,\omega_2)$ and the desired one $H_d(\omega_1,\omega_2)$, i.e.,

$$e(\omega_1, \omega_2) = H_a(\omega_1, \omega_2) - H_d(\omega_1, \omega_2) \ . \tag{10}$$

Let $H_d(\omega_1,\omega_2)$ be unity in the passband and zero in the stopband; it can be shown that the error criterion (8) can be evaluated as :

$$E = x^T Q x - 2(1-\alpha) x^T p_b + (1-\alpha) p_c \tag{11}$$

where

$$Q = \alpha P_s + (1-\alpha) P_a \quad , \tag{12}$$

$$P_s = \iint\limits_{\Omega_s} W(\omega_1, \omega_2) s(\omega_1, \omega_2) s^T (\omega_1, \omega_2) d\omega_1 d\omega_2 \quad , \tag{13}$$

$$P_a = \iint\limits_{\Omega_p} W(\omega_1, \omega_2) s(\omega_1, \omega_2) s^T (\omega_1, \omega_2) d\omega_1 d\omega_2 \quad , \tag{14}$$

$$p_b = \iint\limits_{\Omega_p} W(\omega_1, \omega_2) s(\omega_1, \omega_2) d\omega_1 d\omega_2 \quad , \tag{15}$$

$$p_c = \iint\limits_{\Omega_p} W(\omega_1, \omega_2) d\omega_1 d\omega_2 \quad , \tag{16}$$

$$ss^T = \left(uu^T\right) \otimes \left(vv^T\right) \quad . \tag{17}$$

Let us impose a set of constraints - for example flatness constraints at the origin - on the frequency response of (7). This results in the following set of linear constraints :

$$Cx = K \tag{18}$$

where C is a rectangular matrix and K is a vector.

The vector x of filter coefficients appearing in (7) will be evaluated by solving the constrained optimization problem :

Minimize E of (11) subject to Cx = K .

Using the Lagrange multipliers technique which has been applied to a related 1-D filter design problem [2], one gets the unique solution :

$$x = (1-\alpha)Q^{-1}p_b + Q^{-1}C^T(CQ^{-1}C^T)^{-1}\left[K - (1-\alpha)CQ^{-1}p_b\right] \quad . \tag{19}$$

III. CIRCULARLY SYMMETRIC LOWPASS FILTER

Let ω_p and ω_s be respectively the edges of the pass and stop bands of $H_d(\omega_1,\omega_2)$ of a circularly symmetric lowpass filter. In order to evaluate vector x of (19), one should evaluate the matrices P_s and P_a and vector p_b defined by (13)-(15). Taking the weight function to be unity everywhere, using (17) and (6), and integrating only over the first quadrant because of the circular symmetry, one gets :

$$P_s = P_{s_1} + P_{s_2} \quad , \tag{20}$$

$$P_{s_1} = \int_{\omega_2=0}^{\omega_s} \left[\int_{\omega_1=\sqrt{\omega_s^2-\omega_2^2}}^{\pi} u(\omega_1)u^T(\omega_1)d\omega_1 \right] \otimes v(\omega_2)v^T(\omega_2)d\omega_2 \quad , \tag{21}$$

$$P_{s_2} = \int_0^{\pi} u(\omega_1)u^T(\omega_1)d\omega_1 \otimes \int_{\omega_s}^{\pi} v(\omega_2)v^T(\omega_2)d\omega_2 \quad , \tag{22}$$

$$P_a = \int_{\omega_2=0}^{\omega_p} \left[\int_{\omega_1=0}^{\sqrt{\omega_p^2-\omega_2^2}} u(\omega_1)u^T(\omega_1)d\omega_1 \right] \otimes v(\omega_2)v^T(\omega_2)d\omega_2 \quad , \tag{23}$$

$$p_b = \int_{\omega_2=0}^{\omega_p} \left[\int_{\omega_1=0}^{\sqrt{\omega_p^2-\omega_2^2}} u(\omega_1)d\omega_1 \right] \otimes v(\omega_2)d\omega_2 \quad . \tag{24}$$

