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On Smooth Topological Structures

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ABSTRACT

Abstract

Our main object of this thesis is to investigate smooth structures (topology, uniformity, proximity and topogenity) when some information are known about their smooth structures and vise versa.

Keywords:

Smooth topology, Smooth uniformity, Smooth grill, Smooth proximity and Smooth topogenity.

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Arabic summary

SUMMARY

Summary

A large part of mathematics is based on the notion of a set and on binary logic. Statements are either true or false and an element either belongs to a set or not.

In (1965), Zadeh [111] defined the notion of a fuzzy subset, an element $\mu \in I^X$ was called a fuzzy set in X, with $\mu(x)$ being interpreted as the degree to which x belongs to fuzzy set μ, and the elements $\mu \in I^X$ are generalization of subsets of X. Zadeh showed that the notion of a fuzzy set calculus can be carried over to this larger settings. The notion of a fuzzy set has been used, on the other hand, by computer scientists and engineers to develop the theory of fuzzy logic and hence to design fuzzy logic controllers. The most interesting articles on the applications of fuzzy sets, see, for example, Yager [108] in (1982).

Since then, mathematicians have been attempting to extend fundamental mathematical notions to fuzzy setting, like algebra and topology, replacing subsets by fuzzy subsets and standard notions by analogous fuzzy notions. Since there are many meaningful ways to extend notions, there has been a certain amount of debate on the relative merits of the different fuzzifications of each classical notion.

Topology and some of their related topics, proximity, uniformity and topogenity are extended in the fuzzy mathematics and also which developed itself fuzzy mathematics. In particular, one of the extensions of the notion of a topology was first defined by Chang [14] in (1968) and made an attempt to

develop basic topological notions for such spaces. Since then, many authors as, Wong [105], Hutton [45], Lowen [63], Gougen [36], Pao and Ying [76, 77], and other discussed respectively various aspects of fuzzy topology. In these authors, a fuzzy topology τ on a set X is defined as a classical subsets of the fuzzy power-set I^X. The open sets are the fuzzy subsets that belong to τ, i.e., $\lambda \in I^X$ is open or not open which is a crisp treatments.

The notions of fuzzy proximity introduced and studied by Katasars [50] in (1979) on a set X as a binary relation δ on the collection of the fuzzy subsets of $I^X \times I^X$ satisfying certain axioms. Also, the fuzzy proximity is a crisp relation between fuzzy subsets, i.e., for two fuzzy subsets $\lambda, \mu \in I^X$, $\delta(\lambda, \mu) = 1$ or $\delta(\lambda, \mu) = 0$, which is also a crisp treatment. However, this definition turned out to be unsuccessful, in particular, because of this fact that fuzzy proximities as crisp relations are in a canonical one-to-one corresponding fuzzy proximity induced on X the same crisp topology. In (1989) Morsi [74] found a characterization of the concept of fuzzy proximity introduced by Artico and Moresco [2]. This characterization of Artico-Moresco fuzzy proximities show that these fuzzy proximities are much closely connected with ordinary fuzzy proximities than Katasars fuzzy proximities. Morsi showed that the fuzzy topology is determined by a nieghborhood structure in the sense of Lowen [65], and all these concepts are consistent with Chang fuzzy topologies (or with Lowen fuzzy topologies [64] as a special case of Chang's).

Fuzzy uniformities have two roots tracing back to Lowen [66] and to Hutton [46]. Lowen defined fuzzy uniformities as a fuzzification of the entourage approach to uniformities, while Hutton followed a variation of the covering approach to uniformities.

Mingsheng [71] introduced the concept of a fuzzifying uniformity to developed foundations of the corresponding theory. Fuzzifying uniformities are the uniform counterparts of fuzzifying topologies [68-70].

In fuzzy topology the classic (Csaszar's [18,19]) theory of topogenous structures was reflected mainly in the form of two different theories of fuzzy topogenous structures: both of these theories were developed by Katsaras and Petalas. The first one of these theories worker out in [54, 55] presents a unified approach to these theories of Chang fuzzy topological spaces, Hutton uniform spaces [46] and Katasars fuzzy proximity spaces. The second one, developed in [58, 59], establishes common framework to the theories of Lowen fuzzy topological spaces [63], Lowen-Hohle fuzzy uniform spaces [40, 66] and Artico-Moresco fuzzy proximity spaces [3]. Thus both of these approaches to the fuzzification of the concept of a topogenous structures originate from Chang's concept of a fuzzy topology i.e., realize a fuzzy topology on a set X as an ordinary (crisp) subset τ of the family I^X of fuzzy subsets of X.

It is easy to see that they have always investigated fuzzy objects with crisp methods. For example, fuzziness in the concept of openness of a fuzzy set has not been considered, which seems to be a drawback in the process of fuzzification of the concept of topological spaces. Generally when we extend a mathematical structure to fuzzy sets we have in mind an extension of this mathematical object which will work with fuzzy sets in place of ordinary subsets. we think that it could be more interesting to reformulate the defining axioms themselves in terms of fuzzy logic. In the case of fuzzy logic the truth values is an element in the closed unit interval I and can be called "truth degree" of a particular proposition, in this case an axiom like $P \Rightarrow Q$ as defining a constraint between the truth values p,q (of P,Q resp.). Here we have taken $p \leq q$.

By the ends of eighties and beginning of nineties many mathematician remarked that the fuzziness in these extensions is not enough, since we handle with fuzzy subsets but the handing is crisp.

For this reason many mathematicians try to make a fuzzy treatment for this structures. Šostak in (1985) [93] introduced a new definition of fuzzy topology as an extension of both crisp topology and Chang's fuzzy topology (which we call smooth topology), according to which a smooth topology on a set X is a fuzzy subset of the powerset I^X (i.e, a mapping $\tau : I^X \to I$) satisfying certain axioms. In (1992), smooth topological spaces in Šostak sense were independently redefined by Ramadan [80]. It has been developed in many directions [4, 40, 41, 20-23, 36, 43, 71, 78, 84].

In (1993) Badard, Ramadan and Mashhour [8] introduced the concept of smooth preuniformity and smooth preproximity spaces using the concepts of a gradation of uniformity and a gradation proximity [7]. The concept of smooth pretopogenous structure is introduced by Ramadan [81]. In (1997) Ramadan [82] introduced the concept of smooth filter and some fundamental properties. For more details on smooth topological structures and some related concepts we refer to [30, 31, 32, 43, 67, 100, 191].

As continuation to study of a framework of smooth topological structures, our purpose here to investigate more further the structures (smooth topologies, smooth grills, smooth proximities, smooth uniformities and smooth topogenous) when Some information are known about their fuzzy structure and vise versa.

This thesis includes a preface, five Chapters 0-IV, and a list of Bibliography.

In Chapter 0, we attempt to cover enough of fundamental concepts, definitions and known results concerning our subject to make this thesis to a some what self contained.

In Chapter I, the aim of this chapter is give the notation of the smooth uniform spaces. We study the relations between smooth topology and smooth uniform spaces. The product of smooth uniform spaces is studied

In Chapter II, we introduce the concepts of smooth grills and smooth proximity spaces and we prove some of their properties. The links between smooth proximity, smooth topology and smooth uniformity are given.

In Chapter III, we deal further with the theory of smooth syntopogenous structures. In section 3.1 we introduce the basic concepts, some properties, product and subspaces of the smooth topogenous structures. In section 3.2 we study the links between smooth (semi-) topogenous order and smooth (supra) topology.

In Chapter IV, we investigate further the concept of smooth topogenous spaces compatible with smooth uniform spaces. In section 4.1 we construct smooth topogenous spaces from smooth uniform spaces. In section 4.2 we introduce smooth uniform spaces induced by smooth topogenous spaces.

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Publications

Most of the results of this thesis either have been accepted or submitted for publication as follows:

(1) Smooth uniform spaces, Journal of Korea Fuzzy logic and intelligent Society, 2(1) (2002) 83-86.

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(4) Fuzzy semi-topogenous orders, J. of Fuzzy mathematics (submitted)

(5) On fuzzy syntopogenous structures, J. of Fuzzy mathematics (submitted).

CHAPTER 0

Chapter 0

Introduction

For the sake of fixing notation, we recall some basic definitions. We shall let X be a nonempty and I be the closed unit interval and we let $I_\circ = I - \{0\}$ $= (0,1]$, $I_1 = I - \{1\} = [0,1)$. A fuzzy set in X is an element of the set I^X of all functions from the set X into I, denoted by $\lambda, \mu \in I^X$. A fuzzy set, which assigns to each element in X the value $\alpha, 0 \le \alpha \le 1$, is denoted by $\underline{\alpha}$. For any two fuzzy sets λ and μ, $\underline{1} - \lambda, \lambda \vee \mu, \lambda \wedge \mu,$ and $\lambda \le \mu$ have their usual meanings. We denote the characteristic function of a subset A of 2^X by 1_A. If $\mu \in I^X$ then we define $\mu^\alpha = \{x \in X \setminus \mu(x) > \alpha\}$ and supp$\mu = \{x \in X \setminus \mu(x) > 0\}$. A fuzzy relation on X is a function $u : X \times X \to I$ and $I^{X \times X}$ is the set of all fuzzy relations on X, denoted by $u, v \in I^{X \times X}$.

0.1 Topological structures

Historically, the attempt to develop the fuzzy counterpart of general topology was undertaken by C. L. Chang in 1968 [14] and is called *fuzzy topology*. In the last the fuzzy topologies were defined as certain subsets τ of the power set I^X of fuzzy subsets of X. Thus, to be consistent, they are preferably to be considered as crisp topological type structures on the families of fuzzy sets than fuzzy topologies while the term a fuzzy topology is related to some fuzzy structure of topological type on the fuzzy power sets I^X. For the first time, the idea of such an approach was probably expressed in U. Höhle's paper [40]. However, in that paper, fuzzy topological structures were considered only on the power set 2^X of crisp subsets of X. In more general situations similar ideas in the mid-1980s were independently discussed in [28, 63, 90, 91].

These kinds of topologies are called smooth topologies. In the sequel we a survey of the important concepts and some properties of the fuzzy and smooth topological structures.

0.1.a Fuzzy topological structures

0.1.a.1 Definition [14]

A subset $\tau \subset I^X$ is called a *fuzzy topology* on X if it satisfies for $\mu, \lambda \in I^X$, the following conditions:

(O1) $\underline{1}, \underline{0} \in \tau$,

(O2) If $\mu, \lambda \in \tau$, then $\mu \wedge \lambda \in \tau$,

(O3) If $\mu_i \in \tau, \forall i \in \Gamma$, then $\sup_{i \in \Gamma} \mu_i \in \tau$.

The pair (X, τ) is called a *fuzzy topological space* and the fuzzy set belonging to τ is called *open* in this space.

Soon J. A. Goguen [35] proposed a natural generalization of the preceding definition by substituting L-fuzzy sets for fuzzy sets. Namely, according to J. A. Goguen, *L-fuzzy topology* on X.

0.1.a.2 Definition [14]

Let (X, τ_1) and (Y, τ_2) be two fuzzy topological spaces. A function $f : X \to Y$ is called a *fuzzy continuous* function if

$$\lambda \in \tau_2 \Rightarrow f^{-1}(\lambda) \in \tau_1 \text{ for all } \lambda \in I^X.$$

0.1.a.3 Definition [103]

A fuzzy set μ in a fuzzy topological space is called *close,* if its complement $1 - \mu$ is open. It is clear that the family σ of all closed fuzzy subsets of a given fuzzy topological space has the following properties:

(CO1) $\underline{1}, \underline{0} \in \sigma$,

(CO2) If $\mu, \lambda \in \sigma$, then $\mu \vee \lambda \in \tau$, (CO3) If $\mu_i \in \tau, \forall i \in \Gamma$, then $\inf_{i \in \Gamma} \mu_i \in \tau$.

0.1.a.4 Definition [76]

The *closure*, $\overline{\mu}$ of $\mu \in I^X$ is the intersection of all closed fuzzy subsets containing μ, i.e.,

$$\overline{\mu} = \cap\{1 - \lambda \in \tau \setminus \mu \leq \lambda\}$$

0.1.a.5 Definition [36, 103]

The *closure operator* is a function $^{-} : I^X \to I^X$ satisfying the following conditions:

(1) $\overline{0} = 0$,

(2) $\overline{\mu} \leq \mu$,

(3) $\overline{\mu \vee \lambda} = \overline{\mu} \vee \overline{\lambda}$,

(4) $\overline{\overline{\mu}} = \overline{\mu}$.

The concepts of closeness and closure operator, as well as that of the interior operator $^{\circ} : I^X \to I^X$ where $\mu^{\circ} = \cup\{\lambda \in \tau \setminus \lambda \leq \mu\}$, can be used the characterize the continuity of functions of fuzzy topological spaces. Namely, the following four properties are equivalent for a function $f : X \to Y$ (see, e.g., [103]).

(1) f is continuous,

(2) If $\lambda \in \sigma_Y$ then $f^{-1}(\lambda) \in \sigma_X$,

(3) $f(\overline{\mu}) \leq \overline{f(\mu)}, \forall \mu \in I^X$,

(4) $f^{-1}(\lambda^{\circ}) \leq (f^{-1}(\lambda))^{\circ}, \forall \mu \in I^X$.

0.1.a.6 Definition [92]

Let (X,τ) be a fuzzy topological space and $Y \subset X$. The induced fuzzy topology on Y is defined as $\tau_Y = \{\mu_Y = \mu \backslash_Y : \mu \in \tau\}$, where $\mu \backslash_Y$ denotes the restriction of μ to the set Y.

It is easy to verify that the natural inclusion function $i:(Y,\tau_Y) \to (X,\tau)$ is continuous, in this case, and, moreover, τ_Y can be characterizes the weakest (in the sense of $\subset$) fuzzy topological space on Y for which the inclusion i is continuous.

0.1.a.7 Definition [36]

Let $\{(X_i,\tau_i) \backslash i \in \Gamma\}$ be a family of fuzzy topological spaces, and let $X = \prod_{i \in \Gamma} X_i$ be the product of the corresponding sets and $p_i : X \to X_i$ denote the corresponding projection. Let $P = \{\mu = p_i^{-1}(\lambda_i) \backslash \lambda_i \in \tau_i, i \in \Gamma\}$ and $B = \{\mu_{i_1} \wedge \ldots \wedge \mu_{i_n} \backslash n \in N, \mu_{i_1} \in p\}$, i.e., B is the family of all finite meets of elements from P. The product fuzzy topology τ on X can be defined as the family of all joins of elements from B, i.e., $\tau = \{\gamma = \sup_j \mu^j \backslash \{\mu^j : j \in \Delta\} \subset B\}$.

Similarly, as the standard terminology from general topology to the fuzzy case, one can say that P is a subbase and B is a base for the product fuzzy topology τ.

0.1.b Smooth topological structures

0.1.b.1 Definition [80]

A function $\tau : I^X \to I$ is called a *smooth topology* on X if it satisfies the following conditions:

(SO1) $\tau(\underline{0}) = \tau(\underline{1}) = 1$,

(SO2) $\tau(\mu_1 \wedge \mu_2) \geq \tau(\mu_1) \wedge \tau(\mu_2)$ for each $\mu_1, \mu_2 \in I^X$, (SO3) $\tau(\sup_{i\in\Gamma} \mu_i) \geq \inf_{i\in\Gamma} \tau(\mu_i)$ for any $\{\mu_i\}_{i\in\Gamma} \subset I^X$.

The pair (X,τ) is called a *smooth topological space*. The value $\tau(\mu)$ of a smooth topology τ on X expresses the degree to that μ is open.

0.1.b.2 Definition [40]

A *smooth cotopology* (or a gradation of closeness) is defined as a function $\Im : I^X \to I$ which satisfies:

(SCO1) $\Im(\underline{0}) = \Im(\underline{1}) = 1$,

(SCO2) $\Im(\lambda_1 \vee \lambda_2) \geq \Im(\lambda_1) \wedge \Im(\lambda_2)$ for each $\lambda_1, \lambda_2 \in I^X$,

(SCO3) $\Im(\inf_{i\in\Gamma} \lambda_i) \geq \inf_{i\in\Gamma} \Im(\lambda_i)$ for any $\{\lambda_i\}_{i\in\Gamma} \subset I^X$.

On the set $\tau(X)$ of all smooth topologies on X we can introduce a partial ordering $\leq$ by: $\tau_1 \leq \tau_2$ iff $\tau_2(\mu) \leq \tau_1(\mu)$, for all $\mu \in I^X$. In particular τ_1 is *coarser* than τ_2 (or τ_2 is *finer* than τ_1) iff $\tau_1 \leq \tau_2$. Obviously the function $\tau_{ind} : I^X \to I$ defined by $\tau_{ind}(\mu) = 1, \forall \mu \in I^X$ is the finest smooth topology on X.

0.1.b.3 Remark

If smooth topology τ on X satisfies the following fourth property (ST4) $\tau(I^X) \subset \{0,1\}$ (resp. $\tau : 2^X \to I$), then such smooth topology in the one-to-one way corresponds to a fuzzy topology [14] in Chang's sense (resp. fuzzifying topology [68-70] in Ming's sense)

0.1.b.4 Definition [80]

Let (X,τ_1) and (Y,τ_2) be smooth topological spaces. Let $f : X \to Y$ be a function. Then:

(1) f is called smooth continuous iff $\tau_2(\mu) \leq \tau_1(f^{-1}(\mu))$ for each $\mu \in I^Y$,

(2) f is called smooth open iff $\tau_1(\lambda) \leq \tau_2(f(\lambda))$ for each $\lambda \in I^X$,

(3) f is called smooth closed iff $\tau_1(1-\lambda) \le \tau_2(f(1-\lambda))$ for each $\lambda \in I^X$.

0.1.b.5 Example

Let $\tau : I^X \to I$ be a function defined by:

$$\tau(\mu) = \inf\{\mu(x) : x \in \text{supp}\mu\}$$

and $\tau(\underline{0}) = 1$. Then τ is a smooth topology on X.

0.1.b.6 Proposition [80]

Let (X,τ) be a smooth topological space. For each $\alpha \in I$, let $\tau_\alpha = \{\mu \in I^X \setminus \tau(\mu) \ge \alpha\}$. Then τ_α is a fuzzy topology on X (in the sense of Chang), which the α-level fuzzy topology.

0.1.b.7 Definition [16]

Let (X,τ) be a smooth topological space. A function $C_\tau : I^X \times I_1 \to I^X$ defined by

$$C_\tau(\lambda, r) = \inf\{\rho \in I^X \setminus \lambda \le \rho, \tau(\underline{1} - \rho) \ge r\}.$$

is called *smooth supra closure operator* if it satisfies for each $\lambda, \lambda_1, \lambda_2 \in I^X$ and $r, r_1, r_2 \in I_1$ the following properties:

(1) $C_\tau(\underline{0}, r) = \underline{0}$,

(2) $\lambda \le C_\tau(\lambda, r)$,

(3) If $\lambda_1 \le \lambda_2$, then $C_\tau(\lambda_1, r) \le C_\tau(\lambda_2, r)$,

(4) If $r_1 \le r_2$, then $C_\tau(\lambda, r_1) \le C_\tau(\lambda, r_2)$,

A smooth supra closure operator C_τ is called *smooth closure operator* if it satisfies;

(5) $C_\tau(\lambda_1 \vee \lambda_2, r) = C_\tau(\lambda_1, r) \vee C_\tau(\lambda_2, r)$,

A smooth supra closure operator C_τ is called *topological* if it satisfies;

(6) $C_\tau(C_\tau(\lambda, r), r) = C_\tau(\lambda, r)$.

0.1.b.8 Definition [43]

Let (X,τ) be a smooth topological space. A function $\mathrm{I}_\tau : I^X \times I_1 \to I^X$ defined by

$$\mathrm{I}_\tau(\lambda,r) = \inf\{\nu \in I^X \setminus \nu \leq \lambda, \tau(\nu) \geq r\}.$$

is called *smooth supra interior operator* on X if it satisfies for $\lambda, \lambda_1, \lambda_2 \in I^X$ and $r, r_1, r_2 \in I_1$ the following properties:

I1) $\mathrm{I}_\tau(\underline{1}, r) = \underline{1}$.

I2) $\mathrm{I}_\tau(\lambda, r) \leq \lambda$.

I3) If $\lambda_1 \leq \lambda_2$, then $\mathrm{I}_\tau(\lambda_1, r) \leq \mathrm{I}_\tau(\lambda_2, r)$.

I4) If $r \leq s$, then $\mathrm{I}_\tau(\lambda, r) \leq \mathrm{I}_\tau(\lambda, s)$.

A smooth supra interior operator I_τ is called *smooth supra interior* it satisfies:

I5) $\mathrm{I}_\tau(\lambda_1 \wedge \lambda_2, r) = \mathrm{I}_\tau(\lambda_1, r) \wedge \mathrm{I}_\tau(\lambda_2, r)$.

A smooth interior operator I_τ is called *topological* it satisfies:

T6) $\mathrm{I}_\tau(\mathrm{I}_\tau(\lambda, r), r) = \mathrm{I}_\tau(\lambda, r)$.

0.1.b.9 Definition [62]

Let $\underline{0} \notin \Theta$ be a subset of I^X. A function $B : \Theta \to I$ is called a *base* on X if it satisfies the following conditions:

(B1) $B(\underline{1}) = 1$,

(B2) $B(\lambda_1 \wedge \lambda_2) \geq B(\lambda_1) \wedge B(\lambda_2), \forall \lambda_1 \wedge \lambda_2 \in \Theta$.

A base B always *generates* a smooth topology τ_B on X in the following sense:

0.1.b.10 Theorem [62]

Let B be a base on X. For each $\lambda \in I^X$, we define the function $\tau_B : I^X \to I$ as follows:

$$\tau_B(\lambda) = \begin{cases} \sup\{\inf_{j \in \Delta} B(\lambda_j)\}, & \text{if } \lambda = \sup_{j \in \Delta} \lambda_j, \text{ for } \{\lambda_j\}_{j \in \Delta} \subset \Theta, \\ 1, & \text{if } \lambda = \underline{0}, \\ 0, & \text{otherwise.} \end{cases}$$

Then (X, τ_B) is a smooth topological space.

0.1.b.11 Theorem [62]

Let $\{(X_i, \tau_i) \setminus i \in \Gamma\}$ be a family of smooth topological spaces. X is a set and, for each $i \in \Gamma, f_i : X \to X_i$ is a function. Let

$$\Theta = \{\underline{0} \neq \lambda = \inf_{j=1}^{n} f_{k_j}^{-1}(\lambda_{k_j}) \setminus \tau_{k_j}(\lambda_{k_j}) > 0, \forall k_j \in K\}$$

for every finite index set $K = \{k_1, \ldots, k_n\} \subset \Gamma$. Define the function $B : \Theta \to I$ on X by

$$B(\lambda) = \sup\{\inf_{j=1}^{n} \tau_{k_j}(\lambda_{k_j}) \setminus \lambda = \inf_{j=1}^{n} f_{k_j}^{-1}(\lambda_{k_j})\}.$$

For every finite index set $K = \{k_1, \ldots, k_n\} \subset \Gamma$. Then:

(1) B is a base on X,

(2) The smooth topology τ_B generated by B is the coarsest smooth topology on X which for each $i \in \Gamma, f_i$ is fuzzy continuous,

(3) A function $f : (Y, \tau') \to (X, \tau_B)$ is fuzzy continuous iff for each $i \in \Gamma$, $f_i \circ f : (Y, \tau') \to (X_i, \tau_i)$ is fuzzy continuous.

0.1.c Mingsheng Ying's Fuzzifying topologies:

A specific viewpoint on what can be the subject of fuzzy topology was developed by Mingsheng Ying [68-70]. Contrary to the approaches discussed in the previous sections, all of which could be united under the name of point-set lattice-theoretic fuzzy topology, Mingsheng Ying's theory, based on the semantic analysis of concepts and results of general topology, is to be referred to the so-called model-theoretic fuzzy topology (we make use here of S.E. Rodabaugh's terminology, slightly modified, (see [88, 89]). By means of the semantic method of continuous-valued logic, Mingsheng Ying arrives at the concept of a fuzzifying topology on a set X (which is, in fact, a function $\tau : 2^X \to I$ satisfying the same axioms of smooth topology) and then consistently develops the theory of fuzzifying topologies.

The theory developed up until now [68-70] includes such items as local structure of fuzzifying topologies, their convergence structure axioms of countability, compactness (including a version of the Tychonoff theorem), connectedness, and others. All these concepts appear to be predicates of multivalued logic and can take values from I.

0.1.c.1 Proposition [68]

τ is a fuzzifying topology iff for any $\alpha \in I$, τ_α is a classical topology.

0.1.c.2 Proposition [58]

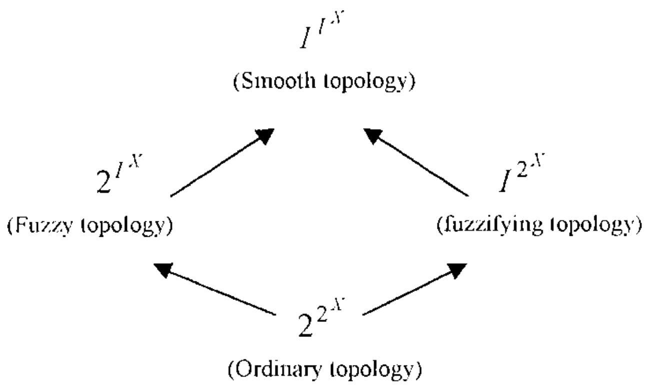
If T is a fuzzifying topology, and $\tau : I^X \to I$ be a function given by:

$$\tau(\mu) = \inf_{\alpha \in I} T(\mu_\alpha).$$

Then τ is a smooth topology.

These observations enables us to reduce the study of certain properties of a smooth topology τ to the study of much simpler objects, the corresponding α-level Chang fuzzy topology τ_α.

There are natural injections between the sets as indicated:



0.1.c.3 Proposition [84]

If T is a fuzzifying topology, and $\tau : I^X \to I$ be a function given by:

$$\tau(\lambda) = \sup_{\alpha \in I} (T(\lambda_\alpha) \wedge \alpha).$$

Then τ is a smooth topology.

0.1.c.4 Proposition [84]

A function $f : (X, \tau_1) \to (Y, \tau_2)$ between smooth topological spaces is a smooth continuous if $f : (X, T_1) \to (Y, T_2)$ is a fuzzifying continuous, i.e.,

$$\tau_2(M) \leq \tau_1(f^{-1}(M)) \text{ for each } M \in 2^Y.$$

0.1.d Smooth filter

The notion of smooth filter [82] is one of the most important concept in smooth topology. We define a smooth filter as a function $F : I^X \to I$ that is a fuzzy subset of the power set I^X rather than a crisp subset. Similar ideas have been discussed by some other authors (Gähler [26], Garcia, Prada and Burton [33], Höhle and Šostak [43] and others) with respect to different structure.

0.1.d.1 Definition [82]

A function $F : I^X \to I$ is called a *smooth filter* on X if it satisfies the following conditions:

(F1) $F(\underline{0}) = 0$,

(F2) $F(\lambda_1 \wedge \lambda_2) \geq F(\lambda_1) \wedge F(\lambda_2)$,

(F3) If $\lambda_1 \leq \lambda_2$ then $F(\lambda_1) \leq F(\lambda_2)$.

A smooth filter is said to be proper if: $F(\underline{1}) = 1$.

Specially, if $F \in 2^{I^X}$, then F is a Fuzzy filter; if $F \in I^{2^X}$, then F is a Fuzzifying filter and if $F \in 2^{2^X}$, then F is a ordinary filter.

On the set $F(X)$ of all smooth filter on X we can introduce a partial ordering $\leq$ by: $F_1 \leq F_2$ iff $F_2(\mu) \leq F_1(\mu)$, for all $\mu \in I^X$. In particular F_1 is *coarser* than F_2 (or F_2 is *finer* than F_1) iff $F_1 \leq F_2$.

0.1.d.2 Proposition [82]

F is a fuzzifying filter iff for any $\alpha \in I$, F_α is a classical filter.

0.1.d.3 Theorem [82]

A function $F': I^X \to I$ is called a *smooth filter* on X if it satisfies the following conditions:

(F1) $F'(\phi) = 0$ and $F'(X) = 1$,

(F2) $F'(M \cap N) \geq F'(M) \wedge F'(N)$,

(F3) If $N \subset M$ then $F'(N) \leq F'(M)$,

(F4) $F': I^X \to I$ is retrieved from its restriction to crisp subsets by the formula:

$$F'(\lambda) = \sup_{\alpha \in I} (\alpha \wedge F(\lambda_\alpha)), \lambda \in I^X$$

0.2 Uniform structures

0.2.a Fuzzy uniform structure

In [66] Lowen defined fuzzy uniformities as a fuzzification of the entourage approach to uniformities,

0.2.a.1 Definition

(1) For each fuzzy relation u on X and for $\lambda \in I^X$, the *image* $u[\lambda]$ of λ with respect to u is the fuzzy subset of X defined by

$$u[\lambda](x) = \sup_{y \in X} (\lambda(y) \wedge u(y,x)), \forall x, y \in X.$$

(2) The *composition* $u \circ v$ of two fuzzy relations u and v on X is the fuzzy relation on X defined by

$$u \circ v(x,y) = \sup_{z \in X} (u(x,z) \wedge v(z,y)), \forall x, y \in X$$

(3) The *symmetric* u^s of u on X is the fuzzy relation on X defined by

$$u^S(x,y) = u(y,x), \forall x, y \in X.$$

0.2.a.2 Definition [66]

A subset $U \subset I^{X \times X}$ is called a *fuzzy uniformity* on X if it satisfies for $u, v \in I^{X \times X}$, the following conditions:

(U1) $\underline{0} \notin U$,

(U2) $u \in U$ and $v \in U$ iff $u \wedge v \in U$,

(U3) $\underline{1} \in U$,

(U4) If $u \in U$, then $1_\Delta \in U$,

(U5) If $u \in U$, then $u^s \in U$,

(U6) If $u \in U$, there exists $v \in U$ such that $v \circ v \leq u$.

The pair (X, U) is said to be a *fuzzy uniform space*.

In [46] Hutton followed a variation of the covering approach to uniformities.

0.2.a.3 Notation

Let X be a set and Ω_X be the set of all functions $\alpha : I^X \to I^X$ such that

(1) $\alpha(\underline{0}) = 0$,

(2) $\alpha(\mu) \geq \mu$,

(3) $\alpha(\sup_{i\in\Gamma} \mu_i) = \sup_{i\in\Gamma} \alpha(\mu_i)$.

0.2.a.4 Remark

For $\alpha_1, \alpha_2 \in \Omega_X$, we define $\mu \in I^X$,

(a) $(\alpha_1 \wedge \alpha_2)(\mu) = \inf\{\alpha_1(\mu_1) \vee \alpha_2(\mu_2) \setminus \mu = \mu_1 \vee \mu_2\}$,

(b) $\alpha^{-1}(\mu) = \inf\{\lambda \in I^X \setminus \alpha(\underline{1} - \lambda) \leq \underline{1} - \mu\}$,

(c) $\alpha_1 \circ \alpha_2(\mu) = \alpha_1(\alpha_2(\mu))$.

Then $\alpha_1 \wedge \alpha_2, \alpha_1 \circ \alpha_2, \alpha_1^{-1} \in \Omega_X$.

0.2.a.5 Lemma

For every $\alpha, \beta, \gamma, \alpha_1, \beta_1 \in \Omega_X$, the following properties hold:

(1) If $\alpha \leq \alpha_1, \beta \leq \beta_1$ then $\alpha \wedge \beta \leq \alpha_1 \wedge \beta_1$,

(2) $\alpha \wedge \beta \leq \alpha, \alpha \wedge \beta \leq \beta$ and $\alpha \wedge \alpha = \alpha$,

(3) $(\alpha^{-1})^{-1} = \alpha$,

(4) $\alpha \leq \beta$ iff $\alpha^{-1} \leq \beta^{-1}$,

(5) Let a function $\alpha_{\underline{1}} : I^X \to I^X$ be define by

$$\alpha_{\underline{1}}(\mu) = \begin{cases} \underline{1} & \text{if } \mu \neq \underline{0}, \\ \underline{0} & \text{if } \mu = \underline{0}. \end{cases},$$

Then $\alpha_{\underline{1}} = \alpha_{\underline{1}}^{-1} \in \Omega_X$ and $\alpha \wedge \alpha_{\underline{1}} = \alpha$

(6) $(\alpha \circ \beta)^{-1} = \beta^{-1} \circ \alpha^{-1}$

(7) $(\alpha \wedge \beta)^{-1} = \alpha^{-1} \circ \beta^{-1}$,

(8) $(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma)$.

0.2.a.6 Definition [46]

A subset U of Ω_X is called a *fuzzy uniformity* on X satisfying for $\alpha, \beta \in \Omega_X$, the following condition:

(FU1) $\alpha \wedge \beta \in U$ iff $\alpha \in U$ and $\beta \in U$,

(FU2) If $\alpha \in U$ and $\beta \leq \alpha$, then $\beta \in U$,

(FU3) If $\alpha \in U$, there exists $\beta \in U$ such that $\beta \circ \beta \leq \alpha$,

(FU4) If $\alpha \in U$, then $\alpha^{-1} \in U$.

The pair (X, U) is said to be a *fuzzy uniform space*.

0.2.b Smooth uniform structure

0.2.b.1 Definition [8]

A function $U : I^{X \times X} \to I$ is called a *smooth quasi-uniformity* on X if it satisfying for $u, w \in I^{X \times X}$, the following conditions:

(SU1) if $u \not\leq 1_\Delta$, then $U(u) = 0$,

(SU2) $U(u \wedge w) = U(u) \wedge U(w)$,

(SU3) $U(\underline{1} \times \underline{1}) = 1$,

(SU4) $U(u) \leq \sup\{U(w) / w \circ w \leq u\}$.

The pair (X, U) is said to be a *smooth quasi-uniform space*.

A smooth quasi-uniformity is said to be *smooth uniformity* if it satisfies:

(SU5) $U(u) \leq U(u^s)$, where $u^s(x, y) = u(y, x)$

0.2.b.2 Definition [8]

Let U_1 and U_2 be smooth uniformity on X. We say U_1 is *finer* than U_2 (or U_2 is coarser than U_1) iff $U_2(u) \leq U_1(u)$ for all $u \in I^{X \times X}$.

0.2.b.3 Theorem [8]

Let (X, U) be a smooth uniform space. For each $\alpha \in I_1$, let $U^\alpha = \{u \in I^{X \times X} \setminus U(u) > \alpha\}$. Then U^α is a fuzzy uniformity on X.

0.2.b.4 Definition [8]

Let (X,U) and (Y,V) be smooth uniform spaces. A function $f: X \to Y$ is said to be *smooth uniform continuous* if

$$V(v) \le U((f \times f)^{-1}(v)),\ \forall v \in I^{Y \times Y}$$

0.2.b.5 Theorem [8]

Let (X,U), (Y,V) and (Z,W) be smooth uniform spaces. If $f: X \to Y$ and $g: Y \to Z$ are smooth uniform continuous, then $g \circ f: X \to Z$ is a smooth uniform continuous.

0.3 Proximity structures

0.3.a Fuzzy proximity structure

0.3.a.1 Definition [50]

A binary relation δ on I^X is called a *fuzzy proximity* on X if δ satisfies the following axioms:

(FP1) If $(\lambda,\mu) \in \delta$, then $(\mu,\lambda) \in \delta$,

(FP2) If $(\lambda,\mu) \in \delta$, then $\lambda \ne \underline{0}$ and $\mu \ne \underline{0}$,

(FP3) If $(\lambda \vee \mu, \nu) \in \delta$, then $(\lambda,\nu) \in \delta$ or $(\mu,\nu) \in \delta$.

(FP4) IF $(\lambda,\mu) \notin \delta$, then there exist $\nu \in I^X$ such that

$$(\lambda,\nu) \notin \delta,\ (\underline{1} - \nu, \mu) \notin \delta$$

(FP5) IF $\lambda \wedge \mu \ne \underline{0}$, then $(\lambda,\mu) \in \delta$.

The pair (X,δ) is said to be a *fuzzy proximity space*.

0.3.a.2 Proposition [50]

Let (X,δ) be a fuzzy proximity space. The function $\mu \to \overline{\mu} = 1 - \sup\{\rho \in I^X \setminus (\rho,\mu) \notin \delta\}$ is a fuzzy closure operator on I^X and the collection $\tau_\delta = \{\mu \in I^X \setminus 1 - \mu = \overline{1 - \mu}\}$ is a fuzzy topology on X.

0.3.a.3 Definition [50]

A function f from a fuzzy proximity (X,δ_1) to a fuzzy proximity (Y,δ_2) is called a *proximity continuous function* (or *proximally continuous*) if $(\lambda,\mu)\in\delta_1$ implies $(f(\lambda),f(\mu))\in\delta_2$ for each $\lambda,\mu\in I^X$. Equivalently, f is a proximity continuous function if $(f^{-1}(\upsilon),f^{-1}(\rho))\in\delta_1$ implies $(\upsilon,\rho)\in\delta_2$ for each $\upsilon,\rho\in I^Y$.

0.3.b Smooth proximity structure

In (1993) Badard, Ramadan and Mashhour [7] introduced the concept of smooth proximity as follows:

0.3.b.1 Definition [7]

A function $\delta: I^X\times I^X\to I$ is called a *smooth quasi-proximity* on X satisfying for $\lambda,\mu,\nu\in I^X$ the following conditions:

(SP1) $\delta(\lambda,\mu)\le\{\sup\lambda(x)\wedge\sup\mu(x)\setminus x\in X\}$,
(SP2) $\delta(\lambda\vee\mu,\nu)=\delta(\lambda,\nu)\vee\delta(\mu,\nu)$,
(SP3) For each $\lambda,\mu\in I^X$, there exists $\rho\in I^X \ni \delta(\lambda,\mu)\ge\{\delta(\lambda,\rho)\vee\delta(1-\rho,\mu)\}$.
(SP4) $\delta(\lambda,\mu)\ge\sup(\lambda\wedge\mu)(x)$.
The pair (X,δ) is said to be a *smooth quasi-proximity space.*

A smooth quasi-proximity space (X,δ) is called a smooth proximity space if

(SP) $\delta=\delta^{-1}$, where $\delta^{-1}(\lambda,\mu)=\delta(\mu,\lambda)$.

Let δ_1 and δ_2 be smooth quasi-proximities on X. We say δ_2 is *finer* than δ_1 (or δ_1 is coarser than δ_2) iff $\delta_2(\lambda,\mu)\le\delta_1(\lambda,\mu)$ for all $\lambda,\mu\in I^X$.

0.3.b.2 Remark

It is clear that:

(i) $\sup(\lambda\wedge\mu)\le\delta(\lambda,\mu)\le(\sup\lambda)\wedge(\sup\mu)$

(ii) $\delta(\underline{\alpha},\underline{\beta}) = \alpha \wedge \beta$

(iii) $\delta(\mu,\mu) = \sup \mu$

(iv) $\delta(\mu,\underline{\alpha}) = \alpha \wedge \sup \mu$

(v) If $\lambda \leq \lambda_1, \mu \leq \mu_1$, then $\delta(\lambda,\mu) \leq \delta(\lambda_1,\mu_1)$.

0.3.b.3 Theorem [7]

Let (X,δ) be a smooth quasi-proximity space. Define for each $r \in I_0$, the family $\delta_r = \{(\lambda,\mu) \in I^X \times I^X \setminus \delta(\lambda,\mu) \geq r\}$, is a fuzzy quasi-proximity on X.

0.4 Topogeneous structures

0.4.a Fuzzy topogeneous structure

The concept of fuzzy topogeneous structure is introduced by Katsaras [53] as follows:

0.4.a.1 Definition [53]

A binary relation η on I^X is a fuzzy semi-topogeneous order on X, if it satisfies the following conditions:

(FT1) $(\underline{1},\underline{1}),(\underline{0},\underline{0}) \in \eta$,

(FT2) If $(\mu,\lambda) \in \eta$, then $\mu \leq \lambda$,

(FT3) If $\mu \leq \mu_1, \lambda_1 \leq \lambda$ and $(\mu_1,\lambda_1) \in \eta$, then $(\mu,\lambda) \in \eta$,

A fuzzy topogeneous order on X is a fuzzy semi-topogeneous order which satisfies the following addition axiom:

(FT4) $(\mu_1 \vee \mu_2,\lambda) \in \eta$ iff $(\mu_1,\lambda) \in \eta$, $(\mu_2,\lambda) \in \eta$ and

$(\mu,\lambda_1 \wedge \lambda_2) \in \eta$ iff $(\mu,\lambda_1) \in \eta$, $(\mu,\lambda_2) \in \eta$.

0.4.a.2 Definition [53]

Let η_1 and η_2 be fuzzy semi-topogeneous order on X, $i \in \Delta$ for an indexed set Δ, then,

(a) η_1 is said to be:

(1) Symmetrical if $(\mu,\lambda)\in\eta_1 \Rightarrow (1-\lambda,1-\mu)\in\eta_1$

(2) Perfect if $(\mu_i,\lambda_i)\in\eta_1 \Rightarrow (\bigcup_{i\in\Delta}\lambda, \bigcup_{i\in\Delta}\mu)\in\eta_1$

(3) Biperfect if it is perfect and $(\mu_i,\lambda_i)\in\eta_1 \Rightarrow (\bigcap_{i\in\Delta}\lambda, \bigcap_{i\in\Delta}\mu)\in\eta_1$

(b) η_1 is *finer* than η_2 (or η_2 is coarser than η_1) iff

$$(\lambda,\mu)\in\eta_1 \Rightarrow (\lambda,\mu)\in\eta_2 \text{ for all } \lambda,\mu\in I^{X\times X}.$$

0.4.a.3 Definition

Let η_1 and η_2 be smooth semi-topogenuous orders on X . η_1 is finer than η_2 (η_2 is coarser than η_2) if $\eta_2 \le \eta_1$

0.4.a.4 Definition [54]

A fuzzy syntopogeneous structure on X is a non-empty family S of fuzzy topogeneous orders on X having the following properties:

(1) S is directed, i.e., for $\eta_1,\eta_2\in S, \exists\eta_3\in S \ni \eta_1,\eta_2\le\eta_3$

(2) For any $\eta\in S$ there exist $\eta_1\in S$ such that $\eta\le\eta_1\circ\eta_1$, where $\circ$ is the composition of relations.

The pair (X,S) is called a *fuzzy syntopogeneous space.* In case S consists of a single topogeneous order, it is called a fuzzy topogeneous structure, and the pair (X,S) is called a fuzzy topogeneous space. S is said to be perfect (resp. biperfect) if each member of S is perfect (resp. biperfect).

0.4.b Smooth topogeneous structure

The concept of smooth topogeneous structure is introduced by Šostak [99] as follows:

0.4.b.1 Definition [99]

A function $\eta: I^X\times I^X\to I$ is called a smooth semi-topogeneous order on X, if it satisfies the following axioms:

$ST1)\ \eta(\underline{1},\underline{1}) = \eta(\underline{0},\underline{0}) = 1,$

$ST2)\ \mu - \lambda \le 1 - \eta(\mu,\lambda)$ for any $\mu, \lambda \in I^X$

$ST3)$ if $\lambda_1 \le \lambda, \mu \le \mu_1 (\lambda, \mu, \lambda_1, \mu_1 \in I^X)$, then $\eta(\lambda,\mu) \le \eta(\lambda_1, \mu_1)$.

0.4.b.2 Proposition [99]

Let η be smooth semi-topogeneous order on X and let the mapping $\eta^s : I^X \times I^X \to I$ defined by

$$\eta^s(\lambda,\mu) = \eta(\underline{1} - \mu, \underline{1} - \lambda), \forall \lambda, \mu \in I^X$$

Then η^s is a smooth semi-topogeneous order on X.

0.4.b.3 Definition [99]

A smooth semi-topogenous order η is called smooth topogenous if

for any $\lambda_1, \lambda_2, \lambda, \mu_1, \mu_2, \mu \in I^X$.

$(ST5)\ \eta(\lambda_1 \vee \lambda_2, \mu) = \eta(\lambda_1, \mu) \wedge \eta(\lambda_1, \mu),$

$(ST6)\ \eta(\lambda, \mu_1 \wedge \mu_2) = \eta(\lambda, \mu_1) \wedge \eta(\lambda_1, \mu_2).$

0.4.b.4 Definition [99]

A smooth semi-topogenous order η is called perfect, if

$(ST7)\ \eta(\bigcup_{i \in \Gamma} \lambda_i, \mu) = \inf_{i \in \Gamma} \eta(\lambda_i, \mu)$, for any $\{\mu, \lambda_i / i \in \Gamma\} \subset I^X$.

A perfect smooth topogenous order η is called biperfect, if

$(ST8)\ \eta(\lambda, \bigcap_{i \in \Gamma} \mu_i) = \inf_{i \in \Gamma} \eta(\lambda, \mu_i)$, for any $\{\lambda, \mu_i : i \in \Gamma\} \subset I^X$.

0.4.b.5 Definition [99]

Let η_1 and η_2 be smooth semi-topogenuous orders on X. η_1 is finer than η_2 (η_2 is coarser than η_2) if $\eta_2 \le \eta_1$

0.4.b.6 Definition [99]

A smooth semi-topogenous order η is called smooth topogenous if for any $\lambda_1, \lambda_2, \lambda, \mu_1, \mu_2, \mu \in I^X$.

$(ST5)\ \eta(\lambda_1 \vee \lambda_2, \mu) = \eta(\lambda_1, \mu) \wedge \eta(\lambda_1, \mu)$,

$(ST6)\ \eta(\lambda, \mu_1 \wedge \mu_2) = \eta(\lambda, \mu_1) \wedge \eta(\lambda_1, \mu_2)$.

0.4.b.7 Definition [99]

A smooth semi-topogenous order η is called perfect, if

$(ST7)\ \eta(\bigcup_{i\in\Gamma} \lambda_i, \mu) = \inf_{i\in\Gamma} \eta(\lambda_i, \mu)$, for any $\{\mu, \lambda_i / i \in \Gamma\} \subset I^X$.

A perfect smooth topogenous order η is called biperfect, if

$(ST8)\ \eta(\lambda, \bigcap_{i\in\Gamma} \mu_i) = \inf_{i\in\Gamma} \eta(\lambda, \mu_i)$, for any $\{\lambda, \mu_i : i \in \Gamma\} \subset I^X$.

0.4.b.8 Theorem [99]

Let $\eta_1, \eta_2 : I^X \times I^X \to I$ be perfect (resp. smooth topogenous, biperfect) smooth semi-topogenous orders on X. Define the composition $\eta_1 \circ \eta_2$ of η_1 and η_2 on X by

$$\eta_1 \circ \eta_2(\lambda, \mu) = \sup_{\nu \in I^X} \{\eta_1(\lambda, \nu) \wedge \eta_2(\nu, \mu)\}$$

Then $\eta_1 \circ \eta_2$ is a perfect (resp. smooth topogenous, biperfect) smooth semi-topogenous orders on X.

0.4.b.9 Definition

A Fuzzy syntopogenous structure on X is a non-empty family Υ_X of smooth topogenous orders on X satisfying the following two conditions:

(S1) Υ_X is directed i.e. given two smooth topogenous orders $\eta_1, \eta_2 \in \Upsilon_X$, there exists a smooth topogenous order $\eta \in \Upsilon_X$ such that $\eta_1, \eta_2 \leq \eta$.

(S2) For every $\eta \in \Upsilon_X$ there exists $\eta_1 \in \Upsilon_X$ such that $\eta \leq \eta_1 \circ \eta_1$.

The pair (X, Υ_X) is called a fuzzy syntopogenous space.

0.4.b.10 Definition

A fuzzy syntopogenous structure Υ_X is called topogenous if Υ_X consisting of a single element. In this case, $\Upsilon_X = \{\eta\}$ is called a fuzzy topogenous structure, denoted by $\Upsilon_X = \{\eta\} = \eta$, and (X, Υ_X) is called fuzzy topogenous space.

A fuzzy syntopogenous structure Υ_X is called *perfect* (resp. biperfect symmetric etc.) if each smooth topogenous order $\eta \in \Upsilon_X$ is perfect (resp. biperfect symmetric etc).

CHAPTER I

Chapter I

Smooth uniform spaces

Smooth uniformities have two roots tracing back to Lowen [66] and to Hutton [46]. In this chapter we introduce the notions of smooth uniform spaces throught the above two roots.