Using (4), one can derive the following expressions for the inner integrals in (21), (23) and (24) and the single integrals in (22) :

$$\left[\int_0^a u(\omega)d\omega\right]_r = \frac{\sin((r-1)a)}{(r-1)} \quad , \mathrm{r} = 1, \dots, \mathrm{M}_1 \tag{25}$$

$$\left[\int_0^a u(\omega)u^T(\omega)d\omega\right]_{r,c} = \frac{\sin((r+c-2)a)}{2(r+c-2)} + \frac{\sin((r-c)a)}{2(r-c)} \quad , r,c = 1,\dots,M_1 \tag{26}$$

$$\left[\int_a^{\pi} u(\omega)u^T(\omega)d\omega\right]_{r,c} = \begin{cases} \pi - a & r = c = 1 \\ \dfrac{\pi}{2} - \dfrac{a}{2} - \dfrac{\sin(2(r-1)a)}{4(r-1)} & r = c = 2,\ldots,M_1 \\ -\dfrac{\sin((r+c-2)a)}{2(r+c-2)} - \dfrac{\sin((r-c)a)}{2(r-c)} & r \neq c\,, r,c = 1,\ldots,M_1 \end{cases} \tag{27}$$

$$\int_0^{\pi} u(\omega)u^T(\omega)d\omega = Diag\{\pi, 0.5\pi, \cdots, 0.5\pi\} \quad . \tag{28}$$

Substituting (25) in (24), the r th element of vector p_b reduces to :

$$\left(p_b\right)_r = \int_0^{\omega_p} \frac{\sin\left((r_1-1)\sqrt{\omega_p^2-\omega_2^2}\right)}{(r_1-1)} \cos((r_2-1)\omega_2)d\omega_2 \tag{29}$$

where the index r is related to the indices r_1 and r_2 by :

$$\mathrm{r} = (\mathrm{r}_1-1)\mathrm{M}_2 + \mathrm{r}_2 \qquad , \mathrm{r}_1 = 1,\ldots,\mathrm{M}_1 \;,\; \mathrm{r}_2 = 1,\ldots,\mathrm{M}_2 \quad . \tag{30}$$

In order to evaluate the integral in (29), the definition of the following function is introduced :

$$F(\lambda,a,b) = \frac{1}{a}\int_0^{\lambda} \sin\left(a\sqrt{\lambda^2-\omega^2}\right)\cos(b\omega)d\omega \tag{31}$$

In the appendix , it will be proved that the integral in (31) has the closed-form expression :

$$F(\lambda,a,b) = \frac{\pi}{2}\frac{\lambda}{\sqrt{a^2+b^2}} J_1\left(\lambda\sqrt{a^2+b^2}\right) \tag{32}$$

where $J_1(x)$ is Bessel function of the first kind of order one. In the same Appendix it will be proved that (32) is also valid in the limiting cases of a = 0 or b = 0 , i.e.

$$F(\lambda,0,b) \equiv \int_0^{\lambda} \sqrt{\lambda^2-\omega^2}\cos(b\omega)d\omega \tag{33-a}$$

$$F(\lambda,0,b) = \frac{\pi}{2}\frac{\lambda}{b} J_1(\lambda b) \tag{33-b}$$

$$F(\lambda,a,0) \equiv \frac{1}{a}\int_0^{\lambda} \sin\left(a\sqrt{\lambda^2-\omega^2}\right)d\omega \tag{34-a}$$

$$F(\lambda,a,0) = \frac{\pi}{2}\frac{\lambda}{a} J_1(\lambda a) \tag{34-b}$$

Defining the function $G(\lambda,a)$ as :

$$G(\lambda,a) \equiv F(\lambda,a,0) \tag{35-a}$$

and using (33-b) and (34-b), one gets :

$$G(\lambda,a) = F(\lambda,0,a) \quad . \tag{35-b}$$

Moreover in the extreme case of a = b = 0 , (31), (33) - (35) reduce to :

$$F(\lambda,0,0) \equiv G(\lambda,0) \equiv \int_0^{\lambda} \sqrt{\lambda^2-\omega^2}d\omega = \frac{\pi}{4}\lambda^2 \quad . \tag{36}$$