1.1 Smooth uniformity by entourage approach

In this section we introduce the definition of smooth uniform spaces which depend on the entourage approach. Some properties, subspace of smooth uniform space and smooth topology induced by a smooth uniform space are studied.

1.1.1 Definition

A function $U : I^{X\times X} \to I$ is called a *smooth uniformity* on X satisfying for $u, v \in I^{X\times X}$, the following conditions:

(SU1) U is a smooth filter on $X \times X$.

(SU2) $U(u) \leq U(u^S)$, where $u^S(x,y) = u(y,x)$

(SU3) $U(u) \leq \sup\{U(v) / v \circ v \leq u\}$, where

$$v \circ u(x,y) = \sup_{z\in X} (u(x,z) \wedge v(z,y))$$

The pair (X, U) is said to be a *smooth uniform space*

Let U_1 and U_2 be smooth uniformities on X. We say U_1 is *finer* than U_2 (or U_2 is coarser than U_1) iff $U_2(u) \leq U_1(u)$ for all $u \in I^{X\times X}$.

1.1.2 Definition

A function $B : I^{X\times X} \to I$ is called a *smooth uniformity base* on X satisfying for $u, v \in I^{X\times X}$, the following conditions:

$$= \sup_{z\in X} \{\sup_{y\in X} (\lambda(y) \wedge u_2(y,z)) \wedge u_1(z,x)\}$$
$$= \sup_{z\in X} \{u_2[\lambda](z) \wedge u_1(z,x)\}$$
$$= u_1[u_2[\lambda]](x).$$

(4) is similar to (3)

(5) Suppose there exist $x \in X$ and $t \in I$ such that

$$(u_1 \wedge u_2)[\lambda_1 \wedge \lambda_2](x) > t > u_1[\lambda_1](x) \wedge u_2[\lambda_2](x).$$

Since $(u_1 \wedge u_2)[\lambda_1 \wedge \lambda_2](x) > t$, there exists $y \in X$ such that

$$(u_1 \wedge u_2)[\lambda_1 \wedge \lambda_2](x) \geq (\lambda_1 \wedge \lambda_2)(y) \wedge (u_1 \wedge u_2)(y,x) > t.$$

It implies

$$t < (\lambda_1 \wedge \lambda_2)(y) \wedge (u_1 \wedge u_2)(y,x)$$
$$\leq \{(\lambda_1)(y) \wedge (u_1 \wedge u_2)(y,x)\} \vee \{(\lambda_2)(y) \wedge (\mu_1 \wedge \mu_2)(y,x)\}$$
$$\leq \{(\lambda_1)(y) \wedge u_1(y,x)\} \vee \{(\lambda_2)(y) \wedge u_2(y,x)\}$$
$$\leq u_1[\lambda_1](x) \wedge u_2[\lambda_2](x).$$

It is a contradiction.

(6) It is proved from:

$$f^{-1}(v[f(\lambda)])(x) = v[f(\lambda)](f(x))$$
$$= \sup_{y\in Y} \{f(\lambda)(y) \wedge v(y, f(x))\}$$
$$= \sup_{z\in X} \{f(\lambda)(f(z)) \wedge v(f(z), f(x))\}$$

$$= \sup_{z\in X} \{\lambda(z) \wedge (f\times f)^{-1}(v)(z,x)\}$$
$$= (f\times f)^{-1}(v)[\lambda](x).$$

(7) and (8) are easily proved.

(9) $(f\times f)^{-1}(v_1) \circ (f\times f)^{-1}(v_2)(x_1,x_2)$

$$= \sup_{z\in X} \{(f\times f)^{-1}(v)(x_1,z) \circ (f\times f)^{-1}(v)(z,x_2)$$

$$= \sup_{z \in X} v(f(x_1), f(z)) \wedge v(f(z), f(x_2))$$

$$\leq \sup_{y \in Y} v(f(x_1), y) \wedge v(y, f(x_2))$$

$$= v \circ v(f(x_1), f(x_2))$$

$$= (f \times f)^{-1}(v \circ v).$$

1.1.6 Theorem

Let (X, U) be smooth uniform space. A function $\tau_U : I^X \to I$ defined by

$$\tau_U(\lambda) = \inf_x \{(1 - \lambda(x)) \vee \sup_{u[x] \leq \lambda} U(u)\},$$

where $u[x](y) = u(y, x)$. Then τ_U is a smooth topology on X.

Proof

(O1) It is easily checked.

(O2) First, we show that

$$(\sup_{u[x] \leq \lambda_1} U(u)) \wedge (\sup_{v[x] \leq \lambda_2} U(v)) \leq (\sup_{w[x] \leq \lambda_1 \wedge \lambda_2} U(w)),$$

suppose there exists $t \in (0,1)$ such that

$$(\sup_{u[x] \leq \lambda_1} U(u)) \wedge (\sup_{v[x] \leq \lambda_2} U(v)) > t > (\sup_{w[x] \leq \lambda_1 \wedge \lambda_2} U(w)).$$

For each $i \in \{1,2\}$, there exists u_i with $u_i[x] < \lambda_i$ such that $U(u_i) > t$. It implies $(u_i \wedge u_2)[x] < \lambda_i \wedge \lambda_2$ and $U(u_i \wedge u_2) \geq U(u_i) \wedge U(u_2) > t$. Hence,

$$(\sup_{w[x] \leq \lambda_1 \wedge \lambda_2} U(w)) > t.$$

It is a contradiction.

Suppose there exist $\lambda_1, \lambda_2 \in I^X$ and $t \in (0,1)$ such that

$$\tau_U(\lambda_1 \wedge \lambda_2) < t < \tau_U(\lambda_1) \wedge \tau_U(\lambda_2).$$

Since $\tau_U(\lambda_1 \wedge \lambda_2) < t$, there exists $x \in X$ such that

$$\tau_U(\lambda_1 \wedge \lambda_2) \le (\underline{1} - (\lambda_1 \wedge \lambda_2)(x)) \vee \sup_{u[x] \le \lambda_1 \wedge \lambda_2} U(u) < t$$

It is implies

$$\underline{1} - (\lambda_1 \wedge \lambda_2)(x) < t. \qquad (A)$$

Since

$$(\sup_{u[x] \le \lambda_1} U(u)) \wedge (\sup_{v[x] \le \lambda_2} U(v)) \le (\sup_{w[x] \le \lambda_1 \wedge \lambda_2} U(w)) < t.$$

By (A)

$$(\underline{1} - \lambda_1(x)) \vee (\sup_{u[x] \le \lambda_1} U(u)) < t \text{ or } (\underline{1} - \lambda_2(x)) \vee (\sup_{v[x] \le \lambda_2} U(v)) < t.$$

It implies

$$\tau_U(\lambda_1) \wedge \tau_U(\lambda_2) < t.$$

It is a contradiction.

(O3) Since I is an infinitely distributive lattice. Then

$$\tau_U(\sup_{j \in J} \lambda_j) = \inf_x \{(\underline{1} - (\sup_{j \in J} \lambda_j)(x)) \vee \sup_{u[x] \le \sup_{j \in J} \lambda_j} U(u)\}$$

$$= \inf_x \{\inf_{j \in J} (\underline{1} - (\lambda_j)(x)) \vee \sup_{u[x] \le \sup_{j \in J} \lambda_j} U(u)\}$$

$$= \inf_j \{\inf_x (\underline{1} - (\lambda_j)(x)) \vee \sup_{u[x] \le \sup_{j \in J} \lambda_j} U(u)\}$$

$$\ge \inf_j \{\inf_x (\underline{1} - (\lambda_j)(x)) \vee \sup_{u[x] \le \lambda_j} U(u)\}$$

$$= \inf_j \tau_U(\lambda_j).$$

1.1.7 Theorem

Let (X,U) and (Y,V) be smooth uniform spaces and $f : X \to Y$ be smooth uniform continuous. Then $f : (X, \tau_U) \to (Y, \tau_V)$ is a smooth continuous.

Proof

First, we show that $f^{-1}(v[f(x)]) = (f \times f)^{-1}(v)[x]$ from:

$$\begin{aligned} f^{-1}(v[f(x)]) &= v([f(x)])(f(z)) \\ &= v(f(z), f(x)) \\ &= (f\times f)^{-1}(v)(z,x) \\ &= (f\times f)^{-1}(v)[x](z). \end{aligned}$$

Thus

$$v[f(x)] \le \lambda \text{ implies } f^{-1}(v[f(x)]) \le (f\times f)^{-1}(v)[x] \le f^{-1}(\lambda).$$

Hence

$$\begin{aligned} \tau_U(\lambda) &= \inf_y \{(\underline{1}-\lambda(y)) \vee \sup_{v[y]\le\lambda} U(v)\} \\ &\le \inf_x \{(\underline{1}-\lambda(f(x))) \vee \sup_{v[f(x)]\le\lambda} U(v)\} \\ &\le \inf_x \{(\underline{1}-(f^{-1}(\lambda)(x))) \vee \sup_{(f\times f)^{-1}(v)[x]\le f^{-1}(\lambda)} U((f\times f)^{-1}(v))\} \\ &\le \tau_U(f^{-1}(\lambda)). \end{aligned}$$

1.1.8 Theorem

Let (X, U) be smooth uniform space. Define the function $C_U : I^X \times I_1 \to I^X$, by

$$C_U(\lambda, r) = \inf\{\mu[\lambda] / U(\mu) > r\}.$$

For each $\lambda, \lambda_1, \lambda_2 \in I^X$ and $r, r_1, r_2 \in I_1$, we have the following properties:

(1) $C_U(\underline{0}, r) = \underline{0}$,

(2) $\lambda \le C_U(\lambda, r)$,

(3) If $\lambda_1 \le \lambda_2$, then $C_U(\lambda_1, r) \le C_U(\lambda_2, r)$,

(4) $C_U(\lambda_1 \vee \lambda_2, r) = C_U(\lambda_1, r) \vee C_\eta(\lambda_2, r)$,

(5) If $r_1 \le r_2$, then $C_U(\lambda, r_1) \le C_U(\lambda, r_2)$,

(6) $C_U(C_U(\lambda, r), r) = C_U(\lambda, r)$.

Proof

(1) Since $u(\underline{0})=\underline{0}$ then $C_U(\underline{0},r)=\underline{0}$.

(2) For $U(u)>0$, by Lemma 1.1.5 (1) $\lambda \le u[\lambda]$ implies $\lambda \le C_U(\lambda,r)$.

(3) and (5) are easily proved.

(4) From (3), we have

$$C_U(\lambda_1 \vee \lambda_2, r) \ge C_U(\lambda_1, r) \vee C_U(\lambda_2, r).$$

Conversely, suppose there exist $\lambda_1, \lambda_2 \in I^X$ and $r \in I$ such that

$$C_U(\lambda_1 \vee \lambda_2, r) \not\le C_U(\lambda_1, r) \vee C_U(\lambda_2, r).$$

There exist $x \in X$ and $t \in I_1$ such that

$$C_U(\lambda_1 \vee \lambda_2, r)(x) > t > C_U(\lambda_1, r)(x) \vee C_U(\lambda_2, r)(x).$$

Since $C_U(\lambda_i, r)(x) < t$, for each $i \in \{1,2\}$, there exists $u_i \in I^{X \times X}$ with $U(u_i) > r$ such that

$$C_U(\lambda_i, r)(x) \le u_i[\lambda_i](x) < t.$$

On the other hand, since $U(u_1 \wedge u_2) > \mathrm{r}$ and from Lemma 1.1.5(5),

$$(u_1 \wedge u_2)[\lambda_1 \wedge \lambda_2] \le u_1[\lambda_1] \wedge u_2[\lambda_2],$$

we have

$$\begin{aligned} C_U(\lambda_1 \vee \lambda_2, r)(x) &\le (u_1 \wedge u_2)[\lambda_1 \wedge \lambda_2](x) \\ &\le u_1[\lambda_1](x) \wedge u_2[\lambda_2](x) \\ &< \mathrm{t}. \end{aligned}$$

It is a contradiction.

(6) Suppose there exist $\lambda \in I^X$ and $r \in I_1$ such that

$$C_U(C_U(\lambda, r), r) \not\le C_U(\lambda, r).$$

There exist $x \in X$ and $t \in I$ such that

$$C_U(C_U(\lambda, r), r)(x) > t > C_U(\lambda, r)(x).$$

Since $C_U(\lambda, r)(x) < t$, there exists $u \in I^{X \times X}$ with $U(u) > r$ such that

$$C_U(\lambda,r)(x) \le u[\lambda](x) < t.$$

On the other hand, since $U(u) > r$, by (SU3), there exists $u_1 \in I^{X\times X}$ such that

$$u_1 \circ u_1 \le u,\ U(u_1) > r.$$

Since $C_U(\lambda,r) \le u_1[\lambda]$, we have

$$\begin{aligned} C_U(C_U(\lambda,r),r) &\le C_U(u_1[\lambda],r) \\ &\le u_1[u_1[\lambda]] \\ &= (u_1 \circ u_1)[\lambda] \\ &\le u[\lambda]. \end{aligned}$$

Thus, $C_U(C_U(\lambda,r),r)(x) \le u[\lambda](x) < t.$

It is a contradiction.

1.1.9 Theorem

Let (X,U) be a smooth uniform space. Define a function $\tau_U : I^X \to I$, by

$$\tau_U(\lambda) = \sup\{r \in I_1 \mid C_U(\underline{1}-\lambda, r) = \underline{1}-\lambda\}.$$

Then τ_U is a smooth topology on X induced by U.

Proof

(O1) Since $C_U(\underline{0},r) = \underline{0}$ and $C_U(\underline{1}) = 1$, for all $r \in I_1$, then

$$\tau_U(\underline{0}) = \tau_U(\underline{1}) = 1.$$

(O2) Suppose there exist $\lambda_1, \lambda_2 \in I^X$ and $t \in (0,1)$ such that

$$\tau_U(\lambda_1 \wedge \lambda_2) < t < \tau_U(\lambda_1) \wedge \tau_U(\lambda_2).$$

Since $\tau_U(\lambda_1) > t$ and $\tau_U(\lambda_2) > t$, there exist $r_1, r_2 > t$ such that

$$\underline{1} - \lambda_i = C_U(\underline{1} - \lambda_i, r_i), i = 1,2.$$

Put $r = r_1 \wedge r_2$. By (4-5) of Theorem 1.1.8, we have

$$C_U(\underline{1} - (\lambda_1 \wedge \lambda_2), r) = \underline{1} - (\lambda_1 \wedge \lambda_2).$$

Consequently, $\tau_U(\lambda_1 \wedge \lambda_2) \ge r > t$. Hence

$$\tau_U(\lambda_1 \wedge \lambda_2) \geq \tau_U(\lambda_1) \wedge \tau_U(\lambda_2).$$

(O3) Suppose there exists a family $\{\lambda_j \in I^X \setminus j \in \Gamma\}$ and $t \in (0,1)$ such that

$$\tau_U(\sup_{j\in\Gamma} \lambda_j) < t < \inf_{j\in\Gamma} \tau_U(\lambda_j).$$

Since $\inf_{j\in\Gamma} \tau_U(\lambda_j) < t$, for each $j \in \Gamma$, there exists $r_j > t$ such that

$$\underline{1} - \lambda_j = C_U(\underline{1} - \lambda_j, r_j).$$

Put $r = \inf_{j\in\Gamma} r_j$. By (4-5) of Theorem 1.1.8, we have

$$C_U(\underline{1} - \sup_{j\in\Gamma} \lambda_j, r) = \underline{1} - \sup_{j\in\Gamma} \lambda_j.$$

Consequently, $\tau_U(\sup_{j\in\Gamma} \lambda_j) \geq r > t$. Hence,

$$\tau_U(\sup_{j\in\Gamma} \lambda_j) \geq \inf_{j\in\Gamma} \tau_U(\lambda_j).$$

1.1.10 Theorem

Let (X,U) and (Y,V) be smooth uniform spaces. Let $f:(X,U) \to (Y,V)$ be smooth uniform continuous. Then:

(1)$f(C_U(\lambda,r)) \leq C_V(f(\lambda),r))$, for each $\lambda \in I^X$.

(2)$C_U(f^{-1}(\mu),r) \leq f^{-1}(C_V(f(\mu),r))$, for each $\mu \in I^Y$.

(3)$f:(X,\tau_U) \to (Y,\tau_V)$ is a smooth continuous.

Proof

(1) Suppose there exist $\lambda \in I^X$ and $r \in I_1$ such that

$$f(C_U(\lambda,r)) \not\leq C_V(f(\lambda),r)).$$

There exists $y \in Y$ and $t \in I_\circ$ such that

$$f(C_U(\lambda,r))(y) > t > C_V(f(\lambda),r))(y),$$

Since $f^{-1}(\{y\}) = \phi$, provides a contradiction that $f(C_U(\lambda,r))(y) = 0$,

$f^{-1}(\{y\}) \neq \phi$, and there exists $x \in f^{-1}(\{y\})$ such that

$$f(C_U(\lambda, r))(y) \geq C_U(\lambda, r)(x) > t > C_V(f(\lambda), r))(f(x)).$$

Since $C_V(f(\lambda), r))(f(x)) < t$, there exists $v \in I^{Y \times Y}$ with $V(v) > r$ such that

$$C_V(f(\lambda), r))(f(x)) \leq v[f(\lambda)](f(x)) < t.$$

On the other hand, since f is a smooth uniform continuous, from Definition 0.2.b.4, we have, $U((f \times f)^{-1}(v)) \geq V(v) > r$.

It implies

$$\begin{aligned} v[f(\lambda)](f(x)) &= (f \times f)^{-1}(v)[\lambda](x) \\ &= \sup_{z \in X} \{\lambda(z) \wedge (f \times f)^{-1}(v)(z, x)\} \\ &\geq C_U(\lambda, r)(x). \end{aligned}$$

Thus, $C_U(\lambda, r)(x) < t$, it is a contradiction.

(2) For each $\mu \in I^Y$ and $r \in I_1$, put $\lambda \in f^{-1}(\mu)$. From (1),

$$f(C_U(f^{-1}(\mu), r)) \leq C_V(f(f^{-1}(\mu)), r) \leq C_V(\mu, r),$$

It implies

$$C_U(f^{-1}(\mu), r) \leq f^{-1}(f(C_U(f^{-1}(\mu), r))) \leq f^{-1}(C_V(\mu, r)).$$

(3) From (2), $C_V(\mu, r) = \mu$ implies $C_U(f^{-1}(\mu), r) = f^{-1}(\mu)$. It is easily proved.

1.2 Product smooth uniformity spaces

In this section we study the concepts of the product and the subspace of the smooth uniform spaces.

1.2.1 Theorem

Let $\{(X_k, V_K) / k \in \Gamma\}$ be a family of smooth uniform spaces, X a set and for each $k \in \Gamma$, $f_k : X \to X_k$ a function. We define, for each $u \in I^{X \times X}$,

$$U(u)=\sup\{\inf_{i=1}^{n} V_{k_i}(u_{k_i})\setminus \inf_{i=1}^{n}(f_{k_i}\times f_{k_i})^{-1}(u_{k_i})\leq u\}$$

where the supremum is taken over every finite index $K=\{k_1,\ldots,k_n\}\subset\Gamma$.

Then:

(1) The structure U is the coarsest smooth uniformity on X for which each f_k is a smooth uniform continuous.

(2) A function $f:(Z,W)\to(X,U)$ is smooth uniform continuous iff for each $k\in\Gamma, f_k\circ f$ is smooth uniform continuous.

(3) $C_U(\lambda\times r)=\inf\{\inf_{i\in\Gamma}^{n}(f_{k_i}\times f_{k_i})^{-1}(v_{k_i}^{-1})[\lambda]\setminus V_{k_i}(v_{k_i})>r,\forall k_i\in K\}$,

where the infimum is taken over every finite index $K=\{k_1,\ldots,k_n\}\subset\Gamma$.

(4) τ_U induced by U coincides with the coarsest smooth topology τ_B on X for which each $f_i:(X,\tau_B)\to(X_i,\tau_{U_i})$ is a smooth continuous.

Proof

(1) First, we will show that U is a smooth uniformity on X.

(SU1) If $U(u)>0$, there exists finite indices $K=\{k_1,\ldots,k_n\}\subset\Gamma$ such that

$$U(u)\geq\inf_{i=1}^{n} V_{k_i}(\mu_{k_i})>0,\ \inf_{i=1}^{n}(f_{k_i}\times f_{k_i})^{-1}(\mu_{k_i})\leq\mu$$

Since $V_{k_i}(u_{k_i})>0$ for each $k_i\in K$, by (SF1), there exists $1_\Delta\in I^{X_{k_i}\times X_{k_i}}$, with $1_\Delta\leq u_{K_i}$. Hence

$$1_\Delta\leq\inf_{i=1}^{n}(f_{k_i}\times f_{k_i})^{-1}(1_\Delta)\leq\inf_{i=1}^{n}(f_{k_i}\times f_{k_i})^{-1}(u_{k_i})\leq\mu..$$

Since $U(\mu\wedge v)\leq U(u)$ and $U(u\wedge v)\leq U(v)$, we have

$$U(u\wedge v)\leq U(u)\wedge U(v)..$$

For any $u,v\in I^{X\times X}$, we will show that

$$U(u\wedge v)\leq U(u)\wedge U(v).$$

If $U(u)=0$ or $U(v)=0$, it is trivial.

If $U(u)>0$ and $U(v)>0$, for $\varepsilon>0$, such that $U(u)\wedge U(v)>\varepsilon>0$, there exist finite indices $K=\{k_1,\dots,k_n\}$ and $L=\{l_1,\dots,l_m\}\subset\Gamma$ such that

$$\inf_{i=1}^{n} V_{k_i}(u_{k_i})\geq U(u)-\varepsilon,\ \inf_{i=1}^{n}(f_{k_i}\times f_{k_i})^{-1}(u_{k_i})\leq u,$$

$$\inf_{j=1}^{m} V_{l_j}(v_{l_j})\geq U(v)-\varepsilon,\ \inf_{j=1}^{m}(f_{l_j}\times f_{l_j})^{-1}(v_{l_j})\leq v.$$

Since $u\wedge v\geq(\inf_{i=1}^{n}(f_{k_i}\times f_{k_i})^{-1}(\mu_{k_i}))\wedge(\inf_{j=1}^{m}(f_{l_j}\times f_{l_j})^{-1}(v_{l_j}))$,

$$\begin{aligned}U(u\wedge v)&\geq(\inf_{i=1}^{n} V_{k_i}(u_{k_i}))\wedge(\inf_{j=1}^{m} V_{l_j}(v_{l_j}))\\&\geq U(u)\wedge U(v)-\varepsilon.\end{aligned}$$

Since ε is arbitrary, this gives the desired result.

(SU2) Suppose that there exist $u\in I^{X\times X}$ and $r\in(0,1)$ such that

$$U(u^s)<r<U(u).$$

Since $U(u)>r$, by the definition of U, there exist a finite index $K=\{k_1,\dots,k_n\}\subset\Gamma$ such that

$$U(u)\geq\inf_{i=1}^{n} V_{k_i}(u_{k_i})>r,\ \inf_{i=1}^{n}(f_{k_i}\times f_{k_i})^{-1}(u_{k_i})\leq u.$$

For each $k_i\in K$ by (SU2), since $V_{k_i}(u_{k_i})>r$ for each $k_i\in K$,

$$V_{k_i}(u^s_{k_i})\geq V_{k_i}(u_{k_i})>r.$$

It follows that

$$\begin{aligned}\inf_{i=1}^{n}(f_{k_i}\times f_{k_i})^{-1}(u^s_{k_i})&\leq\inf_{i=1}^{n}((f_{k_i}\times f_{k_i})^{-1}(u_{k_i}))^s\\&\leq u^s\end{aligned}$$

Hence,

$$U(u^s) \geq \inf_{i=1}^{n} V_{k_i}(u^s_{k_i}) > r.$$

It is a contradiction.

(SU3) For $u \in I^{X \times X}$. We will show that

$$\sup\{U(u_1) / u_1 \circ u_1 \leq u\} \geq U(u).$$

If $U(u) = 0$. It is trivial.

Suppose that there exist $u \in I^{X \times X}$ and $r \in (0,1)$ such that

$$\sup\{U(u_1) / u_1 \circ u_1 \leq u\} < r < U(u).$$

Since $U(u) > r$, by the definition of U, there exists a finite index $K = \{k_1, \ldots, k_n\} \subset \Gamma$ such that

$$U(u) \geq \inf_{i=1}^{n} V_{k_i}(v_{k_i}) > r, \quad \inf_{i=1}^{n} (f_{k_i} \times f_{k_i})^{-1}(v_{k_i}) \leq \mu.$$

For each $k_i \in K$ by (SU3),

$$\sup\{V_{k_i}(w) / w \circ w \leq v_{k_i}\} \geq V_{k_i}(v_{k_i}).$$

Since $V_{k_i}(v_{k_i}) > r$ for each $k_i \in K$, there exist $w_{k_i} \in I^{X_{k_i} \times X_{k_i}}$, and $r_i \in (0,1]$ such that

$$w_{k_i} \circ w_{k_i} \leq v_{k_i}, \; V_{k_i}(w_{k_i}) \geq r_i > r.$$

Put $w = \inf_{i=1}^{n} (f_{k_i} \times f_{k_i})^{-1}(w_{k_i})$. For each $k_i \in K$, we have

$$w \circ w \leq (f_{k_i} \times f_{k_i})^{-1}(w_{k_i}) \circ (f_{k_i} \times f_{k_i})^{-1}(w_{k_i})$$

Hence

$$\begin{aligned} w \circ w &\leq \inf_{i=1}^{n} (f_{k_i} \times f_{k_i})^{-1}(w_{k_i}) \circ (f_{k_i} \times f_{k_i})^{-1}(w_{k_i}). \\ &\leq \inf_{i=1}^{n} (f_{k_i} \times f_{k_i})^{-1}(w_{k_i} \circ w_{k_i}) \\ &\leq \inf_{i=1}^{n} (f_{k_i} \times f_{k_i})^{-1}(v_{k_i}) \leq \mu. \end{aligned}$$

Then we have $w \circ w \leq \mu$ and

$$U(w) \geq \inf_{i=1}^{n} V_{k_i}(w_{k_i}) \geq \inf_{i=1}^{n} r_i > r$$

Hence, $\sup\{U(u_1)/u_1 \circ u_1 \leq u\} > r$. It is a contradiction.

Second, it is easily proved that, by the definition of U, for all $k \in \Gamma$,

$$U((f_k \times f_k)^{-1}(v_k)) \geq V_k(v_k), \forall v_k \in I^{X_k \times X_k}.$$

Hence, each $f_k : (X, U) \to (X, V_k)$ is smooth uniform continuous.

Finally, if $f_k : (X, U') \to (X, V_k)$ is smooth uniform continuous, that is, $U'((f_k \times f_k)^{-1}(v)) \geq V_k(v), \forall k \in \Gamma$, then it is proved that $U' \geq U$ from the following:

$$U(u) = \sup\{\inf_{i=1}^{n} V_{k_i}(v_{k_i}) / \inf_{i=1}^{n} (f_{k_i} \times f_{k_i})^{-1}(v_{k_i}) \leq u\}$$

$$\leq \sup\{\inf_{i=1}^{n} U'((f_{k_i} \times f_{k_i})^{-1}(v_{k_i})) / \inf_{i=1}^{n} (f_{k_i} \times f_{k_i})^{-1}(v_{k_i}) \leq u\}$$

$$= \sup\{U'(\inf_{i=1}^{n} ((f_{k_i} \times f_{k_i})^{-1}(v_{k_i}))) / \inf_{i=1}^{n} (f_{k_i} \times f_{k_i})^{-1}(v_{k_i}) \leq u\}$$

$$\leq U'(u), \forall \mu \in I^X$$

(2) Let f be smooth uniform continuous. From Theorem 0.2.b.5 and (1), the composition of smooth uniform continuous functions is a smooth uniform continuous function.

Conversely, suppose that $f : (Z, W) \to (X, U)$ is not smooth uniform continuous. There exists $u \in I^{X \times X}$ such that

$$W((f \times f)^{-1}(u)) < U(u).$$

By the definition of U, there exists a finite index $K = \{k_1, \ldots, k_n\} \subset \Gamma$ such that

$$W((f \times f)^{-1}(u)) < \inf_{i=1}^{n} V_{k_i}(v_{k_i}) \leq U(u), \ \inf_{i=1}^{n} (f_{k_i} \times f_{k_i})^{-1}(v_{k_i}) \leq u.$$

On the other hand, for each $k_i \in K$ and $f_{k_i} \circ f$ is a smooth uniform continuous, we have

$$V_{k_i}(v_{k_i}) \leq W(((f_{k_i} \circ f) \times (f_{k_i} \circ f))^{-1}(v_{k_i}))$$
$$= W((f \times f)^{-1} \circ (f_{k_i} \times f_{k_i})^{-1}(v_{k_i})).$$

It follows that

$$\inf_{i=1}^{n}(V_{k_i}(v_{k_i})) \leq \inf_{i=1}^{n} W(((f_{k_i} \circ f) \times (f_{k_i} \circ f))^{-1}(v_{k_i}))$$
$$\leq W(\inf_{i=1}^{n}((f \times f)^{-1} \circ (f_{k_i} \times f_{k_i})^{-1}(v_{k_i})))$$
$$= W(((f \times f)^{-1}(\inf_{i=1}^{n}(f_{k_i} \times f_{k_i})^{-1}(v_{k_i})))$$
$$\leq W((f \times f)^{-1}(u)).$$

It is a contradiction.

(3) From Theorem 1.1.8, we only show that

$$\inf\{u[\lambda]/U(u) > r\} = \inf\{\inf_{i=1}^{n}(f_{k_i} \times f_{k_i})^{-1}(v_{k_i})[\lambda]/V_{k_i}(v_{k_i}) > r, \forall k_i \in K\}.$$

Where the infimum is taken over every finite index $K = \{k_1, \ldots, k_n\} \subset \Gamma$.

Since

$$U((f_{k_i} \times f_{k_i})^{-1}(v_{k_i}) \geq V_{k_i}(v_{k_i}) > r,$$

we have

$$\inf\{u[\lambda]/U(u) > r\} \leq \inf\{\inf_{i=1}^{n}(f_{k_i} \times f_{k_i})^{-1}(v_{k_i})[\lambda]/V_{k_i}(v_{k_i}) > r\}.$$

Conversely, suppose that

$$\inf\{u[\lambda]/U(u) > r\} \ngeq \inf\{\inf_{i=1}^{n}(f_{k_i} \times f_{k_i})^{-1}(v_{k_i})[\lambda]/V_{k_i}(v_{k_i}) > r\}.$$

There exists $x \in X$ such that

$$\inf\{u[\lambda]/U(u) > r\}(x) < \inf\{\inf_{i=1}^{n}(f_{k_i} \times f_{k_i})^{-1}(v_{k_i})[\lambda]/V_{k_i}(v_{k_i}) > r\}(x).$$

There exists $u \in I^{X \times X}$ with $U(u) > r$ such that

$$u[\lambda](x) < \inf\{\inf_{i=1}^{n} (f_{k_i} \times f_{k_i})^{-1}(v_{k_i})[\lambda] / V_{k_i}(v_{k_i}) > r\}(x).$$

Since $U(u) > r$, there exists a finite index $K = \{k_1, ..., k_n\} \subset \Gamma$ such that

$$U(u) \geq \inf_{i=1}^{n} V_{k_i}(v_{k_i}) > r, \ \inf_{i=1}^{n} (f_{k_i} \times f_{k_i})^{-1}(v_{k_i}) \leq u.$$

It implies

$$\begin{aligned}\inf\{\inf_{i=1}^{n} (f_{k_i} \times f_{k_i})^{-1}(v_{k_i})[\lambda] / V_{k_i}(v_{k_i}) > r\} &\leq \inf_{i=1}^{n} (f_{k_i} \times f_{k_i})^{-1}(v_{k_i})[\lambda] \\ &\leq u[\lambda].\end{aligned}$$

It is a contradiction.

(4) Suppose there exists $\lambda \in I^X$ such that

$$\tau_B(\lambda) < \tau_U(\lambda).$$

By the definition of τ_U from Theorem 1.1.9 there exists $r_\circ \in I_\circ$ such that

$$C_U(\underline{1} - \lambda, r_\circ) = \underline{1} - \lambda \text{ and}$$

$$\tau_B(\lambda) < r_\circ \leq \tau_U(\lambda).$$

Since $C_U(\underline{1} - \lambda, r_\circ) = \underline{1} - \lambda$, we have

$$\begin{aligned}\underline{1} - \lambda &= C_U(\underline{1} - \lambda, r_\circ) \\ &= \inf\{u[\underline{1} - \lambda] \setminus U(u) > r_\circ\} \\ &= \inf\{\inf_{i=1}^{n} (f_{k_i} \times f_{k_i})^{-1}(v_{k_i})[\underline{1} - \lambda] / U_{k_i}(v_{k_i}) > r_\circ\}\end{aligned}$$

where the infimum is taken over every finite index $K = \{k_1, ..., k_n\} \subset \Gamma$

From Lemma 1.1.5(6), since

$$(f_{k_i} \times f_{k_i})^{-1}(v_{k_i})[\underline{1} - \lambda] = f_{k_i}^{-1}(v_{k_i}[f_{k_i}(\underline{1} - \lambda)]),$$

we have

$$\inf\{(f_{k_i} \times f_{k_i})^{-1}(v_{k_i})[\underline{1} - \lambda] / U_{k_i}(v_{k_i}) > r_\circ\} = f_{k_i}^{-1}(C_{U_{k_i}}(f_{k_i}(\underline{1} - \lambda), r_\circ)).$$

It follows that

$$\lambda = \underline{1} - \inf\{\inf_{i=1}^{n} f_{k_i}^{-1}(C_{U_{k_i}}(f_{k_i}(\underline{1}-\lambda), r_\circ))\}$$
$$= \sup\{\sup_{i=1}^{n} f_{k_i}^{-1}(\underline{1} - C_{U_{k_i}}(f_{k_i}(\underline{1}-\lambda), r_\circ))\}.$$

where the first supremum is taken over every finite index $K = \{k_1, \ldots, k_n\} \subset \Gamma$.

Since $C_{U_{k_i}}(f_{k_i}(\underline{1}-\lambda), r_\circ) = C_{U_{k_i}}(C_{U_{k_i}}(f_{k_i}(\underline{1}-\lambda), r_\circ), r_\circ)$ from Theorem 1.1.8 (6), we have

$$\tau_{U_{k_i}}(\underline{1} - C_{U_{k_i}}(f_{k_i}(\underline{1}-\lambda), r_\circ)) \geq r_\circ.$$

Put $\mu_i = f_{k_i}^{-1}(\underline{1} - C_{U_{k_i}}(f_{k_i}(\underline{1}-\lambda), r_\circ))$. From Theorem 0.1.b.14 we have

$$B(u_i) \geq \tau_{U_{k_i}}(\underline{1} - C_{U_{k_i}}(f_{k_i}(\underline{1}-\lambda), r_\circ)) \geq r_\circ.$$

It implies $\tau_B(\sup_{i=1}^{n} u_i) \geq r_\circ$. By the definition of τ_B from Theorem 0.1.b.14 we have

$$\tau_B(\lambda) \geq \inf \tau_B(\sup_{i=1}^{n} \mu_i) \geq r_\circ.$$

It is a contradiction. Therefore, $\tau_B(\lambda) \geq \tau_U(\lambda), \forall \lambda \in I^X$.

We will show that $\tau_B(\lambda) \geq \tau_U(\lambda), \forall \lambda \in I^X$, equivalently, the identity function $id_X : (X, \tau_U) \to (X, \tau_B)$ is smooth continuous. We only show that $f_i \circ id_X : (X, \tau_U) \to (X_i, \tau_{U_i})$ is smooth continuous from Theorem 0.1.b.14 (3). It is obvious from Theorem 1.1.12.

From Theorem 1.1.12 we can define subspaces and products in the obvious way.

1.2.2 Definition

Let (X,U) be a smooth uniform space and A a subset of X. The pair (A,U_A) is said to be a *subspace* of (X,U) if it is endowed with the coarsest smooth uniformity structure induced by the inclusion function.

Let A be a nonempty subset of X and let $u \in I^{A\times A}$. We define the extension of u to $X\times X$, denoted by $u_{X\times X}$ by:

$$u_{X\times X}(x,y) = \begin{cases} u(x,y), & \text{if } x,y \in A, \\ 1, & \text{otherwise.} \end{cases}$$

1.2.3 Theorem

Let (X,U) be a smooth uniform space and A be a nonempty subset of X. The function $V: I^{A\times A} \to I$ is defined by

$$V(u) = U(u_{X\times X}), \forall u \in I^{A\times A}$$

Then $V = U_A$.

Proof

Let $i: A \to X$ be an inclusion function. Since $U_A(u) = \sup\{U(v) \setminus (i\times i)^{-1}(v) \le u\}$ from Theorem 1.2.1 and $(i\times i)^{-1}(u_{X\times X}) = u$, we have $V \le U_A$.

Suppose there exists $u \in I^{A\times A}$ such that $V(u) < U_A(u)$. There exists $v \in I^{X\times X}$ with $(i\times i)^{-1}(v) \le u$ such that

$$U(u_{X\times X}) = V(u) < U(v) \le U_A(u).$$

Since $v \le u_{X\times X}, U(u_{X\times X}) \ge U(v)$. It is a contradiction.

1.2.4 Definition

Let $\{(X_i,U_i)/i\in\Gamma\}$ be a family of smooth uniform spaces. The coarsest smooth uniformity structure $U = \otimes U_i$ on $X = \prod_{i\in\Gamma} X_i$ induced by the collection

$\{\pi_i : X \to X_i / i \in \Gamma\}$ of projections is called the *product smooth uniformity structure* of $\{V_i / i \in \Gamma\}$, and (X,U) is called the product smooth uniform space.

As an immediate consequence of Theorem 1.2.1, we have

1.2.5 Corollary

Let $(X_k, U_k)_{k\in\Gamma}$ be smooth uniform spaces. Let $X = \prod_{k\in\Gamma} X_k$ be a set and for each $k \in \Gamma$, $\pi_k : X \to X_k$ a projection. The structure $U = \otimes U_k$ is defined by, for each $u \in I^{X\times X}$,

$$U(u) = \sup\{\inf_{i=1}^{n} U_{k_i}(\nu_{k_i}) \setminus \inf_{i=1}^{n} (\pi_{k_i} \times \pi_{k_i})^{-1}(\nu_{k_i}) \le u\},$$

where the supremum is taken over every finite index $K = \{k_1, \ldots, k_n\} \subset \Gamma$.

Then:

(1) U is the coarsest smooth uniformity on X for which π_k is smooth uniform continuous.

(2) A function $f : (Z,W) \to (X,U)$ is a smooth uniform continuous iff $\pi_k \circ f$ is a smooth uniform continuous.

(3) The smooth topology τ_U induced by U coincides with the product smooth topology τ_B on X for which each $\pi_i : (X, \tau_B) \to (X_i, \tau_{U_i})$ is a smooth continuous.

1.2.6 Corollary

Let $(X, U_i)_{i\in\Gamma}$ be smooth uniform spaces. We define, for $u \in I^{X\times X}$,

$$U(u) = \sup\{\inf_{i=1}^{n} U_{k_i}(\nu_{k_i}) / \inf_{i=1}^{n} \nu_{k_i} \le u\},$$

where the supremum is taken over every finite index $K = \{k_1, \ldots, k_n\} \subset \Gamma$.

Then:

(1) The structure U is the coarsest smooth uniformity on X finer than U_i.

(2) The smooth topology τ_U induced by U coincides with the product smooth topology τ_B on X of a family $\{(X_i, \tau_{U_i})\}_{i \in \Gamma}$.

1.3 Smooth uniformity by covering approach

In this section we introduce the definition of smooth uniform spaces which depend on covering approach where the conditions (FQU3) and (FU) are defined in a somewhat different view of Samanta [91]. We define a bases for a smooth uniform spaces and we study the smooth topology induced by a smooth uniform space.

1.3.1 Definition

A unction $U : \Omega_X \to I$ is said to be a *smooth quasi-uniformity* on X if it satisfying the following the conditions:

(FQU1) For $\alpha, \beta \in \Omega_X$, we have $U(\alpha \wedge \beta) \geq U(\alpha) \wedge U(\beta)$,

(FQU2) If $\alpha \leq \beta$ then $U(\alpha) \leq U(\beta)$,

(FQU3) For every $\alpha \in \Omega_X$ we have $U(\alpha) \leq \sup_{\beta \circ \beta \leq \alpha} U(\beta)$

(FQU4) There exists $\alpha \in \Omega_X$ such that $U(\alpha) = 1$.

The pair (X, U) is said to be a *smooth quasi-uniform space.*

A smooth quasi-uniform space (X, U) is said to be a *smooth uniform space* if the following condition is satisfied:

(FU) for every $\alpha \in \Omega_X$ we have $U(\alpha) \leq \sup_{\beta \leq \alpha^{-1}} U(\beta)$

Let U_1 and U_2 be smooth (quasi-)uniformities on X. We say U_1 is *finer* than U_2 (or U_2 is *coarser* than U_1) iff $U_2(\alpha) \leq U_1(\alpha)$ for all $\alpha \in \Omega_X$.

1.3.2 Remark

(1) Let (X,U) be a smooth quasi-uniform space. By (FQU1), (FQU2) and Lemma 1.3.1 (2), we have $U(\alpha \wedge \beta) = U(\alpha) \wedge U(\beta)$.

(2) Let (X,U) be a smooth quasi-uniform space. By Lemma 0.2.a.5 (5) and (FQU4), since $\alpha \le \alpha_{\underline{1}}$ for all $\alpha \in \Omega_X$, we have $U(\alpha_{\underline{1}}) = 1$.

(3) If (X,U) is a smooth quasi-uniform space, then, By (FU) and (FQU3), we have $U(\alpha) = \sup_{\beta \le \alpha^{-1}} U(\beta)$. From Lemma 0.2.a.5 (3), we have

$U(\alpha^{-1}) = U(\alpha)$.

1.3.5 Definition

Let Θ_X be a subset of Ω_X. A function $B : \Theta_X \to I$ is said to be *base* for a smooth quasi-uniformity on X if it satisfies the following conditions:

(FQB1) For $\alpha, \beta \in \Theta_X$, we have $B(\alpha \wedge \beta) \ge B(\alpha) \wedge B(\beta)$.

(FQB2) For every $\alpha \in \Theta_X$ we have $B(\alpha) \le \sup_{\beta \circ \beta \le \alpha} B(\beta)$.

(FQU3) There exists $\alpha \in \Theta_X$ such that $B(\alpha) = 1$.

The pair (X,B) is called a *smooth quasi-uniform base.*

A smooth quasi-uniform space (X,B) is said to be a *smooth uniform base* if the following condition is satisfied:

(FB) for every $\alpha \in \Theta_X$, we have $B(\alpha) \le \sup_{\beta \le \alpha^{-1}} B(\beta)$

1.3.6 Remark

(1) Let (X,U) be smooth uniform space. For each $r \in I_1$, let

$$U^r = \{\alpha \in \Omega_X \setminus U(\alpha) > r\}.$$

Then U^r is a Hutton fuzzy uniformity on X.

(2) Every smooth (quasi-)uniform space (X,U) is a smooth (quasi-)uniform base in the sense of $\Theta_X = \Omega_X$.

A base B always generates a smooth (quasi-) uniformity U_B on X in following theorem.

1.3.7 Theorem

Let (X,B) be a smooth (quasi-) uniform base. Define, for every $\alpha \in \Omega_X$,

$$U_B(\alpha)=\begin{cases}\sup\limits_{\beta\leq\alpha} B(\beta) & \text{if } \{\beta\in\Theta_X \setminus \beta\leq\alpha\}\neq\phi,\\ 0 & \text{otherwise.}\end{cases}$$

Then U_β is a smooth (quasi-)uniformity on X generated by B.

Proof

(FQU1) For any $\alpha,\beta\in\Omega_X$, we will show that

$$U_B(\alpha\wedge\beta)\geq U_B(\alpha)\wedge U_B(\beta)$$

If $U_B(\alpha)=0$ or $U_B(\beta)=0$, is trivial.

If $U_B(\alpha)\neq 0$ and $U_B(\beta)\neq 0$, for $\varepsilon>0$ such that $U_B(\alpha)\wedge U_B(\beta)>\varepsilon$, there exist $\alpha_1,\beta_1\in\Theta_X$ such that

$$B(\alpha_1)\geq U_B(\alpha)-\varepsilon, \alpha_1\leq\alpha,$$
$$B(\beta_1)\geq U_B(\beta)-\varepsilon, \beta_1\leq\beta,$$

Since $\alpha_1\wedge\beta_1\leq\alpha\wedge\beta$, we have

$$\begin{aligned}U_B(\alpha\wedge\beta)&\geq B(\alpha_1\wedge\beta_1)\\&\geq B(\alpha)\wedge B(\beta)\\&\geq U_B(\alpha)\wedge U_B(\beta)-\varepsilon\end{aligned}$$

Since ε is arbitrary, this gives the desired result.

(FQU2) It is easily proved from the definition of U_B.

(FQU3) If $U_B(\alpha)=0$, then there exists the identity function $E\in\Omega_X$ with $E\circ E\leq\alpha$ such that $U_B(E)\geq 0$.

Suppose that there exists $\gamma\in\Omega_X$ and $r\in(0,1)$ such that

$$\sup\{U_B(\alpha)\setminus\alpha\circ\alpha\leq\gamma\}<r<U_B(\gamma).$$

By the definition of U_B, there exists $\gamma_1\leq\gamma$ such that

$$U_B(\gamma) \geq B(\gamma_1) > r.$$

Since $\sup\{B(\alpha_1) \setminus \alpha_1 \circ \alpha_1 \leq \gamma_1\} \geq B(\gamma_1) > r$ from (FQB2), there exists $\varphi \in \Theta_X$ such that $\varphi \circ \varphi \leq \gamma_1$ and $B(\varphi) > r$.

It follows

$$\sup\{U_B(\alpha) \setminus \alpha \circ \alpha \leq \gamma\} \geq B(\varphi) > r.$$

It is a contradiction. Hence,

$$\sup\{U_B(\alpha) \setminus \alpha \circ \alpha \leq \gamma\} \geq U_B(\gamma).$$

(FU) It is similar to (FQU3).

1.3.8 Definition

Let (X, B) and (X, B') be smooth (quasi-) uniform bases. We say B' is finer than B, denoted by $B' \geq B$, iff for any $B(\alpha) > 0$ and $\varepsilon > 0$, there exists $\beta \leq \alpha$ such that $B'(\beta) \geq B(\alpha) - \varepsilon$.

1.3.9 Theorem

Let (X, B) and (X, B') be smooth (quasi-) uniform bases for (X, U) and (X, U'), respectively. Then $U \leq U'$ iff $B \leq B'$.

Proof

For any $B(\alpha) > 0$, since $U \leq U'$, we have

$$U'(\alpha) \geq U(\alpha) \geq B(\alpha).$$

From Theorem 1.3.7 of the definition of U', for $\varepsilon > 0$ there exists $\beta \leq \alpha$ such that

$$B'(\beta) \geq U'(\alpha) - \varepsilon \geq B(\alpha) - \varepsilon.$$

Hence, $B' \leq B$.

Conversely, suppose that there exist $\alpha \in \Omega_X$ and $r \in (0,1)$ such that

$$U(\alpha) > r > U'(\alpha).$$

By the definition of U, there exists $\beta \leq \alpha$ such that

$$U(\alpha) \geq B(\beta) > r > U'(\alpha).$$

Since $B \leq B'$, for $B(\beta) > r$ and $\varepsilon = B(\beta) - r$, there exists $\gamma \leq \beta$ such that

$$B'(\gamma) \geq (B(\beta) - \varepsilon) = r.$$

Hence, $U'(\alpha) \geq U'(\gamma) \geq B'(\gamma) \geq r$.

It is a contradiction. Therefore $U \leq U'$.

1.3.10 Lemma

Define $U_\rho : I^X \to I^X$ as follows.

$$U_\rho(\lambda) = \begin{cases} \underline{0} & \text{if } \lambda = \underline{0} \\ \rho & \text{if } \underline{0} \neq \lambda \leq \rho, \\ \underline{1} & \text{otherwise.} \end{cases}$$

Then:

(1) $U_\rho \in \Omega_X$,

(2) $(U_\rho)^{-1} = U_{\underline{1}-\rho}$,

(3) $U_\rho \circ U_\rho = U_\rho$ and $(U_\rho \wedge U_\mu) \circ (U_\rho \wedge U_\mu) = (U_\rho \wedge U_\mu)$.