After the above digression, Eq (29) can be expressed using definition (31) as :

$$\left(p_b\right)_r = F(\omega_p, r_1-1, r_2-1) \quad . \tag{37}$$

Similarly substitute (26) in (23) to obtain :

$$\left(P_a\right)_{r,c} = \int_0^{\omega_p}\left[\frac{\sin\left((r_1+c_1-2)\sqrt{\omega_p^2-\omega_2^2}\right)}{2(r_1+c_1-2)}+\frac{\sin\left((r_1-c_1)\sqrt{\omega_p^2-\omega_2^2}\right)}{2(r_1-c_1)}\right]\cos((r_2-1)\omega_2)\cos((c_2-1)\omega_2)d\omega_2 \tag{38}$$

and upon using (31) one gets :

$$\begin{aligned}\left(P_a\right)_{r,c} = \frac{1}{4}\big[&F(\omega_p,r_1+c_1-2,r_2+c_2-2)+F(\omega_p,r_1+c_1-2,r_2-c_2)\\ &+F(\omega_p,r_1-c_1,r_2+c_2-2)+F(\omega_p,r_1-c_1,r_2-c_2)\big]\end{aligned} \tag{39}$$

where the index c is related to the indices c_1 and c_2 by :

$$c = (c_1-1)M_2+c_2 \qquad , \ c_1 = 1,\ldots,M_1 \ , \ c_2 = 1,\ldots,M_2 \ . \tag{40}$$

In order to evaluate the elements of matrix P_{s_1} of (21) by utilizing (27), three subcases arise :

(a) For $r_1 \neq c_1$:

$$\left(P_{s_1}\right)_{r,c} = \int_0^{\omega_s}\left[-\frac{\sin\left((r_1+c_1-2)\sqrt{\omega_s^2-\omega_2^2}\right)}{2(r_1+c_1-2)}-\frac{\sin\left((r_1-c_1)\sqrt{\omega_s^2-\omega_2^2}\right)}{2(r_1-c_1)}\right]\cos((r_2-1)\omega_2)\cos((c_2-1)\omega_2)d\omega_2 \tag{41}$$

and using (31), one gets :

$$\begin{aligned}\left(P_{s_1}\right)_{r,c} = -\frac{1}{4}\big[&F(\omega_s,r_1+c_1-2,r_2+c_2-2)+F(\omega_s,r_1+c_1-2,r_2-c_2)\\ &+F(\omega_s,r_1-c_1,r_2+c_2-2)+F(\omega_s,r_1-c_1,r_2-c_2)\big]\end{aligned} \tag{42}$$

(b) For $r_1 = c_1$, $r_1,c_1 > 1$:

$$\left(P_{s_1}\right)_{r,c} = \int_0^{\omega_s}\left[\frac{\pi}{2}-\frac{1}{2}\sqrt{\omega_s^2-\omega_2^2}-\frac{\sin\left(2(r_1-1)\sqrt{\omega_s^2-\omega_2^2}\right)}{4(r_1-1)}\right]\cos((r_2-1)\omega_2)\cos((c_2-1)\omega_2)d\omega_2 \tag{43}$$

and using (31) and (35-b) , one gets :

$$\begin{aligned}\left(P_{s_1}\right)_{r,c} = \frac{1}{4}\big[&S(\omega_s,r_2+c_2-2)+S(\omega_s,r_2-c_2)-G(\omega_s,r_2+c_2-2)-G(\omega_s,r_2-c_2)\\ &-F(\omega_s,2(r_1-1),r_2+c_2-2)-F(\omega_s,2(r_1-1),r_2-c_2)\big]\end{aligned} \tag{44}$$

where the function $S(\lambda,a)$ has been defined by :

$$S(\lambda,a) \equiv \pi\int_0^{\lambda}\cos(a\omega)d\omega = \pi\frac{\sin(a\lambda)}{a} \tag{45}$$

and is valid for the limiting case of a = 0, i.e.

$$S(\lambda,0) = \pi\lambda \ . \tag{46}$$

(c) For $r_1 = c_1 = 1$:

$$\left(P_{s_1}\right)_{r,c} = \int_0^{\omega_s}\left(\pi-\sqrt{\omega_s^2-\omega_2^2}\right)\cos((r_2-1)\omega_2)\cos((c_2-1)\omega_2)d\omega_2 \tag{47}$$

and using (35-b) and (45), one gets :

$$\left(P_{s_1}\right)_{r,c} = \frac{1}{2}\left[S(\omega_s,r_2+c_2-2)+S(\omega_s,r_2-c_2)-G(\omega_s,r_2+c_2-2)-G(\omega_s,r_2-c_2)\right] \tag{48}$$

IV. AN ILLUSTRATIVE EXAMPLE

A lowpass circular filter is designed using the results of the previous section to approximate an ideal filter with $\omega_p = 0.5\pi$ and $\omega_s = 0.6\pi$ rad. The support of the impulse response is such that $N_1 = N_2 = 45$. Figure 1 is the perspective plot of the frequency response of a filter designed using 25 constraints (24 flatness constraints at the origin and one normalization constraint) and $\alpha = 0.2$

It should be mentioned that the IMSL routines were used for computing the vector x of (19). For a small number of constraints no numerical instability problems arose; however such problems are expected to arise for a large number of constraints .