Proof

(1), (2) and $U_\rho \circ U_\rho = U_\rho$ of (3) are easily proved.

Since

$$U_\rho \wedge U_\mu(\lambda) = \begin{cases} \underline{0} & \text{if } \lambda = \underline{0}, \\ \mu \wedge \rho & \text{if } \underline{0} \neq \lambda \leq \mu \wedge \rho \\ \mu & \text{if } \lambda \leq \mu, \lambda \not\leq \rho \\ \rho & \text{if } \lambda \not\leq \mu, \lambda \leq \rho \\ \mu \vee \rho & \text{if} \lambda \leq \mu \vee \rho, \lambda \not\leq \mu, \lambda \not\leq \rho \\ \underline{1} & \text{otherwise} \end{cases}$$

we have $(U_\rho \wedge U_\mu) \circ (U_\rho \wedge U_\mu) = (U_\rho \wedge U_\mu)$.

1.3.11 Example

Define B and B' on X as follows:

$$B(\alpha)=\begin{cases}1 & \text{if } \alpha=\alpha_1, \\ \frac{1}{2} & \text{if } \alpha=\alpha_\rho\end{cases}$$

and

$$B'(\alpha)=\begin{cases}1 & \text{if } \alpha=\alpha_1, \\ \frac{2}{3} & \text{if } \alpha=\alpha_\rho\wedge\alpha_\mu.\end{cases}$$

From Lemma 1.3.10, B and B' are smooth uniform bases on X. From Definition 1.3.8 we have $B\leq B'$.

From Theorem 1.3.7 we obtain the followings:

$$U_B(\alpha)=\begin{cases}1 & \text{if } \alpha=\alpha_1, \\ \frac{1}{2} & \text{if } \alpha_\rho\leq\alpha<\alpha_1 \\ 0 & \text{otherwise}\end{cases}$$

and

$$U_{B'}(\alpha)=\begin{cases}1 & \text{if } \alpha=\alpha_1, \\ \frac{2}{3} & \text{if } \alpha_\rho\wedge\alpha_\mu.\leq\alpha<\alpha_1 \\ 0 & \text{otherwise}\end{cases}$$

Then $U_B\leq U_{B'}$.

1.3.12 Theorem

Let $U:\Omega_X\to I$ be a smooth quasi-uniformity on X. Define a function

$C_U: I^X\times I_1\to I^X$, by

$$C_U(\lambda,r)=\inf\{\alpha(\lambda)/U(\alpha)>r\}.$$

For each $\lambda,\lambda_1,\lambda_2\in I^X$ and $r,r_1,r_2\in I_1$, we have the following statements hold:

(1) $C_U(\underline{0},r)=\underline{0}$,

(2) $\lambda\leq C_U(\lambda,r)$,

(3) If $\lambda_1 \le \lambda_2$, then $C_U(\lambda_1, r) \le C_U(\lambda_2, r)$,

(4) $C_U(\lambda_1 \vee \lambda_2, r) = C_U(\lambda_1, r) \vee C_U(\lambda_2, r)$,

(5) If $r_1 \le r_2$, then $C_U(\lambda, r_1) \le C_U(\lambda, r_2)$,

(6) $C_U(C_U(\lambda, r), r) = C_U(\lambda, r)$.

1.2.13 Theorem

Let $U : \Omega_X \to I$ be a smooth quasi-uniformity on X. Define a function $\tau_U : I^X \to I$, by

$$\tau_U(\lambda) = \sup\{r \in I_1 \mid C_U(\underline{1} - \lambda, r) = \underline{1} - \lambda\}.$$

Then τ_U is a smooth topology on X induced by U.

1.3.14 Definition

Let (X, U) and (Y, V) be smooth quasi-uniform spaces. A function $f : X \to Y$ is said to be *smooth quasi-uniform continuous* if

$$V(\alpha) \le U(f^{\leftarrow}(\alpha)), \forall \alpha \in \Omega_Y$$

where $f^{\leftarrow}(\alpha)(\lambda) = f^{-1}(\alpha(f(\lambda)))$ for all $\lambda \in I^X$.

From Theorem 1.3.7, we easily prove the following theorem.

1.3.15 Theorem

Let (X, B_1) and (Y, B_2) be smooth quasi-uniform bases. If $B_2(\alpha) \le B_1(f^{\leftarrow}(\alpha)), \forall \alpha \in \Omega_Y$ then $f : (X, U_{B_1}) \to (Y, U_{B_2})$ is smooth quasi-uniform continuous.

1.3.16 Theorem

Let $(X, U), (Y, V)$ and (Z, W) be smooth quasi-uniform spaces. If $f : (X, U) \to (Y, V)$ and $g : (Y, V) \to (Z, W)$ are smooth uniform continuous, then $g \circ f : (X, U) \to (Z, W)$ is smooth quasi-uniform continuous.

Proof

It follows that for each, $\alpha \in \Omega_Z$,

$$\begin{aligned} U((f \circ g)^{\leftarrow}(\alpha)) &= (f^{\leftarrow}(g^{\leftarrow}(\alpha))) \\ &\geq V(g^{\leftarrow}(\alpha)) \\ &\geq W(\alpha). \end{aligned}$$

1.3.17 Theorem

Let (X,U) and (Y,V) be smooth quasi-uniform spaces. Let $f : X \to Y$ be smooth quasi-uniform continuous. Then:

(1)$f(C_U(\lambda,r)) \leq C_V(f(\lambda),r))$, for each $\lambda \in I^X$,

(2)$C_U(f^{-1}(\mu),r) \leq f^{-1}(C_V(f(\mu),r))$, for each $\mu \in I^Y$,

(3)$f : (X,\tau_U) \to (Y,\tau_V)$ is smooth continuous.

Proof

The proof is similar to the proof of Theorem 1.1.10.

CHAPTER II

Chapter II

Smooth grills and smooth proximity spaces

In this chapter, we introduce the notions of smooth grills and smooth proximity spaces with a somewhat different point in [1,5,7,30] and investigate some properties of them specially we make a characterization of smooth proximity by smooth grill. The subspaces of smooth proximity spaces and the relationships among smooth proximities, smooth topologies and smooth uniformities are studied.

2.1 Smooth grills

In this section we introduced the concept of smooth grill with the notions (homogeneous, weakly stratified, stratified and strongly stratified). We give some example to show that weakly stratified smooth grill and stratified smooth grill are independent notions. Finally we study the relation between smooth grills and fuzzifying grills.

2.1.1 Definition

A nonzero function $S : I^X \to I$ is called a *smooth stack* on X if it satisfying the following condition:

$(SS1)$ if $\mu \leq \lambda$ then $S(\mu) \leq S(\lambda)$.

2.1.2 Definition

A nonzero function $G : I^X \to I$ is called a *smooth grill* if it satisfying the following conditions:

$(SG1) G(\underline{0}) = 0$,

$(SG2) G(\mu \vee \nu) \leq G(\mu) \vee G(\nu)$,

$(SG3)$ if $\nu \leq \mu$ then $G(\nu) \leq G(\mu)$.

2.1.3 Remark

In the above definition, the conditions (SG2) and (SG3) are equivalent to the following condition:

$(SG)\ G(\mu \vee \nu) = G(\mu) \vee G(\nu)$.

2.1.4 Notation

For a set X, $S(X)$ and $\Gamma(X)$ denote, respectively, the sets of all smooth stacks and smooth grills on X. Of course, the requiring that G be nonzero is equivalent to requiring that $G(\underline{1}) > 0$.

2.1.5 Theorem

For $S \in S(X)$ we define, $\theta_S : I^X \to I$ by

$$\theta_S(\nu) = 1 - S(\underline{1} - \nu)$$

For $\{S_i : i \in I\} \subset S(X)$, we have the following properties:

(1) $\theta_{S_i} \in S(X)$

(2) if $S_1 \leq S_2$ then $\theta(S_1) \leq \theta(S_2)$,

(3) $\theta_{S_i} \circ \theta_{S_i} = \theta_{S_i}$ for each $i \in \Gamma$,

(4) $\theta(\sup_{i\in\Gamma} S_i) = \inf_{i\in\Gamma} \theta(S_i)$,

(5) $\theta(\inf_{i\in\Gamma} S_i) = \sup_{i\in\Gamma} \theta(S_i)$.

The proof is straightforward because $\sup_{i\in\Gamma} S_i, \inf_{i\in\Gamma} S_i \in S(X)$. .

2.1.6 Definition

Let $G : I^X \to I$ be a nonzero function and $\mu \in I^X$. We define $\langle G \rangle : I^X \to I$ by

$$\langle G \rangle(\mu) = \sup_{\nu \leq \mu} G(\nu).$$

A function $G: I^X \to I$ is called a *smooth grill base* on *X* satisfying the following conditions:

$(SGB1) G(\underline{0}) = 0,$

$(SGB2) \langle G\rangle(\mu \vee \nu) \leq G(\mu) \vee G(\nu), \forall \mu, \nu \in I_\circ$

Naturally, a smooth grill is a smooth grill base.

2.1.8 Theorem

If a function $G: I^X \to I$ is smooth grill base, then $\langle G\rangle$ is a s-grill.

Proof.

The condition (SG_1) is easily checked. From the definition of $\langle G\rangle$, we have

$$\langle G\rangle(\mu \vee \nu) \leq G(\mu) \vee G(\nu) \leq \langle G\rangle(\mu) \vee \langle G\rangle(\nu).$$

Suppose there exist $\mu, \nu \in I^X$ and $t \in I_\circ$ such that

$$\langle G\rangle(\mu \vee \nu) < t < \langle G\rangle(\mu) \vee \langle G\rangle(\nu) \qquad \text{(A)}$$

If $G(\mu) > t$, there exists $\mu_1 \in I^X$ with $\underline{0} \neq \mu_1 \leq \mu$ such that

$$\langle G\rangle(\mu) \geq \langle G\rangle(\mu_1) > t.$$

Thus $\langle G\rangle(\mu \vee \nu) \geq \langle G\rangle(\mu_1) > t$. It is a contradiction for the equation (A).

Similarly, if $G(\nu) > t$, it is contradiction. Hence,

$$\langle G\rangle(\mu \vee \nu) \geq \langle G\rangle(\mu) \vee \langle G\rangle(\nu).$$

Then $\langle G\rangle$ is a smooth grill.

2.1.8 Definition

If *G* is a smooth grill base on *X*, we define the characteristic, denoted by $c(G)$, of *G* by

$$c(G) = \sup_{\nu \in I^X} G(\nu).$$

It follows from definition that $c(G) > 0$. Just as for smooth grill we have the following lemma:

2.1.9 Lemma

If a function $G : I^X \to I$ is a smooth grill base on X, then

$$c(\langle G \rangle) = c(G).$$

Proof

$$\begin{aligned} c(\langle G \rangle) &= \sup_{\nu \in I^X} \langle G \rangle(\nu) \\ &= \sup_{\nu \in I^X} (\sup_{\mu \le \nu} G(\mu)) \\ &= \sup_{\mu \in I^X} G(\mu) = c(G) \end{aligned}$$

2.1.10 Definition

If G is a smooth grill (base) on X with $c(G) = c$ then for $0 \le \alpha \le c$, we define the (upper) α-cut grill (base), denoted by G^α, associated with G by

$$G^\alpha = \{\nu \in I^X : G(\nu) > \alpha\}.$$

and, for $0 < \alpha \le c$, we define the (lower) α-cut grill (base), denoted by G_α, associated with G by

$$G_\alpha = \{\nu \in I^X : G(\nu) \ge \alpha\}.$$

2.1.11 Theorem

If G is a *smooth grill* (base) on X with $c(G) = c$, and:

(1) $0 \le \alpha < c$, then G^α is a fuzzy grill (base) on X.

(2) $0 < \alpha \le c$, then G_α is a fuzzy grill (base) on X.

Proof

Let G be *smooth grill* on X. $G(\underline{1}) = c > \alpha$, implies $\underline{1} \in G^\alpha$. Thus $G^\alpha \ne \phi$. Since $G(\underline{0}) = 0$, then $\underline{0} \notin G^\alpha$. $\mu \vee \nu \in G^\alpha$ implies $\alpha < G(\mu \vee \nu) \le G(\mu) \vee G(\nu)$ implies, $\alpha < G(\mu)$ or $\alpha < G(\nu)$ implies $\mu \in G^\alpha$ or $\underline{\nu} \in G^\alpha$.

Finally, if $\underline{\nu} \in G^\alpha$ and $\nu \le \mu$ then $\alpha < G(\nu) \le G(\mu)$, and Hence,, $\mu \in G^\alpha$. To prove (FGB$_2$) let $\nu_1 < \nu_2 \vee \nu_3$ and $\nu_1 \in G^\alpha$. Then $\alpha < \langle G \rangle(\nu_1) \le \langle G \rangle(\nu_2 \vee \nu_3) \le G(\nu_2) \vee G(\nu_3)$ and Hence,, $\alpha \le G(\nu_2)$ or $\alpha \le G(\nu_3)$ then $\nu_2 \in G^\alpha$ or $\nu_3 \in G^\alpha$. The proof of (2) is clearly and left to the reader.

2.1.12 Lemma

If G is a *smooth grill* (base) on X with $c(G) = c$, then for $0 \le \alpha \le \beta < c$,

$$G_c \le G^\beta \le G^\alpha \le G^0.$$

The proof is straightforward.

2.1.13 Definition

Let G be a smooth grill on X with $c(G) = c > 0$. For $\mu \in I^X$, define

$$S_G(\mu) = \{\alpha \in (0, c] : \mu \in G^\alpha\}$$

Then $S_G(\mu) = \phi$ or $S_G(\mu)$ is the interval of the form $(0, \beta]$.

The proof is straightforward.

2.1.14 Theorem

If G is a smooth grill on X with $c(G) = c > 0$. We define $SG : I^X \to I$ by

$$SG(\mu) = c \wedge (\sup\{\alpha \in S_G(\mu)\}).$$

Then SG is smooth grill.

Proof

(SG1) Since $\underline{0} \notin G^\alpha$. for $0 < \alpha \le c$, $S_G(\underline{0}) = \phi$ Hence, $SG(\underline{0}) = 0$

(SG$_2$) and (SG$_3$) are proved from:

$$SG(\lambda \vee \mu) = c \wedge (\sup\{\alpha \in S_G(\lambda \vee \mu)\})$$

$$= c \wedge (\sup\{\alpha \in S_G(\lambda)\} \vee (\sup\{\alpha \in S_G(\mu)\}))$$

$= (c \wedge (\sup\{\alpha \in S_G(\lambda)\} \vee (c \wedge \sup\{\alpha \in S_G(\mu)\}))$

$= SG(\lambda) \vee SG(\mu).$

2.1.15 Definition

(1) A smooth grill is called *weakly stratified* if and only if it satisfies

$$\forall \alpha \in I,\ G(\underline{\alpha}) \geq \alpha .$$

(2) A smooth grill is called *homogenous* if and only if it satisfies

$$\forall \alpha \in I,\quad G(\underline{\alpha}) = \alpha .$$

(3) A smooth grill is called *stratified* if and only if it satisfies

$$\forall \alpha \in I,\ \forall \lambda \in I^X,\quad G(\underline{\alpha} \vee \lambda) \leq \alpha \vee G(\lambda) .$$

(4) A smooth grill is called *strongly stratified* if and only if it satisfies

$$\forall \lambda \in I^X,\quad G(\lambda = \sup_{\alpha \in I} (\alpha \wedge G(1_{\lambda_\alpha}))$$

2.1.16 Remark

If G is a homogeneous smooth grill then G is stratified and weakly stratified.

2.1.17 Theorem

Let G is a smooth grill base on X such that $G(\underline{1}) = 1$. If G is a strongly stratified, then it is homogeneous.

Proof

Since $G(\underline{1}) = 1$, we have

$$G(1_{(\underline{\beta})_\alpha}) = \begin{cases} 1 & \text{if } 0 \leq \alpha \leq \beta \\ 0 & \text{if } \alpha > \beta \end{cases}$$

Thus,

$$\forall \beta \in I, G(\underline{\beta}) = \sup_{\alpha \in I} (\alpha \wedge G(1_{(\underline{\beta})_\alpha})) = \beta.$$

2.1.18 Example

(1) A function $S : I^X \to I$ defined by $S(\nu) = \sup_{x \in X} \nu(x)$ is a strongly stratified smooth grill on X. By Theorem 2.1.17 and remark 2.1.16, S is homogeneous, stratified and weakly stratified.

(2) A function $S : I^X \to I$ defined by $S(\nu) = \sup_{x \in X} \frac{1}{2} \nu(x)$ is a stratified smooth grill on X. S is neither homogeneous nor stratified because $S(\underline{0.4}) = 0.2$. Furthermore, it is not strongly stratified because:

$$\sup_{\alpha \in I} (\alpha \wedge G(1_{(0,4)_\alpha})) = 0.4 \neq S(\underline{0.4}).$$

(3) A function $S : I^X \to I$ defined by $S(\nu) = \min\{1, \sup_{x \in X} 2\nu(x)\}$ is a weakly stratified smooth grill on X. S is not stratified because

$$0.8 = S(\underline{0.4} \vee \underline{0.3}) > (0.4 \vee G(\underline{0.3})) = 0.6 .$$

Furthermore, it is neither homogeneous nor strongly stratified.

(4) A function $S : I^X \to I$ defined by

$$S(\nu) = \begin{cases} 1 & \text{if } \nu = 0 \\ 0 & \text{if } \nu \neq 0 \end{cases}$$

is a weakly stratified and stratified smooth grill on X. It is neither homogeneous nor strongly stratified.

(5) From (2) and (3), the weakly stratified smooth grills and stratified smooth grill are the independent notions.

2.1.19. Definition

A nonzero function $g : 2^X \to I$ is a *fuzzifying grill* on X if and only if it satisfying the following conditions:

(fg1) $g(\phi) = 0$,
(fg2) $g(A \cup B) \leq g(A) \vee g(B)$,
(fg3) if $A \subset B$ then $G(A) \leq G(B)$.

2.1.20 Theorem

Let X be a set and $\Delta(X)$ denote the collection of fuzzifying grills on X. Let $\Sigma(X)$ denote the collection of strongly stratified smooth grills on X. For $g \in \Delta(X)$, let $G^g : I^X \to I$ be defined by

$$G^g(\lambda) = \sup_{\alpha \in I} (\alpha \wedge g(\lambda_\alpha)).$$

For $G \in \Sigma(X)$, let $g^G : 2^X \to I$ be defined by

$$g^G(A) = G(1_A).$$

For $g \in \Delta(X)$, let $\psi : \Delta(X) \to \Sigma(X)$ be defined by

$$\Psi(g) = G^g.$$

For $G \in \Sigma(X)$, let $\Phi : \Sigma(X) \to \Delta(X)$ be defined by

$$\Phi(G) = g^G.$$

Then it has the following properties:

(1) $G^g \in \Sigma(X)$,

(2) $g^G \in \Delta(X)$,

(3) $\Psi \circ \Phi = 1_{\Sigma(X)}$, that is $G^{g^G} = G$, for each $G \in \Sigma(X)$,

(4) $\Phi \circ \Psi = 1_{\Delta(X)}$, that is $g^{G^g} = g$, for each $g \in \Delta(X)$,

(5) Ψ is bijection .

Proof.

(1) To prove $G^g \in \Sigma(X)$, the axiom (SG1) follows from (fg1), since $\underline{0}_\alpha = \phi$. The axiom (SG3) follows from (fg3) and the fact that $\lambda \leq \mu$ implies $\lambda_\alpha \subset \mu_\alpha, \forall \alpha \in I$.

(SG2) Suppose there exist $\lambda, \mu \in I^X$ such that

$$G^g(\lambda \vee \mu) > G^g(\lambda) \vee G^g(\mu).$$

There exists $t \in I_\circ$ such that

$$G^g(\lambda \vee \mu) > t > G^g(\lambda) \vee G^g(\mu). \qquad \text{(B)}$$

From the definition of G^g , There exists $\alpha \in I_\circ$ such that

$$G^g(\lambda \vee \mu) \geq \alpha \wedge g((\lambda \vee \mu)_\alpha) > t$$

Since $(\lambda \vee \mu)_\alpha = \lambda_\alpha \vee \mu_\alpha$ and $g((\lambda \vee \mu)_\alpha) = g(\lambda_\alpha) \vee g(\mu_\alpha)$, we have

$$\begin{aligned} G^g(\lambda) \vee G^g(\mu) &\geq \{\alpha \wedge g(\lambda_\alpha)\} \vee \{\alpha \wedge g(\mu_\alpha)\} \\ &= \alpha \wedge g((\lambda \vee \mu)_\alpha) \\ &> t. \end{aligned}$$

It is a contradiction for the equation (B). Thus,

$$G^g(\lambda \vee \mu) \leq G^g(\lambda) \vee G^g(\mu).$$

Now, to prove G^g is strongly stratified, since $G^g(1_{\lambda_\alpha}) = g(\lambda_\alpha)$ for each $\lambda \in I^X$, we have

$$\begin{aligned} G^g(\lambda) &= \sup_{\alpha \in I} (\alpha \wedge g(\lambda_\alpha)) \\ &= \sup_{\alpha \in I} (\alpha \wedge G^g(1_{\lambda_\alpha})). \end{aligned}$$

Thus, $G^g \in \Sigma(X)$.

(2) The axioms (fg1) and (fg3) follow from (SG1) and (SG3), respectively. The axiom (fg2) follows (SG2) because

$$\begin{aligned} g^G(A) \vee g^G(B) &= G(1_A) \vee G(1_B) \\ &\geq G(1_A \vee 1_B) \\ &= G(1_{A \cup B}) \\ &= g^G(A \cup B). \end{aligned}$$

Thus, $g^G \in \Delta(X)$.

(3) Because G is strongly stratified, it follows that

$$G^{g^G}(\lambda) = \sup_{\alpha\in I}(\alpha \wedge g^G(\lambda_\alpha))$$
$$= \sup_{\alpha\in I}(\alpha \wedge G(1_{\lambda_\alpha}))$$
$$= G(\lambda).$$

(4) Since g is a fuzzifying grill on X, it follows that

$$g^{G^g}(A) = G^g(1_A)$$
$$= \sup_{\alpha\in I}(\alpha \wedge g((1_A)_\alpha))$$
$$= g(A).$$

(5) Follows from (3) and (4).

2.2 Smooth proximity spaces

This section consists of three parts: definitions and general properties of the smooth proximity, smooth topologies induced by smooth proximities and smooth quasi-proximity induced by smooth quasi-uniformity.

2.2.a Definitions and general properties

2.2.a.1 Definition

A function $\delta : I^X \times I^X \to I$ is called a *smooth quasi-proximity* on X, if it satisfies the following axioms:

(SQP1) $\delta(\underline{1},\underline{0}) = \delta(\underline{0},\underline{1}) = 0$,

(SQP2) if $\delta(\lambda,\mu) \neq 1$, then $\lambda \leq 1-\mu$,

(SQP3) (1) $\delta(\lambda,\mu\vee\nu) = \delta(\lambda,\nu)\vee\delta(\lambda,\nu)$,

(2) $\delta(\lambda\vee\mu,\nu) = \delta(\lambda,\nu)\vee\delta(\mu,\nu)$,

(SQP4) $\delta(\lambda,\mu) \geq \inf_{\rho\in I^X}\{\delta(\lambda,\rho)\vee\delta(\underline{1}-\rho,\mu)\}$.

The pair (X,δ) is said to be *smooth quasi-proximity space*.

A smooth quasi-proximity space (X,δ) is called *smooth proximity space* if it satisfies:

(SP5) $\delta = \delta^{-1}$, where $\delta^{-1}(\lambda,\mu) = \delta(\mu,\lambda)$.

A smooth proximity space (X,δ) is called *principal* if it satisfies:

(PSP) $\delta(\sup_{j\in J}\lambda_j,\mu)\le\sup_{j\in J}\delta(\lambda_j,\mu)$

2.2.a.2 Remark

(1) A smooth proximity space (X,δ) is called *basic smooth proximity space* if it satisfies the conditions (SP1-SP3) and (SP5).

(2) Let (X,δ) be a smooth quasi-proximity space. Then the structure δ^{-1} is a smooth quasi-proximity on X.

(3) Let (X,δ) be a smooth proximity space, then for each $r\in(0,1]$ the family $\delta_r=\{(\lambda,\mu)\in I^X\times I^X\setminus\delta(\lambda,\mu)\ge r\}$ is a fuzzy proximity space on X.

Now we identify the relation δ on I^X with the mapping $\delta: I^X\to(I)^{I^X}$ such that

$$\delta_\lambda(\mu)=\delta(\mu,\lambda).$$

It is clearly that, δ_λ is a smooth grill on X.

Let δ be smooth basic proximity on a set X and let G be a smooth grill on X. Then we define, $\underline{e}: M(X)\times\Gamma(X)\to\Gamma(X)$ as follows:

$$\underline{e}(\delta,G)(\lambda)=\inf_{\mu\in I^X}(\delta(\mu,\lambda)\vee G(\mu)),$$

where $M(X)$ is the set of all smooth proximity spaces.

2.2.a.3 Theorem

(i) $\underline{e}(\delta,G)\in G(X)$

(2) $\underline{e}(\delta,G)\supset G$

Proof.

Since for $\alpha\in I$, we have $\delta(\mu,\underline{0})=0$ and $G(\underline{0})=0$, then $\underline{e}(\delta,G)(\underline{0})=0$.

$$\underline{e}(\delta,G)(\lambda\vee\mu)=\inf_{\nu\in I^X}(\delta(\nu,\lambda\vee\mu)\vee G(\lambda\vee\mu))$$
$$=\inf_{\nu\in I^X}((\delta(\nu,\lambda)\vee\delta(\nu,\mu))\vee(G(\lambda)\vee G(\mu)))$$
$$=\inf_{\nu\in I^X}((\delta(\nu,\lambda)\vee G(\lambda))\vee(\delta(\nu,\mu)\vee G(\mu)))$$
$$=\inf_{\nu\in I^X}(\delta(\nu,\lambda)\vee G(\lambda))\vee\inf_{\nu\in I^X}(\delta(\nu,\mu)\vee G(\mu))$$
$$=\underline{e}(\delta,G)(\lambda)\vee\underline{e}(\delta,G)(\mu).$$

Therefore, $\underline{e}(\delta,G)\in G(X)$.

Second, since, $G(\lambda)\le\delta(\mu,\lambda)\vee G(\lambda)$ for all $\mu\in I^X$, then

$$G(\lambda)\le\inf_{\mu\in I^X}\delta(\mu,\lambda)\vee G(\lambda)=\underline{e}(\delta,G)(\lambda).$$

Therefore, $\underline{e}(\delta,G)\supset G$.

2.2.a.4 Theorem

A smooth basic proximity is a smooth proximity iff $\underline{e}(\delta,\delta_\lambda)=\delta_\lambda$ for each $\forall\lambda\in I^X$.

Proof

Since δ_λ is smooth grill on X and Hence, by Theorem 2.2.a.3 we have $\underline{e}(\delta,G)\ge G$. Before proceeding further let us note that for some $\upsilon\in I^X$ can be expressed as $\underline{1}-\upsilon$ and by symmetry of δ and $\lambda\in\delta_\mu$ iff $\mu\in\delta_\lambda$. Now, by definition of smooth basic proximity δ on X is a smooth proximity iff $\delta(\lambda,\upsilon)\vee\delta(1-\upsilon,\mu)\le\delta(\lambda,\mu)$. Since

$$e(\delta,\delta_\lambda)(\mu)=\inf_{\upsilon}(\delta(\upsilon,\mu)\vee\delta_\lambda(\upsilon))$$
$$\le\delta(1-\upsilon,\mu)\vee\delta(\lambda,\upsilon)$$
$$=\delta(\lambda,\upsilon)\vee\delta(1-\upsilon,\mu)$$
$$\le\delta(\lambda,\mu)$$
$$=\delta_\lambda(\mu).$$

Then, $e(\delta,\delta_\lambda) \subset \delta_\lambda$, $e(\delta,\delta_\lambda) = \delta_\lambda$.

We will construct the coarsest smooth quasi-proximity on X finer than δ_1 and δ_2.

2.2.a.5 Theorem

Let (X,δ_1) and (X,δ_2) be smooth quasi-proximity spaces. We define, for all $\lambda,\mu \in I^X$

$$\delta_1 \cup \delta_2(\lambda,\mu) = \inf\{\sup_{j,k}(\delta_1(\lambda_j,\mu_k) \wedge \delta_2(\lambda_j,\mu_k))\}.$$

Where for every finite families $(\lambda_j),(\mu_k)$ such that $\lambda = \sup \lambda_j$ and $\mu = \sup \mu_k$.

Then the structure $\delta_1 \cup \delta_2$ is the coarsest smooth quasi-proximity on X finer than δ_1 and δ_2.

Let (X,δ) be a smooth quasi-proximity space. For each $\lambda,\mu \in I^X$ we define

$$\delta^*(\lambda,\mu) = \delta \cup \delta^{-1}(\lambda,\mu)$$

By the above theorem, we can easily prove that (X,δ^*) is a smooth proximity space.

2.2.a.6 Definition

Let (X,δ_1) and (Y,δ_2) be smooth quasi-proximity spaces. A function $f: X \to Y$ is a *smooth quasi-proximity (proximity) continuous* if it satisfies

$$\delta_1(\mu,\upsilon) \le \delta_2(f(\mu),f(\upsilon)) \text{ for every } \mu,\upsilon \in I^X$$

Equivalently,

$$\delta_1(f^{-1}(\lambda),f^{-1}(\rho)) \le \delta_2(\lambda,\rho) \text{ for every } \lambda,\rho \in I^Y.$$

Using the above definition, we can easily prove the following lemma.

2.2.a.7 Lemma

If a function $f:(X,\delta_1)\to(Y,\delta_2)$ is a smooth quasi-proximity continuous, then:

(a) $f:(X,\delta_1^{-1})\to(Y,\delta_2^{-1})$ is a smooth quasi-proximity continuous.

(b) $f:(X,\delta_1^{*})\to(Y,\delta_2^{*})$ is a smooth quasi-proximity continuous.

2.2.a.8 Theorem

Let (Y,δ) be a smooth proximity space, X a set and $f:X\to Y$ a function. We define $\delta_f:I^X\times I^X\to I$ by

$$\delta_f(\lambda,\mu)=\delta(f(\lambda),f(\mu))$$

Then:

(1) The structure δ_f is the coarsest smooth proximity on X for which f is smooth proximity continuous.

(2) A function $g:(Z,\delta^*)\to(X,\delta_f)$ is smooth proximity continuous iff $f\circ g$ is smooth proximity continuous.

Proof

(1) First, we will show that δ_f is a smooth proximity on X.

(SQP1) Since $\delta_f(\underline{1},\underline{0})=\delta(f(\underline{1}),f(\underline{0}))\le\delta(\underline{1},\underline{0})=0$. Similarly $\delta_f(\underline{0},\underline{1})=0$.

(SQP2) Let $\delta_f(\lambda,\mu)=\delta(f(\lambda),f(\mu))\neq 1$. Then $f(\lambda)\le\underline{1}-f(\mu)$ implies

$$\lambda\le f^{-1}(f(\lambda))\le f^{-1}(\underline{1}-f(\mu))\le\underline{1}-f^{-1}(f(\mu))\le\underline{1}-\mu.$$

(SQP3) and (SP) are trivial.

(SQP4) Suppose $\delta_f(\lambda,\mu)\ngeq\inf_{\rho\in I^X}\{\delta_f(\lambda,\rho)\vee\delta_f(\underline{1}-\rho,\mu)\}$.

There exists $r\in(0,1)$ such that

$$\delta_f(\lambda,\mu)<r<\inf_{\rho\in I^X}\{\delta_f(\lambda,\rho)\vee\delta_f(\underline{1}-\rho,\mu)\}.$$

Since $r>\delta_f(\lambda,\mu)=\delta(f(\lambda),f(\mu))$, by (P4), there exists $\gamma\in I^Y$ such that

Since $r > \delta_f(\lambda,\mu) = \delta(f(\lambda), f(\mu))$, by (P4), there exists $\gamma \in I^Y$ such that

$$\delta(f(\lambda),\gamma) \vee \delta(\underline{1}-\gamma, f(\mu)) < r$$

It implies

$$\begin{aligned}&\delta_f(\lambda, f^{-1}(\gamma)) \vee \delta_f(f^{-1}(\underline{1}-\gamma),\mu)\\ &= \delta(f(\lambda), f(f^{-1}(\gamma))) \vee \delta(f(f^{-1}(\underline{1}-\gamma)), f(\mu))\\ &\leq \delta(f(\lambda),\gamma) \vee \delta(\underline{1}-\gamma, f(\mu)) < r\end{aligned}$$

Thus, $\inf_{\rho \in I^X} \{\delta_f(\lambda,\rho) \vee \delta_f(\underline{1}-\rho,\mu)\} < r$.

It is contradiction for (E). Hence,

$$\delta_f(\lambda,\mu) \geq \inf_{\rho \in I^X} \{\delta_f(\lambda,\rho) \vee \delta_f(\underline{1}-\rho,\mu)\}.$$

From the definition of δ_f, f is smooth proximity continuous.

Let $f:(X,\delta') \to (Y,\delta)$ be smooth proximity continuous. Since

$$\delta'(\lambda,\mu) = \delta(f(\lambda), f(\mu)) = \delta_f(\lambda,\mu),$$

δ_f is coarser than δ'.

(2) Let g be smooth proximity continuous. So,

$$\delta^*(\lambda,\mu) \leq \delta_f(g(\lambda), g(\mu)) = \delta(f(g(\lambda)), f(g(\mu)))$$

Hence, $f \circ g$ is smooth proximity continuous.

Let $f \circ g$ be smooth proximity continuous.

$$\delta^*(\lambda,\mu) \leq \delta(f(g(\lambda)), f(g(\mu))) = \delta_f(g(\lambda), g(\mu))$$

Then g is smooth proximity continuous.

2.2.a.9 Definition

Let $(X_i,\delta_i)_{i\in\Delta}$ be a family of smooth quasi-proximity spaces. Let X be a set and, for each $i \in \Delta, f_i : X \to X_i$ a function. The *initial structure* δ is the coarsest smooth quasi-proximity on X with respect to which for each $i \in \Delta, f_i$ is a smooth quasi-proximity continuous function.

2.2.a.10 Definition

Let (X,δ) be a smooth quasi-proximity and A be a subset of X. The pair (A,δ_A) is said to be a *subspace* of (X,δ) if it is endowed with the initial smooth quasi-proximity structure with respect to the inclusion function.

2.2.a.11 Definition

Let X be the product $\prod_{i\in\Delta} X_i$ of the family $\{(X_i,\delta_i)\setminus i\in\Delta\}$ of smooth quasi-proximity spaces. An initial smooth quasi-proximity structure $\delta=\otimes\delta_i$ on X with respect to all the projections $\pi_i : X \to X_i$ is called the *product smooth quasi-proximity structure* of $\{\delta_i \setminus i\in\Delta\}$ and $(X,\otimes\delta_i)$ is called the *product smooth quasi-proximity space*.

2.2.a.12 Corollary

Let $(X_i,\delta_i)_{i\in\Delta}$ be a family of smooth quasi-proximity spaces. Let $X=\prod_{i\in\Delta} X_i$ be a set and, for each $i\in\Delta,\pi_i : X\to X_i$ a function. The structure $\delta=\otimes\delta_i$ on X is defined by

$$\delta(\lambda,\mu)=\inf\{\sup_{i,k}\inf_{i\in\Delta}\delta_i(\pi_i(\lambda_j),\pi_i(\mu_k))\}.$$

Where for every finite families $(\lambda_j),(\mu_k)$ such that $\lambda=\sup_{j=1}^{n}\lambda_j,\mu=\sup_{k=1}^{m}\mu_k$

Then:

(1) δ is the coarsest smooth quasi-proximity on X with respect to which for each $i\in\Delta,\pi_i$ is a smooth quasi-proximity continuous.

(2) A function $f:(Y,\delta')\to(X,\delta)$ is a smooth quasi-proximity continuous iff each $\pi_i\circ f:(Y,\delta')\to(X_i,\delta_i)$ is a smooth quasi-proximity continuous.

On the other hand, since

$$\delta(\rho_1 \wedge \rho_2, \underline{1} - (\lambda_1 \wedge \lambda_2)) \leq \delta(\rho_1, \underline{1} - \lambda_1) \vee \delta(\rho_2, \underline{1} - \lambda_2)$$
$$< 1 - r.$$

we have $I_\delta(\lambda_1 \wedge \lambda_2, r) \geq (\rho_1 \wedge \rho_2) > t$. It is a contradiction.

(6) Since $I_\delta(\lambda, r) \leq \lambda$, $I_\delta(I_\delta(\lambda, r), r) \leq I_\delta(\lambda, r)$.

Suppose $I_\delta(I_\delta(\lambda, r), r) \not\geq I_\delta(\lambda, r)$. There exists $x \in X$ and $t \in (0,1)$ Such that

$$I_\delta(I_\delta(\lambda, r), r) < t < I_\delta(\lambda, r). \qquad \text{(C)}$$

Since $I_\delta(\lambda, r)(x) > t$, there exists $\rho \in I^X$ with $\delta(\rho, \underline{1} - \lambda) > 1 - r$ such that

$$I_\delta(\lambda, r)(x) \geq \rho(x) > t. \qquad \text{(D)}$$

Since $\inf_{\gamma \in I^X} \{\delta(\rho, \gamma) \vee \delta(\underline{1} - \gamma, \underline{1} - \lambda)\} \leq \delta(\rho, \underline{1} - \lambda) < 1 - r$ from (SQP4), there exists $\gamma \in I^X$ such that

$$\delta(\rho, \gamma) < 1 - r, \; \delta(\underline{1} - \gamma, \underline{1} - \lambda) < 1 - r.$$

Since $\delta(\underline{1} - \gamma, \underline{1} - \lambda) < 1 - r$, we have $\rho \leq I_\delta(\lambda, r) \geq \underline{1} - \gamma$. Thus,

$$\delta(\rho, \underline{1} - I_\delta(\lambda, r)) \leq \delta(\rho, \gamma) < 1 - r.$$

By the definition of $I_\delta(I_\delta(\lambda, r), r)$, $I_\delta(I_\delta(\lambda, r), r) \geq \rho$.

It is a contradiction for (C) and (D).

2.2.b.2 Theorem

Let δ be a smooth proximity on X. Define a function $\tau_\eta : I^X \to I$ by

$$\tau_\delta(\lambda) = \sup\{r \in I_1 / I_\delta(\lambda, r) = \lambda\}.$$

Then τ_δ is a smooth topology on X induced by δ.

Proof

The proof is similar to the proof of Theorem 1.1.9.

2.2.b.3 Theorem

Let (X,τ) be a smooth topological space. Define a function $\delta_\tau : I^X \times I^X \to I$ as follows:

$$\delta_\tau(\lambda,\mu) = \begin{cases} 1 - \sup\{\tau(\upsilon) \setminus \upsilon \in \Phi_{\lambda,\mu}\} & \text{if } \Phi_{\lambda,\mu} \neq \phi, \\ 1 & \text{if } \Phi_{\lambda,\mu} = \phi. \end{cases}$$

Where $\Phi_{\lambda,\mu} = \{\upsilon \in I^X \setminus \lambda \le \upsilon \le \underline{1} - \mu\}$.

Then we have the following properties:

(1) δ_τ is a principal smooth proximity on X,

(2) If δ is a principal smooth proximity on X, then $\delta \le \delta_{\tau_\delta}$,

(3) $\tau_{\delta_\tau} = \tau$.

Proof

(1) (SQP1) and (SQP2) are obvious.

(SQP3) From Remark 2.2.2 (1), we have

$$\delta_\tau(\lambda,\rho_1 \vee \rho_2) \ge \delta_\tau(\lambda,\rho_2) \vee \delta_\tau(\lambda,\rho_2).$$

Suppose there exists $r \in (0,1)$ such that

$$\delta_\tau(\lambda,\rho_1 \vee \rho_2) > r > \delta_\tau(\lambda,\rho_2) \vee \delta_\tau(\lambda,\rho_2).$$

Since $\delta_\tau(\lambda,\rho_i) < r$ for each $i \in \{1,2\}$, there exists $\upsilon_i \in I^X$ with $\lambda \le \upsilon_i \le \underline{1} - \rho_i$ such that $\tau(\upsilon_i) > 1 - r$. Since $\lambda \le \upsilon_1 \wedge \upsilon_2 \le (\underline{1} - \rho_1) \wedge (\underline{1} - \rho_2)$ and $\tau(\upsilon_1 \wedge \upsilon_2) > 1 - r$ we have

$$\delta_\tau(\lambda,\rho_1 \vee \rho_2) \le 1 - \tau(\upsilon_1 \wedge \upsilon_2) < r.$$

It is a contradiction.

(SQP4) Suppose there exists $r \in (0,1)$ such that

$$\delta_\tau(\lambda,\rho) < r < \inf_{\gamma \in I^X} \{\delta_\tau(\lambda,\gamma) \vee \delta_\tau(\underline{1} - \gamma,\rho)\}.$$

Since $\delta_\tau(\lambda,\rho)<r$ there exists $\upsilon\in I^X$ with $\lambda\le\upsilon\le\underline{1}-\rho$ such that $\tau(\upsilon)>1-r$. Since $\lambda\le\upsilon\le\underline{1}-\rho$ we have

$$\delta_\tau(\lambda,\underline{1}-\upsilon)\le 1-\tau(\upsilon)<r,\delta_\tau(\upsilon,\rho)<r.$$

So

$$\inf_{\gamma\in I^X}\{\delta_\tau(\lambda,\gamma)\vee\delta_\tau(\underline{1}-\gamma,\rho)\}<r.$$

It is a contradiction.

(SP5) Suppose there exists $r\in(0,1)$ such that

$$\delta(\sup_{j\in J}\lambda_j,\rho)>r>\sup_{j\in J}\delta(\lambda_j,\rho).$$

Since $\sup_{j\in J}\delta(\lambda_j,\rho)<r$ implies $\delta(\lambda_j,\rho)<r$ for each j there exists $\upsilon_j\in I^X$ with $\lambda_j\le\upsilon_j\le\underline{1}-\rho_j$ such that $\tau(\upsilon_j)>1-r$. Since

$$\sup_j\lambda_j\le\sup_j\upsilon_j\le\underline{1}-\rho.$$

We have

$$\delta_\tau(\sup_j\lambda_j,\rho)\le 1-\tau(\sup_j\upsilon_j)\le\sup_j(1-\tau(\upsilon_j))\le r.$$

It is a contradiction.

(2) Suppose $\delta\not\le\delta_{\tau_\delta}$. There exists $\lambda,\mu\in I^X$ and $r\in(0,1)$ such that

$$\delta(\lambda,\mu)>r>\delta_{\tau_\delta}(\lambda,\mu).$$

By the definition of δ_{τ_δ}, there exists $\rho\in I^X$ and $s\in(0,1)$ with $\lambda\le\rho\le\underline{1}-\mu$ such that $\tau_\delta(\rho)\ge 1-s>1-r$, $I(\rho,1-s)=\rho$,

Since δ is principal and

$$\rho=I(\rho,1-s)=\sup\{\lambda_i\setminus\delta(\lambda_i,\underline{1}-\rho)<s\},\delta(\rho,\rho)\le\sup_i\delta(\lambda_i,\underline{1}-\rho)<s<r$$

Since $\lambda\le\rho\le\underline{1}-\mu$, we have

$$\delta(\lambda,\mu)\le\delta(\rho,\underline{1}-\rho)<r.$$

It is a contradiction.

(3) Suppose there exists $\lambda \in I^X$ and $r \in (0,1)$ such that

$$\tau_{\delta_\tau}(\lambda) < r < \tau(\lambda).$$

Since $\tau(\lambda) > r$, we have $\delta_\tau(\lambda, \underline{1} - \lambda) = 1 - \tau(\lambda) < 1 - r$. So $I_{\delta_\tau}(\lambda, r) = \lambda$.

Thus $\tau_{\delta_\tau}(\lambda) \geq r$. It is a contradiction. Thus $\tau_{\delta_\tau}(\lambda) \geq \tau(\lambda)$.

Suppose there exists $\lambda \in I^X$ and $s \in (0,1)$ with $I_{\delta_\tau}(\lambda, s) = \lambda$ such that

$$\tau_{\delta_\tau}(\lambda) \geq s > \tau(\lambda).$$

Since $\lambda = \sup\{\rho_i \setminus \delta_\tau(\rho_i, \underline{1} - \lambda) < 1 - s\}$, by the definition of $\delta_\tau(\rho_i, \underline{1} - \lambda)$, for each i, there exists υ_i with $\rho_i \leq \upsilon_i \leq \lambda$ such that $\tau(\upsilon_i) > s$. Thus,

$$\lambda = \sup_i \rho_i \leq \sup_i \upsilon_i \leq \lambda \text{ implies } \lambda = \sup_i \upsilon_i.$$

So,

$$\tau(\lambda) = \tau(\sup_i \upsilon_i) \geq \inf_i \tau(\upsilon_i) \geq s.$$

It is a contradiction.

Thus $\tau_{\delta_\tau}(\lambda) \leq \tau(\lambda)$.

2.2.b.4 Example

Let $X = \{a,b,c\}$ be a set. Define a function $\delta : I^X \times I^X \to I$ as follows:

$$\delta(\lambda,\mu) = \begin{cases} 0 & \text{if } \lambda = \underline{0} \text{ or } \mu = \underline{0}, \\ \dfrac{2}{3} & \text{if } \chi_{\{a\}} \geq \lambda \neq \underline{0} \text{ or } \chi_{\{b\}} \geq \mu \neq \underline{0}, \\ 1 & \text{otherwise.} \end{cases}$$

where χ_A is a characteristic function for A. Then δ is a principal smooth proximity on X. From Theorem 2.2.1, we obtain $I_\delta : I^X \times I_1 \to I$ as follows:

$$I_\delta(\lambda,\mu)=\begin{cases}\underline{1} & \text{if } \lambda=\underline{1}, r\in I_1,\\ \chi_{\{a\}} & \text{if } \chi_{\{a,c\}}\le\lambda\ne\underline{1}, 0\le r<\frac{1}{3},\\ \underline{0} & \text{otherwise.}\end{cases}$$

From Theorem 2.2.2, we can obtain a smooth topology $\tau_\delta : I^X \to I$ as follows

$$\tau_\delta(\lambda)=\begin{cases}1 & \text{if } \lambda=\underline{0} \text{ or } \underline{1},\\ 0 & \text{otherwise.}\end{cases}$$

From Theorem 2.2.3 (1), we can obtain a principal smooth proximity $\delta_{\tau_\delta} : I^X \times I^X \to I$ as follows:

$$\delta_{\tau_\delta}(\lambda,\mu)=\begin{cases}0 & \text{if } \lambda=\underline{0} \text{ or } \mu=\underline{0},\\ 1 & \text{otherwise.}\end{cases}$$

Hence, $\delta \le \delta_{\tau_\delta}$ but $\delta \ne \delta_{\tau_\delta}$.

2.2.b.5 Theorem

Let (X,δ) be smooth proximity space. Define a function $\Im^1_\delta : I^X \to I$ by:

$$\Im^1_\delta(\lambda)=\inf_x\{(1-\delta(x,\lambda))\vee\lambda(x)\}.$$

Then $\Im^1_\delta$ is a smooth cotopology on X.

Proof

(CO1) Clear.

(CO2) $\Im^1_\delta(\lambda\vee\mu)=\inf_x\{(1-\delta(x,\lambda\vee\mu))\vee(\lambda\vee\mu)(x)\}$

$$=\inf_x\{(1-(\delta(x,\lambda)\vee\delta(x,\mu)))\vee(\lambda\vee\mu)(x)\}$$

$$=\inf_x\{(1-\delta(x,\lambda))\wedge(1-\delta(x,\mu)))\vee(\lambda\vee\mu)(x)\}$$

$$\geq \inf_x \{(1-\delta(x,\lambda)) \vee \lambda(x)\} \wedge \{(1-\delta(x,\mu)) \vee \mu(x)\}$$

$$= \inf_x \{(1-\delta(x,\lambda)) \vee \lambda(x)\} \wedge \inf_x \{(1-\delta(x,\mu)) \vee \mu(x)\}$$

$$= \Im^1_\delta(\lambda) \wedge \Im^1_\delta(\mu).$$

(CO3) Since I is an infinitely distributive lattice, we have

$$\Im^1_\delta(\inf_j \lambda_i) = \inf_x \{(1-\delta(x,\inf_j \lambda_i)) \vee (\inf_j \lambda_i(x))\}$$

$$\geq \inf_x \{(1-\delta(x,\lambda_i)) \vee (\inf_j \lambda_i(x))\}$$

$$= \inf_j \inf_x \{(1-\delta(x,\lambda_i)) \vee \lambda_i(x)\}$$

$$= \inf_j \Im^1_\delta(\lambda_i).$$

2.2.b.6 Theorem

Let (X,δ_1) and (Y,δ_2) be smooth proximity spaces. if $f: X \to Y$ is a smooth proximity continuous, then $f:(X,\tau_{\delta_1}) \to (Y,\tau_{\delta_2})$ is smooth continuous.