V. CONCLUSION

Circularly symmetric lowpass FIR filters have been designed using closed-form expressions derived based on a constrained optimization problem formulation. The minimization criterion represents a measure of the error between the ideal and actual frequency responses. The constraints can represent flatness conditions at the origin.

APPENDIX

Proof of Eq (32) :

Changing the variable of integration in (31) according to :

$$\omega = \lambda \sin\phi \tag{B1}$$

one gets :

$$F(\lambda, a, b) = \frac{1}{a}\lambda \int_0^{0.5\pi} \sin(a\lambda\cos\phi)\cos(b\lambda\sin\phi)\cos\phi d\phi \tag{B2}$$

Applying the following formula from the table of definite and infinite integrals [3] :

$$\int_0^{0.5\pi} \cos\phi \sin(a\cos\phi)\cos(b\sin\phi)d\phi = \frac{\pi}{2}\frac{a}{\sqrt{a^2+b^2}} J_1\left(\sqrt{a^2+b^2}\right) \tag{B3}$$

one immediately obtains (32).

Proof of Eq (33-b) :

Changing the variable of integration in (33-a) according to (B1), one gets :

$$F(\lambda, 0, b) = \lambda^2 \int_0^{0.5\pi} \cos^2\phi \cos(b\lambda\sin\phi)d\phi \tag{B4}$$

In order to evaluate this integral, one starts by the following formula [4] :

$$\exp(jx\sin\phi) = \sum_{n=-\infty}^{\infty} J_n(x)\exp(jn\phi) \tag{B5}$$

and by taking the real parts, one gets :

$$\cos(x\sin\phi) = \sum_{n=-\infty}^{\infty} J_n(x)\cos(n\phi) \ . \tag{B6}$$

Therefore

$$\int_0^{0.5\pi} \cos^2\phi\cos(x\sin\phi)d\phi = \sum_{n=-\infty}^{\infty} J_n(x) \int_0^{0.5\pi} \cos(n\phi)\cos^2\phi d\phi \tag{B7}$$

Since

$$\int_0^{0.5\pi} \cos(n\phi)\cos^2\phi d\phi = \begin{cases} (-1)^{0.5(n+1)} \dfrac{2}{n(n^2-4)} & \text{, n odd} \\ \dfrac{\pi}{4} & \text{, n} = 0 \\ \dfrac{\pi}{8} & \text{, n} = \pm 2 \\ 0 & \text{, n even , n} \neq 0 \text{, n} \neq \pm 2 \end{cases} \tag{B8}$$

and

$$J_n(x) = (-1)^n J_{-n}(x) \tag{B9}$$

and using the recurrence formula, (B7) reduces to :

$$\begin{aligned} \int_0^{0.5\pi} \cos^2\phi\cos(x\sin\phi)d\phi &= \frac{\pi}{4}\left(J_0(x) + J_2(x)\right) \\ &= \frac{\pi}{2}\frac{J_1(x)}{x} \end{aligned} \ . \tag{B10}$$

Therefore (B4) can be expressed as :

$$F(\lambda,0,b) = \frac{\pi}{2}\frac{\lambda}{b} J_1(\lambda b) \ . \tag{B11}$$

Proof of Eq(34-b) :

Changing the variable of integration in (34-a) according to (B1), one gets :

$$F(\lambda,a,0) = \frac{\lambda}{a}\int_0^{0.5\pi} \sin(a\lambda\cos\phi)\cos\phi d\phi \tag{B12}$$

and consequently

$$F(\lambda,a,0) = \frac{\pi}{2}\frac{\lambda}{a} J_1(\lambda a) \ . \tag{B13}$$

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Figure Captions

Fig. 1 : A perspective plot of the frequency response of the designed filter for the case of $\alpha = 0.2$.