Proof

For each $\rho \in I^Y$, we have

$$\Im^1_{\delta_1}(f^{-1}(\rho)) = \inf_x \{(1-\delta_1(x,f^{-1}(\rho))) \vee f^{-1}(\rho)(x)\}$$

$$\geq \inf_x \{(1-\delta_1(x,f^{-1}(\rho))) \vee f^{-1}(\rho)(x)\}$$

$$\geq \inf_x \{(1-\delta_2(f(x),f(f^{-1}(\rho)))) \vee \rho(f(x))\}$$

$$\geq \inf_y \{(1-\delta_2(y,f(f^{-1}(\rho)))) \vee \rho(y)\}$$

$$= \Im^1_{\delta_2}(\rho).$$

2.2.b.7 Theorem. Let (X,δ) be smooth proximity space. Define a function $\tau_\delta^2 : I^X \to I$ by $\tau_\delta^2(\lambda) = 1 - \delta(\lambda, 1-\lambda)$.Then τ_δ^2 is a smooth topology on X. Conversely Let (X,τ) be smooth topological space. Defined a function $\delta_\tau : I^X \times I^X \to I$ by

$$\delta_\tau(\lambda,\mu) = 1 - \sup\{\tau(\upsilon) \setminus \lambda \le \upsilon \le 1-\mu\}.$$

Then δ_τ is a smooth proximity on X.

Proof. Only we prove the first part since the second part is trivial.

(O1) It is obvious.

(O2) For any $\lambda,\mu \in I^X$, we have,

$$\begin{aligned}\tau_\delta^2(\lambda \wedge \mu) &= 1 - \delta(\lambda \wedge \mu, 1-(\lambda \wedge \mu)) \\ &= 1 - \delta(\lambda \wedge \mu, (1-\lambda) \vee (1-\mu)) \\ &= 1 - \{\delta(\lambda \wedge \mu, 1-\lambda) \vee \delta(\lambda \wedge \mu, 1-\mu)\} \\ &= (1 - \delta(\lambda \wedge \mu, 1-\lambda)) \wedge (1 - \delta(\lambda \wedge \mu, 1-\mu)) \\ &\ge (1 - \delta(\lambda, 1-\lambda)) \wedge (1 - \delta(\mu, 1-\mu)) \\ &= \tau_\delta^2(\lambda) \wedge \tau_\delta^2(\mu).\end{aligned}$$

(O3) For each family $\{\lambda_j \setminus j \in J\} \subset I^X$, we obtain

$$\begin{aligned}\tau_\delta^2(\sup_j \lambda_j) &= 1 - \delta(\sup_j \lambda_j, 1 - \sup_j \lambda_j) \\ &= 1 - \delta(\sup_j \lambda_j, \inf_j (1-\lambda_j)) \\ &\ge 1 - \sup_j \delta(\lambda_j, \inf_j (1-\lambda_j)) \\ &= \inf_j (1 - \delta(\lambda_j, \inf_j (1-\lambda_j))) \\ &\ge \inf_j (1 - \delta(\lambda_j, 1-\lambda_j)) \\ &= \inf_j \tau_\delta^2(\lambda_j).\end{aligned}$$

Thus τ_δ^2 is a smooth topology on X.

2.2.b.8 Theorem. Let (X,δ_1) and (Y,δ_2) be smooth proximity spaces. A function $f: X \to Y$ is a smooth proximity continuous iff $f:(X,\tau^2_{\delta_1}) \to (Y,\tau^2_{\delta_2})$ is a smooth continuous.

Proof. For each $\rho \in I^Y$,

$$\begin{aligned}\tau^2_{\delta_1}(f^{-1}(\rho)) &= 1-\delta_1(f^{-1}(\rho),1-f^{-1}(\rho))\\ &= 1-\delta_1(f^{-1}(\rho),f^{-1}(1-\rho))\\ &\geq 1-\delta_2(f(f^{-1}(\rho)),f(f^{-1}(1-\rho)))\\ &\geq 1-\delta_2(\rho,1-\rho)\\ &= \tau^2_{\delta_2}(\rho)\end{aligned}$$

For each $\lambda,\mu \in I^Y$,

$$\begin{aligned}\delta_{\tau_2}(\lambda,\mu) &= 1-\sup\{\tau_2(\upsilon) \setminus \lambda \leq \upsilon \leq 1-\mu\}\\ &\geq 1-\sup\{\tau_1(f^{-1}(\upsilon) \setminus f^{-1}(\lambda) \leq f^{-1}(\upsilon) \leq 1-f^{-1}(\mu)\}\\ &= \delta_{\tau_1}(f^{-1}(\lambda),f^{-1}(\mu))\end{aligned}$$

2.2.b.9 Example

Let $X=\{a,b,c\}$ be a set. Define a smooth proximity $\delta: I^X \times I^X \to I$ as follows:

$$\delta(\lambda,\mu)=\begin{cases}0 & \text{if} \lambda=0 \text{ or } \mu=0,\\ \dfrac{2}{3} & \text{if } 0\neq\lambda\leq t_{\chi_{\{a\}}}, 0\neq\mu\leq 1-t_{\chi_{\{a\}}}, t\in(0,1),\\ 1 & \text{otherwise.}\end{cases}$$

where χ_A is a characteristic function for A. Then δ is not a principal smooth proximity on X because

$$1=\delta(\chi_{\{a\}},\chi_{\{b,c\}}) > \sup_{t\in(0,1)} \delta(t_{\{a\}},\chi_{\{b,c\}})=\frac{2}{3}.$$

We can obtain $\tau^2_\delta : I^X \to I$ as follows:

$$\tau_\delta^2(\lambda)=\begin{cases}1 & \text{if} \lambda=\underline{0} \text{ or } \underline{1},\\ \frac{1}{3} & \text{if } \lambda=t_{\chi_{\{a\}}}, t\in(0,1),\\ 0 & \text{otherwise.}\end{cases}$$

Then

$$0=\tau_\delta^2(\chi_{\{a\}})=\tau_\delta^2(\sup_{t\in(0,1)} t_{\chi_{\{a\}}})<\sup_{t\in(0,1)}\tau_\delta^2(t_{\chi_{\{a\}}})=\frac{1}{3}.$$

Thus, τ_δ^2 is not a smooth topology on X.

2.2.b.10 Theorem

Let (X,δ) be a principal fuzzifying proximity space.

Then

(1) $\Im_\delta^1(A)=\inf_{x\in A^c}(1-\delta(A,\{x\}))$,

(2) $\tau_\delta^1=\tau_\delta^2$.

Proof

(1) By Theorem 2.2.5, we have

$$\begin{aligned}\Im_\delta^1(A)&=\inf_{x\in X}\{(1-\delta(\{x\},A))\vee\chi_A(x)\}\\ &=\inf_{x\in A}\{(1-\delta(\{x\},A))\vee\chi_A(x)\}\wedge\inf_{x\in A^c}\{(1-\delta(\{x\},A))\vee\chi_A(x)\}\\ &=\inf_{x\in A^c}(1-\delta(\{x\},A)).\end{aligned}$$

(2) For each $A\in 2^X$, we have

$$
\begin{aligned}
\tau_\delta^1(A) = \Im_\delta^1(A^c) \\
&= \inf_{x\in A}(1-\delta(\{x\},A^c)) \\
&= 1-\sup_{x\in A}\delta(\{x\},A^c) \\
&= 1-\delta(\bigcup_{x\in A}\{x\},A^c) \\
&= 1-\delta(A,A^c) \\
&= \tau_\delta^2.
\end{aligned}
$$

2.2.b.11 Theorem

If a function $f:(X,\delta_1)\to(Y,\delta_2)$ is a smooth quasi-proximity continuous, then:

(a) $I_{\delta_1}(f^{-1}(\rho),r)\geq f^{-1}(I_{\delta_2}(\rho),r))$, for each $\rho\in I^Y$ and $r\in[0,1)$.

(b) $f:(X,\tau_{\delta_1})\to(Y,\tau_{\delta_2})$ is a smooth continuous,

(c) $f:(X,\tau_{\delta_1^{-1}})\to(Y,\tau_{\delta_2^{-1}})$ is a smooth continuous,

(d) $f:(X,\tau_{\delta_1^*})\to(Y,\tau_{\delta_2^*})$ is a smooth continuous.

Proof

(a) Since f is a smooth proximity continuous, we have

$$
\begin{aligned}
f^{-1}(I_{\delta_2}(\rho),r)) &= f^{-1}(\sup\{\lambda\in I^Y \setminus \delta_2(\lambda,\underline{1}-\rho)<1-r\}) \\
&\leq \sup\{f^{-1}(\lambda)\in I^X \setminus \delta_1(f^{-1}(\lambda),\underline{1}-f^{-1}(\rho))<1-r\}) \\
&\leq \sup\{\gamma\in I^X \setminus \delta_1(\gamma,\underline{1}-f^{-1}(\rho))<1-r\}) \\
&= I_{\delta_1}(f^{-1}(\rho),r).
\end{aligned}
$$

(b) Suppose that f is not a smooth continuous. Then there exists $\lambda\in I^Y$ such that $\tau_{\delta_2}(\lambda)>\tau_{\delta_2}(f^{-1}(\lambda))$. Hence, there exists $r\in I$ such that

$$\tau_{\delta_2}(\lambda) > r > \tau_{\delta_2}(f^{-1}(\lambda))$$

Since $\tau_{\delta_2}(\lambda) > r$ for some $c > r$ then

$$\lambda = i_{\delta_2}(\lambda, c) = \sup\{\rho \setminus \delta_2(\rho, \underline{1} - \lambda) < c\}$$

Since f is a smooth quasi-proximity continuous, by Lemma 1.1.5, then

$$\begin{aligned} f^{-1}(\lambda) &= \sup\{f^{-1}(\rho) \setminus \delta_2(\rho, \underline{1} - \lambda) < c\} \\ &\leq \sup\{f^{-1}(\rho) \setminus \delta_1(f^{-1}(\rho), \underline{1} - f^{-1}(\lambda)) < c\} \\ &\leq i_{\delta_1}(f^{-1}(\lambda), c). \end{aligned}$$

So, by Theorem 2.2.1, we have $i_{\delta_1}(f^{-1}(\lambda), c) = f^{-1}(\lambda)$

It follows that $\tau_{\delta_1}(f^{-1}(\lambda) \geq c > r$. It is a contradiction.

(c) and (d) are easy from Lemma 2.1.12 and (a).

2.2.c Smooth quasi-proximity induced by smooth quasi-uniformity

In this part we show that every smooth quasi-uniform space induce smooth quasi-proximity space.

2.2.c.1 Theorem

Let (X, U) be a smooth quasi-uniform space. Define, for all $\lambda, \rho \in I^X$,

$$\delta_U(\lambda, \rho) = \begin{cases} 1 - \sup\{U(u) \setminus u \in \Theta_{\lambda,\rho}\} & \text{if } \Theta_{\lambda,\rho} \neq \phi, \\ 1 & \text{if } \Theta_{\lambda,\rho} = \phi. \end{cases}$$

Where $\Theta_{\lambda,\rho} = \{u \in I^{X \times X} \setminus u[\lambda] \leq \underline{1} - \rho\}$. Then:

(1) (X, δ_U) is a smooth quasi-proximity space.

(2) $\tau_U = \tau_{\delta_U}$.

Proof

(1) We will show that δ_U is a smooth proximity on X.

(SQU1) Since $u[\underline{0}] = \underline{0}$ and $u[\underline{1}] = \underline{1}$ for $U(u) = 1$, we have $\delta_U(\underline{0}, \underline{1}) = 0$ and $\delta_U(\underline{1}, \underline{0}) = 0$.

It follows that $U(u_1 \wedge u_2) > 1 - r, (u_1 \wedge u_2)[\lambda_1 \vee \lambda_2] \leq \underline{1} - (\rho)$

Hence,, we have $\delta_U(\lambda_1 \vee \lambda_2, \rho) < r$. It is a contradiction. Therefore

$$\delta_U(\lambda_1, \rho) \vee \delta_U(\lambda_2, \rho) \geq \delta_U(\lambda_1 \vee \lambda_2, \rho)$$

(SQP4) Suppose there exist $\lambda, \rho \in I^X$ and $r \in (0,1)$ such that

$$\delta_U(\lambda, \rho) < r < \inf_{\gamma \in I^X} \{\delta_U(\lambda, \gamma) \vee \delta_U(\underline{1} - \gamma, \rho)\}$$

Since $\delta_U(\lambda, \rho) < r$, there exists $u \in I^{X \times X}$ with $U(u) > 1 - r, u[\lambda] \leq \underline{1} - \rho$. From (SQU4), there exists $v \in I^{X \times X}$ such that $v \circ v \leq u$ and $U(v) > 1 - r$.

Since $v[\lambda] \leq v[\lambda]$ and $v[v[\lambda]] \leq \underline{1} - \rho$, there exists $\underline{1} - v[\lambda] \in I^X$ such that

$$\delta_U(\lambda, \underline{1} - \mathrm{v}[\lambda]) < r \text{ and } \delta_U(\mathrm{v}[\lambda], \rho) < r.$$

So,

$$\inf_{\gamma \in I^X} \{\delta_U(\lambda, \gamma) \vee \delta_U(\underline{1} - \gamma, \rho)\} < r.$$

It is a contradiction.

Thus

$$\delta_U(\lambda, \rho) \geq \inf_{\gamma \in I^X} \{\delta_U(\lambda, \gamma) \vee \delta_U(\underline{1} - \gamma, \rho)\}$$

(2) We only show that $I_U = I_{\delta_U}$.

Let $\rho \in I^X$ such that $u[\rho] \leq \lambda$ and $U(u) > r$. Then

$$\delta_U(\rho, \underline{1} - \lambda) \leq 1 - U(u) < 1 - r$$

Hence,

$$I_U(\lambda, r) \leq I_{\delta_U}(\lambda, r)$$

Let $\rho \in I^X$ such that $\delta_U(\rho, \underline{1} - \lambda) < 1 - r$. By the definition of $\delta_U(\rho, \underline{1} - \lambda)$, there exists $u \in I^{X \times X}$ such that $U(u) > r$ and $u(\rho) \leq \lambda$. Hence, $I_U(\lambda, r) \geq I_{\delta_U}(\lambda, r)$.

2.2.c.2 Theorem

Let (X,U) and (Y,V) be smooth uniformity spaces. if" $f:X\to Y$ is a smooth uniformity continuous, then $f:(X,\delta_U)\to(Y,\delta_V)$ is smooth proximity continuous.

Proof

Suppose there exist $\lambda,\rho\in I^X$ and $r\in(0,1)$ such that

$\delta_U(\lambda,\rho)>r>\delta_V(f(\lambda),f(\rho))$.

Since $\delta_V(f(\lambda),f(\rho))<r$, there exists $v\in I^{Y\times Y}$ such that

$$V(v)>1-r, v[f(\lambda)]\leq \underline{1}-f(\rho)$$

Since, $v[f(\lambda)]\leq\underline{1}-f(\rho)$ by Lemma 1.1.5, implies

$$(f\times f)^{-1}(v)[\lambda]=f^{-1}(v)[f(\lambda)]\leq f^{-1}(\underline{1}-f(\rho))\leq\underline{1}-\rho$$

Since f is smooth uniform continuous, $(f\times f)^{-1}(v)\geq V(v)>1-r$

Thus $\delta_U(\lambda,\rho)<r$. It is a contradiction.

2.3.3 Theorem

Let (X,U) be smooth uniform space. Then $\tau_U^1=\tau_{\delta_U}^1$.

Proof

Since $\tau_{\delta_U}^1(\lambda)=\inf_x\{(1-\delta_U(x,\underline{1}-\lambda))\vee(\underline{1}-\lambda(x))\}$

$$=\inf_x\{(\underline{1}-\lambda(x))\vee\sup_{u[x]\leq\lambda}U(u)\}=\tau_U^1$$

CHAPTER III

Chapter III

Smooth syntopogenous structures

In this chapter we introduce the concept of smooth syntopogenous structures. The relation between smooth (semi-) topogenous order and smooth (supra) topology is studied.

3.1 General definitions and basic properties

In the sequel we define the concept of smooth semi-topogenous order which the condition (ST2) is defined in a somewhat different view of Šostak (See Definition 0.4.b.1.) and we study the product of smooth topogenous spaces.

3.1.1 Definition

A function $\eta : I^X \times I^X \to I$ is called a smooth semi-topogeneous order on X, if it satisfies the following axioms:

$(ST1)$ $\eta(\underline{1},\underline{1}) = \eta(\underline{0},\underline{0}) = 1$,

$(ST2)$ if $\eta(\mu,\lambda) \neq 0$, then $\lambda \leq \mu$,

$(ST3)$ if $\lambda \leq \lambda_1, \mu_1 \leq \mu$, then $\eta(\lambda_1,\mu_1) \leq \eta(\lambda,\mu)$.

3.1.2 Example

We define $\eta_1, \eta_2 : I^X \times I^X \to I$ as follows

$$\eta_1(\lambda,\mu) = \begin{cases} 1, & \text{if } \lambda = \underline{0} \text{ or } \mu = \underline{1}, \\ \frac{2}{3}, & \text{if } \underline{0} \neq \lambda \leq \mu \neq \underline{1}, \\ 0, & \text{otherwise.} \end{cases}$$

$$\eta_2(\lambda,\mu) = \inf_{x \in X} \{(\underline{1} - \lambda)(x) \vee \mu(x)\}.$$

Then η_1 is a smooth semi-topogenous order but not a Šostak smooth semi-topogenous order because ,

$$\frac{2}{3} = \eta_1(\underline{0.4}, \underline{0.5}) \not\leq (\underline{1} - \underline{0.4})(x) \vee \underline{0.5}(x) = 0.6.$$

Moreover, η_2 is a Šostak smooth semi-topogenous order but not a smooth semi-topogenous order because,

$$0.5 = \eta_1(\underline{0.5}, \underline{0.4}), \underline{0.5} \not\leq \underline{0.4}.$$

The following proposition is easily proved from the above definition.

3.1.3 Proposition

Let η be smooth semi-topogeneous order on X and let the functionping $\eta^s : I^X \times I^X \to I$ defined by

$$\eta^s(\lambda, \mu) = \eta(\underline{1} - \mu, \underline{1} - \lambda), \forall \lambda, \mu \in I^X$$

Then η^s is a smooth semi-topogeneous order on X.

3.1.4 Definition

A smooth semi-topogenous order η is called smooth topogenous if for any $\lambda_1, \lambda_2, \lambda, \mu_1, \mu_2, \mu \in I^X$.

(ST5) $\eta(\lambda_1 \vee \lambda_2, \mu) = \eta(\lambda_1, \mu) \wedge \eta(\lambda_1, \mu)$,

(ST6) $\eta(\lambda, \mu_1 \wedge \mu_2) = \eta(\lambda, \mu_1) \wedge \eta(\lambda_1, \mu_2)$.

3.1.5 Definition

Let (X, η_1) and (Y, η_2) be smooth topogenous spaces. A function $f : X \to Y$ is said to be smooth topogenous continuous if

$$\eta_2(\lambda, \mu) \leq \eta_1(f^{-1}(\lambda), f^{-1}(\mu)), \forall \lambda, \mu \in I^Y.$$

3.1.6 Theorem

Let (X, η_1), (Y, η_2) and (Z, η_3) be smooth topogenous spaces. If $f : X \to Y$ and $g : Y \to Z$ are smooth topogenous continuous, then $g \circ f : X \to Z$ is a smooth topogenous continuous.

3.1.7 Definition

Let Υ_X be a fuzzy biperfect syntopogenous structure on X (Definition 0.4.b.9). A function $S:\Upsilon_X \to I$ is called a *smooth syntopogenous structure* on X satisfying for $\eta,\eta_1,\eta_2 \in \Upsilon_X$, the following condition

(T1) there exist $\eta \in \Upsilon_X$ such that $S(\eta)=1$

(T2) $S(\eta_1)\wedge S(\eta_2) \le \sup\limits_{\eta_1,\eta_2 \le \eta} S(\eta)$

(T3) $S(\eta) \le \sup\limits_{\eta_1 \circ \eta_1 \ge \eta} \varsigma(\eta_1)$.

The pair (X,S) is said to be a *smooth syntopogenous space.*

A smooth syntopogenous space (X,S) is said to be a smooth symmetric syntopogenous space if it satisfies

$(ST) S(\eta) \le \sup\limits_{\zeta \ge \eta^S} S(\zeta)$.

3.1.8 Theorem

Let $(X_i,\eta_i)_{i\in\Gamma}$ be a family of smooth topogenous spaces. Let X be a set and, for each $i\in\Gamma, f_i : X \to X_i$ a function. Define the function $\eta : I^X \times I^X \to I$ on X by

$$\eta(\lambda,\mu)=\sup\{\inf_{j,k} \eta_i(f_i(\lambda_j),\underline{1}-f_i(\underline{1}-\mu_k))\}$$

where for every finite families $\{\lambda_j \setminus \lambda = \sup\limits_{j=1}^{n} \lambda_j\}$ and $\{\mu_k \setminus \mu = \sup\limits_{k=1}^{m} \mu_k\}$.

Then:

(1) η is the coarsest smooth topogenuous structure on X with respect to which for each $i\in\Gamma, f_i$ is smooth topogenous continuous.

(2) A function $f:(Y,\eta') \to (X,\eta)$ is smooth topogenous continuous

iff each $f_i \circ f : (Y, \eta') \to (X_i, \eta_i)$ is smooth topogenous continuous.

(3) If $(X_i, \eta_i)_{i \in \Gamma}$ is symmetric for each $i \in \Gamma$, then (X, η) is symmetric.

Proof

(1) First, we will show that η is smooth topogenuous structure on X .

(ST3) Is easily proved.

(ST1) From (ST3), it is easily proved from:

$$\eta(\underline{0},\underline{0}) = \eta_i(f_i(\underline{0}), \underline{1} - f_i(\underline{1})) = \eta_i(\underline{0},\underline{0}) = 1,$$

$$\eta(\underline{1},\underline{1}) = \eta_i(f_i(\underline{1}), \underline{1} - f_i(\underline{0})) = \eta_i(\underline{1},\underline{1}) = 1.$$

(ST2) If $\eta(\lambda, \mu) > 0$, there are finite families (λ_j) and (μ_k) such that $\lambda = \sup \lambda_j$ and $\mu = \inf \mu_k$ with

$$\eta(\lambda, \mu) \geq \{\inf_{j,k} \eta_i(f_i(\lambda_j), \underline{1} - f_i(\underline{1} - \mu_k))\} > 0.$$

It follows that for any j,k there exists an $i_{jk} \in \Gamma$ such that

$$\eta_{i_{jk}}(f_{i_{jk}}(\lambda_j), \underline{1} - f_{i_{jk}}(\underline{1} - \mu_k))\}.$$

It implies $\lambda_j \leq \lambda_k$

(ST4) For any $\lambda, \mu, \upsilon \in I^X$, we will show that

$$\eta(\lambda, \mu \wedge \upsilon) \geq \eta(\lambda, \mu) \wedge \eta(\lambda, \upsilon).$$

Suppose that there exist $\lambda, \mu, \upsilon \in I^X$ and $t \in (0,1)$ such that

$$\eta(\lambda, \mu \wedge \upsilon) < t < \eta(\lambda, \mu) \wedge \eta(\lambda, \upsilon).$$

Since $\eta(\lambda, \mu) > t$ and $\eta(\lambda, \upsilon) > t$, there are finite families $\{\lambda_j \setminus \lambda = \sup \lambda_j\}$, $\{\lambda'_m \setminus \lambda = \sup \lambda'_m\}$, $\{\mu_k \setminus \mu = \inf \mu_k\}$ and $\{\upsilon_l \setminus \upsilon = \inf \upsilon_l\}$

such that

$$\inf_{j,k}(\sup_{i\in\Gamma}\eta_i(f_i(\lambda_j),\underline{1}-f_i(\underline{1}-\mu_k)))>t,$$

$$\inf_{m,l}(\sup_{i\in\Gamma}\eta_i(f_i(\lambda'_m),\underline{1}-f_i(\underline{1}-\upsilon_l)))>t.$$

It follows that $\lambda=\sup_{j,m}(\lambda_j\wedge\lambda'_m)$ and $\mu\wedge\upsilon=(\inf\mu_k)\wedge(\inf\upsilon_l)$. Since

$$\sup_{i\in\Gamma}\eta_i(f_i(\lambda_j),\underline{1}-f_i(\underline{1}-\mu_k))\le\sup_{i\in\Gamma}\eta_i(f_i(\lambda_j\wedge\lambda'_m),\underline{1}-f_i(\underline{1}-\mu_k)),$$

$$\sup_{i\in\Gamma}\eta_i(f_i(\lambda'_m),\underline{1}-f_i(\underline{1}-\upsilon_l))\le\sup_{i\in\Gamma}\eta_i(f_i(\lambda_j\wedge\lambda'_m),\underline{1}-f_i(\underline{1}-\upsilon_l)).$$

We have

$$\begin{aligned}\eta(\lambda,\mu\vee\upsilon)&>(\inf_{j,k}\sup_{i\in\Gamma}\eta_i(f_i(\lambda_j),\underline{1}-f_i(\underline{1}-\mu_k)))\\&\wedge(\inf_{m,l}\sup_{i\in\Gamma}\eta_i(f_i(\lambda'_m),\underline{1}-f_i(\underline{1}-\upsilon_l)))\\&\ge(\inf_{j,k}\sup_{i\in\Gamma}\eta_i(f_i(\lambda_j\wedge\lambda'_m),\underline{1}-f_i(\underline{1}-\mu_k)))\\&\wedge(\inf_{m,l}\sup_{i\in\Gamma}\eta_i(f_i(\lambda_j\wedge\lambda'_m),\underline{1}-f_i(\underline{1}-\upsilon_l)))\\&>t.\end{aligned}$$

It is a contradiction.

Similarly, we have $\eta(\lambda\vee\rho,\mu)\le\eta(\lambda,\mu)\wedge\eta(\rho,\mu)$.

(S2) We will show that $\eta\le\eta\circ\eta$.

Suppose there exist $\lambda,\mu\in I^X$ and $r\in(0,1)$ such that

$$\eta(\lambda,\mu)>r>\eta\circ\eta(\lambda,\mu).$$

Since $\eta(\lambda,\mu)>r$, then there are finite families

$\{\lambda_j\setminus\lambda=\sup_{j=1}^{p}\lambda_j\}$ and $\{\mu_k\setminus\mu=\sup_{k=1}^{q}\mu_k\}$ with

$$\eta(\lambda,\mu)\ge\{\inf_{j,k}\sup_{i\in\Gamma}\eta_i(f_i(\lambda_j),\underline{1}-f_i(\underline{1}-\mu_k))>r.$$

It follows that for any j,k there exists an $i_{jk}\in\Gamma$ such that

$$\eta_{i_{jk}}(f_{i_{jk}}(\lambda_j),\underline{1}-f_{i_{jk}}(\underline{1}-\mu_k))>r.$$

Since $\eta_{i_{jk}}$ is a smooth topogenous structure on $X_{i_{jk}}$ and $\eta_{i_{jk}} \le \eta_{i_{jk}} \circ \eta_{i_{jk}}$,

there exists $\rho_{jk} \in I^{X_{i_{jk}}}$ such that

$$\begin{aligned}&\eta_{i_{jk}} \circ \eta_{i_{jk}} (f_{i_{jk}}(\lambda_j), \underline{1} - f_{i_{jk}}(\underline{1} - \mu_k)) \\ &\ge \eta_{i_{jk}} (f_{i_{jk}}(\lambda_j), \rho_{jk}) \wedge \eta_{i_{jk}} (\rho_{jk}, \underline{1} - f_{i_{jk}}(\underline{1} - \mu_k)) \\ &> r.\end{aligned}$$

Since $\underline{1} - f_{i_{jk}}(\underline{1} - f_{i_{jk}}^{-1}(\rho_{jk})) \ge \rho_{jk}$ and $f_{i_{jk}}(f_{i_{jk}}^{-1}(\rho_{jk})) \le \rho_{jk}$, we have

$$\eta_{i_{jk}} (f_{i_{jk}}(\lambda_j), \underline{1} - f_{i_{jk}}(\underline{1} - f_{i_{jk}}^{-1}(\rho_{jk})) \ge \eta_{i_{jk}} (f_{i_{jk}}(\lambda_j), \rho_{jk}) > r,$$

$$\eta_{i_{jk}} (f_{i_{jk}}(f_{i_{jk}}^{-1}(\rho_{jk})), \underline{1} - f_{i_{jk}}(\underline{1} - \mu_k)) \ge \eta_{i_{jk}} (\rho_{jk}, \underline{1} - f_{i_{jk}}(\underline{1} - \mu_k)).$$

Put

$$\rho_j = \inf_{k=1}^{q} f_{i_{jk}}^{-1}(\rho_{jk}), \quad \rho = \sup_{j=1}^{p} \rho_j.$$

Then, by the definition of η, we have $\eta(\lambda, \rho_j) > r$. Using (ST3), we have

$$\eta(\lambda, \rho) > r.$$

In a similar way, since

$\eta_{i_{jk}} (f_{i_{jk}}(\rho_j)), \underline{1} - f_{i_{jk}}(\underline{1} - \mu_k)) \ge \eta_{i_{jk}} (\rho_{jk}, \underline{1} - f_{i_{jk}}(\underline{1} - \mu_k))r$ we have $\eta(\rho, \mu) > r$. Thus

$$\eta \circ \eta(\lambda, \mu) \ge \eta(\lambda, \rho) \wedge \eta(\rho, \mu) > r.$$

It is a contradiction.

Second, from the definition of η, for two families $\{f_i^{-1}(\lambda)\}$ and, $\{f_i^{-1}(\mu)\}$, we have

$$\begin{aligned}\eta(f_i^{-1}(\lambda), f_i^{-1}(\mu)) &\ge \eta_i (f_i(f_i^{-1}(\lambda)), \underline{1} - f_i(\underline{1} - f_i^{-1}(\mu))). \\ &\ge \eta_i(\lambda, \mu).\end{aligned}$$

Thus, for each $i \in \Gamma, f_i : (X, \eta) \to (X_i, \eta_i)$ is a smooth topogenous continuous.

Let $f_i : (X,\eta') \to (X_i,\eta_i)$ be smooth topogenous continuous. For every $i \in \Gamma$, we have

$$\eta(\lambda,\mu) = \sup\{\inf_{j,k} \sup_{i\in\Gamma} \eta_i(f_i(\lambda_j), \underline{1} - f_i(\underline{1} - \mu_k))\}$$

$$\leq \sup\{\inf_{j,k} \sup_{i\in\Gamma} \eta'(f_i^{-1}(f_i(\lambda_j)), f_i^{-1}(\underline{1} - f_i(\underline{1} - \mu_k)))\}$$

$$= \eta'(\lambda,\mu),$$

we have

$$\eta(\lambda,\mu) \leq \eta'(\lambda,\mu),\ \forall \lambda,\mu \in I^X.$$

(2) Necessity of the composition condition is clear since the composition of smooth topogenous continuous functions is smooth topogenous continuous.

Conversely, suppose f is not a smooth topogenous continuous function. Then there exists $\lambda,\mu \in I^X$ and $r \in (0,1)$ such that

$$\eta'(f^{-1}(\lambda), f^{-1}(\mu)) < r < \eta(\lambda,\mu).$$

Since $\eta(\lambda,\mu) > r$, therefore there are finite families $(\lambda_j),(\mu_k)$ such that

$\{\lambda_j \setminus \lambda = \sup_{j=1}^{p} \lambda_j\}$ and $\{\mu_k \setminus \mu = \sup_{k=1}^{q} \mu_k\}$ and

$$\eta(\lambda,\mu) \geq \inf_{j,k} \eta_i(f_i(\lambda_j), \underline{1} - f_i(\underline{1} - \mu_k)) > r.$$

It follows that for any j,k there exists an $i_{jk} \in \Gamma$ such that

$$\eta_{i_{jk}}(f_{i_{jk}}(\lambda_j), \underline{1} - f_{i_{jk}}(\underline{1} - \mu_k)) > r.$$

On the other hand, since $f_i \circ f$ is smooth topogenous continuous and $f_i(f(f^{-1}(\lambda_j))) \leq f_i(\lambda_j)$,

$$r < \inf_{j,k} \eta_{i_{jk}}(f_{i_{jk}}(\lambda_j), \underline{1} - f_{i_{jk}}(\underline{1} - \mu_k))$$

$$\leq \inf_{j,k} \eta'((f_{i_{jk}} \circ f)^{-1} f_{i_{jk}}(\lambda_j), (f_{i_{jk}} \circ f)^{-1}(\underline{1} - f_{i_{jk}}(\underline{1} - \mu_k)))$$

$$\leq \inf_{j,k} \eta'(f^{-1}(\lambda_j), f^{-1}(\mu_k))$$

$$= \eta'(f^{-1}(\lambda), f^{-1}(\mu)).$$

It is a contradiction.

(3) For every finite families $\{\lambda_j \setminus \lambda = \sup_{j=1}^{p} \lambda_j\}$ and $\{\mu_k \setminus \mu = \sup_{k=1}^{q} \mu_k\}$,

$$\eta(\lambda, \mu) = \sup\{\inf_{j,k} \sup_{i \in \Gamma} \eta_i(f_i(\lambda_j), \underline{1} - f_i(\underline{1} - \mu_k))\}$$

$$= \sup\{\inf_{j,k} \sup_{i \in \Gamma} \eta_i^s(f_i(\lambda_j), \underline{1} - f_i(\underline{1} - \mu_k))\}$$

$$= \sup\{\inf_{j,k} \sup_{i \in \Gamma} \eta_i(f_i(\underline{1} - \mu_k), \underline{1} - f_i(\lambda_j))\}$$

$$= \eta(\underline{1} - \mu, \underline{1} - \lambda)$$

$$= \eta^s(\lambda, \mu).$$

3.1.9 Definition

Let (X, η) be a smooth topogenous spaces and A be a subset of X. The pair (A, η_A) is said to be a *subspace* of (X, η) if it is endowed with the initial smooth topogenous spaces with respect to the inclusion function.

3.1.10 Definition

Let X be the product $\prod_{i \in \Delta} X_i$ of the family $\{(X_i, \eta_i) \setminus i \in \Delta\}$ of smooth topogenous spaces. An initial smooth topogenous spaces $\eta = \otimes \eta_i$ on X

with respect to all the projections $\pi_i : X \to X_i$ is called the *product smooth topogenous spaces* of $\{\eta_i \setminus i \in \Delta\}$ and $(X, \otimes\eta_i)$ is called the *product smooth topogenous spaces*.

3.1.11 Corollary

Let $(X_i, \eta_i)_{i\in\Delta}$ be a family of smooth topogenous spaces. Let $X = \prod_{i\in\Delta} X_i$ be a set and, for each $i \in \Delta, \pi_i : X \to X_i$ a functionping. The structure $\eta = \otimes\eta_i$ on X is defined by

$$\eta(\lambda, \mu) = \inf\{\sup_{i,k} \inf_{i\in\Delta} \eta_i(\pi_i(\lambda_j), \pi_i(\mu_k))\}.$$

where for every finite families $(\lambda_j), (\mu_k)$ such that $\lambda = \sup_{j=1}^{n} \lambda_j, \mu = \sup_{k=1}^{m} \mu_k$

Then:

(1) η is the coarsest smooth topogenous on X with respect to which for each $i \in \Delta, \pi_i$ is a smooth topogenous continuous.

(2) A function $f : (Y, \eta') \to (X, \eta)$ is a smooth topogenous continuous iff each $\pi_i \circ f : (Y, \eta') \to (X_i, \eta_i)$ is a smooth topogenous continuous.

3.2 Smooth (semi-) topogenous order and a smooth (supra) topology

In the sequel we present some relations between Smooth (semi-) topogenous order and a smooth (supra) topology.

3.2.1 Theorem

Let η be a smooth semi-topogenous order on X. Define a functionping $I_\eta : I^X \times I_1 \to I$, by

$$\mathrm{I}_{\eta}(\lambda,r)=\sup\{\mu\in I^{X}\setminus\eta(\mu,\lambda)>r\}.$$

Then we have the following properties:

(1) I_{η} is a smooth supra interior operator.

(2) If η satisfies ST6), I_{η} is smooth interior operator

(3) If η satisfies ST5), $\mathrm{I}_{\eta^{s}}$ is smooth interior operator

(4) If $\eta\leq\eta\circ\eta$, for each $\lambda\in I^{X}$ and $r\in I_{\circ}$,

$$\mathrm{I}_{\eta}(\mathrm{I}_{\eta}(\lambda,r),r)=\mathrm{I}_{\eta}(\lambda,r).$$

(5) If η is a smooth topogenous structure, I_{η} is topological smooth interior operator.

Proof

(1) (I1) Since $\eta(\underline{1},\underline{1})=1$, $\mathrm{I}_{\eta}(\underline{1},r)=\underline{1}$.

I2) Since $\eta(\mu,\lambda)\neq 0,\mu\leq\lambda$ implies $\mathrm{I}_{\eta}(\lambda,r)\leq\lambda$.

(I3) and (I4) are easily proved.

(2) From (I3), we have

$$\mathrm{I}_{\eta}(\lambda_1\wedge\lambda_2,r)\leq\mathrm{I}_{\eta}(\lambda_1,r)\wedge\mathrm{I}_{\eta}(\lambda_2,r).$$

Conversely, suppose there exist $\lambda_1,\lambda_2\in I^{X}$ and $r\in I_1$ such that

$$\mathrm{I}_{\eta}(\lambda_1\wedge\lambda_2,r)\ngeq\mathrm{I}_{\eta}(\lambda_1,r)\wedge\mathrm{I}(\lambda_2,r).$$

There exists $x\in X$ and $t\in I_1$ such that

$$\mathrm{I}_{\eta}(\lambda_1\wedge\lambda_2,r)<t<\mathrm{I}_{\eta}(\lambda_1,r)\wedge\mathrm{I}(\lambda_2,r).$$

Since $\mathrm{I}_{\eta}(\lambda_i,r)(x)>t$, for each $i\in\{1,2\}$, there exists $\mu_i\in I^{X}$ with $\eta(\mu_i,\lambda_i)>r$ such that

$$\mathrm{I}_{\eta}(\lambda_i,r)(x)\geq\mu_i(x)>t.$$

On the other hand, since

$$I_\eta(I_\eta(\lambda,r),r) < t < I_\eta(\lambda,r).$$

Since $I_\eta(\lambda,r)(x) > t$, there exists $\mu \in I^X$ with $\eta(\mu,\lambda) > r$ such that

$$I_\eta(\lambda,r)(x) \geq \mu(x) > t.$$

Since $\eta \leq \eta \circ \eta$, we have

$$r < \eta(\mu,\lambda) \leq \eta \circ \eta(\mu,\lambda).$$

Since $\eta \circ \eta(\mu,\lambda) > r$, there exists $\rho \in I^X$ such that

$$\eta \circ \eta(\mu,\lambda) \geq \eta(\mu,\rho) \wedge \eta(\rho,\lambda) > r.$$

Hence,

$$\mu \leq I_\eta(\rho,r),\ \rho \leq I_\eta(\lambda,r).$$

Thus

$$I_\eta(I_\eta(\lambda,r),r)(x) \geq \mu(x) > t.$$

It is a contradiction.

(5) It is trivial from (2) and (4).

3.2.2 Theorem

Let η be a smooth semi-topogenous order. Define a function $\tau_\eta : I^X \to I$ by

$$\tau_\eta(\lambda) = \sup\{r \in I_1 / I_\eta(\lambda,r) = \lambda\}.$$

Then

(1) τ_η is a smooth supra topology on X induced by η.

(2) If η satisfies (ST6), then τ_η is a smooth topology on X.

(3) If η is a perfect, then $\tau_\eta(\lambda) = \eta(\lambda,\lambda)$ for each $\lambda \in I^X$.

Proof

(1) (O1) Since $I_\eta(\underline{0},r) = \underline{0}$ and $I_\eta(\underline{1},r) = \underline{1}$, for all $r \in I_1$, then

$$\tau_\eta(\underline{0}) = \tau_\eta(\underline{1}) = 1.$$

(O2) Suppose there exists a family $\{\lambda_j \in I^X \setminus j \in \Gamma\}$ and $t \in (0,1)$ such that

$$\tau_\eta(\sup_{j\in\Gamma} \lambda_j) < t < \inf_{j\in\Gamma} \tau_\eta(\lambda_j).$$

Since $\inf_{j\in\Gamma} \tau_\eta(\lambda_j) > t$, for each $j \in \Gamma$, there exists $r_j > t$ such that

$$\lambda_j = \mathrm{I}_\eta(\lambda_j, r_j).$$

Put $r = \inf_{j\in\Gamma} r_j$. By Theorem 3.2.1, we have

$$\begin{aligned} \mathrm{I}_\eta(\sup_{j\in\Gamma} \lambda_j, r) &\geq \sup_{j\in\Gamma} \mathrm{I}_\eta(\lambda_j, r) \\ &\geq \sup_{j\in\Gamma} \mathrm{I}_\eta(\lambda_j, r) \\ &= \sup_{j\in\Gamma} \lambda_j. \end{aligned}$$

So,

$$\mathrm{I}_\eta(\sup_{j\in\Gamma} \lambda_j, r) = \sup_{j\in\Gamma} \lambda_j. \text{Consequently, } \tau_\eta(\sup_{j\in\Gamma} \lambda_j) \geq r > t.$$

It is a contradiction. Hence,

$$\tau_\eta(\sup_{j\in\Gamma} \lambda_j) \geq \inf_{j\in\Gamma} \tau_\eta(\lambda_j)$$

Thus, τ_η is a smooth supra topology on X.

(2) Suppose there exist $\lambda_1, \lambda_2 \in I^X$ and $t \in (0,1)$ such that

$$\tau_\eta(\lambda_1 \wedge \lambda_2) < t < \tau_\eta(\lambda_1) \wedge \tau_\eta(\lambda_2).$$

Since $\tau_\eta(\lambda_i) > t$, for each $i \in \{1,2\}$, there exists $r_i > t$ such that

$$\lambda_i = \mathrm{I}_\eta(\lambda_i, r_i)(x).$$

Put $r = r_1 \wedge r_2$. By Theorem 3.2.1, we have

$$\begin{aligned} \mathrm{I}_\eta(\lambda_1 \wedge \lambda_2, r) &= \mathrm{I}_\eta(\lambda_1, r) \wedge \mathrm{I}_\eta(\lambda_2, r) \\ &\geq \mathrm{I}_\eta(\lambda_1, r_1) \wedge \mathrm{I}_\eta(\lambda_2, r_2) \\ &= \lambda_1 \wedge \lambda_2. \end{aligned}$$

Consequently, $\tau_\eta(\lambda_1 \wedge \lambda_2) \geq r > t$. It is a contradiction. Hence,

$$\tau_\eta(\lambda_1 \wedge \lambda_2) \geq \tau_\eta(\lambda_1) \wedge \tau_\eta(\lambda_2).$$

Thus, τ_η is a smooth topology on X.

(3) Suppose there exists $\lambda \in I^X$ and $r \in (0,1)$ such that

$$\tau_\eta(\lambda) < r < \eta(\lambda,\lambda).$$

Since $\eta(\lambda,\lambda) > r, \lambda = \mathrm{I}_\eta(\lambda,r)$. Thus $\tau_\eta(\lambda) \geq r$. It is a contradiction.

Suppose there exists $\mu \in I^X$ and $s \in (0,1)$ such that

$$\tau_\eta(\mu) > s > \eta(\mu,\mu).$$

There exists $r_\circ \in (0,1]$ with $\mu = \mathrm{I}_\eta(\mu,r_\circ)$ and $r_\circ > s$. Since

$\mathrm{I}_\eta(\mu,r_\circ) = \sup\{\rho \in I^X \setminus \eta(\rho,\mu) > r_\circ\}$. and η is perfect, $\eta(\mu,\mu) \geq r_\circ > s$.

It is a contradiction.

3.2.3 Theorem

Let I be a smooth supra interior operator on X. Define a function $\eta_\mathrm{I} : I^X \times I^X \to I$ by

$$\eta_\mathrm{I}(\lambda,\mu) = \sup\{r \setminus \lambda \leq \mathrm{I}(\mu,r)\}.$$

Then:

(1) η_I is a perfect smooth semi-topogenous order on X such that $\mathrm{I}_{\eta_\mathrm{I}}(\lambda,r) \leq \mathrm{I}(\lambda,r)$ and $\mathrm{I}_{\eta_\mathrm{I}}(\lambda,r-\varepsilon) \geq \mathrm{I}(\lambda,r)$ for each $\lambda \in I^X, r \in I_1$ and $\varepsilon > 0$.

(2) If I is a smooth interior operator on X, then η_I is a smooth topogenous order on X.

(3) If $\mathrm{I}(\mathrm{I}(\lambda,r),r) = \mathrm{I}(\lambda,r)$ for each $\lambda \in I^X, r \in I_1$, then $\eta_\mathrm{I} \precsim \eta_\mathrm{I} \circ \eta_\mathrm{I}$.

(4) If I is a topological smooth interior operator on X, then η_I is a smooth topogenous structure on X.

(5) If η is a semi-topogenous order, then $\eta \leq \eta_{\mathrm{I}_\eta}$.

(6) If η is a perfect semi-topogenous order, then $\eta = \eta_{\mathrm{I}_\eta}$.

Proof

(1) (T1) Since $\underline{1} = \mathrm{I}(\underline{1}, r)$ and $\underline{0} = \mathrm{I}(\underline{0}, r)$ for all $r \in I_1$, then

$$\eta_\mathrm{I}(\underline{1},\underline{1}) = \eta_\mathrm{I}(\underline{0},\underline{0}) = 1.$$

(T2) if $\eta_\mathrm{I}(\lambda,\mu) > 0$, there exists $r \in I_\circ$ with $\lambda \le \mathrm{I}(\mu, r)$ such that $\eta_\mathrm{I}(\lambda,\mu) \ge r > 0$. Thus, $\lambda \le \mathrm{I}(\mu,r) \le \mu$.

(T3) Let $\lambda \le \lambda_1, \mu_1 \le \mu$ and $\lambda_1 \le \mathrm{I}(\mu_1, r)$. Then $\lambda \le \mathrm{I}(\mu, r)$. Hence, $\eta_\mathrm{I}(\lambda_1,\mu_1) \le \eta_\mathrm{I}(\lambda,\mu)$. From (ST3) we only show that

$$\eta_\mathrm{I}(\sup_{i\in\Gamma} \lambda_i, \mu) \ge \inf_{i\in\Gamma} \eta_\mathrm{I}(\lambda_i, \mu).$$

Suppose there exist λ_i, μ and $r \in (0,1)$ such that

$$\eta_\mathrm{I}(\sup_{i\in\Gamma} \lambda_i, \mu) < r < \inf_{i\in\Gamma} \eta_\mathrm{I}(\lambda_i, \mu).$$

Since $\eta_\mathrm{I}(\lambda_i, \mu) > r$ for each $i \in \Gamma$, there exists r_i with $r_i > r$ such that

$$\lambda_i \le \mathrm{I}(\mu, r_i).$$

Put $r_\circ = \inf_{i\in\Gamma} r_i$. Then

$$\lambda_i \le \mathrm{I}(\mu, r_i) \le \mathrm{I}(\mu, r_\circ).$$

It implies

$$\sup_{i\in\Gamma} \lambda_i \le \mathrm{I}(\mu, r_\circ).$$

Thus, $\eta_\mathrm{I}(\sup_{i\in\Gamma} \lambda_i, \mu) \ge r_\circ \ge r$. It is a contradiction

Thus, η_I is a perfect smooth semi-topogenous order on X

Since $\eta_\mathrm{I}(\mu,\lambda) > r$ then $\mu \le \mathrm{I}(\lambda, r)$. It implies $\mathrm{I}_{\eta_\mathrm{I}}(\lambda, r) \le \mathrm{I}(\lambda, r)$.

Since $\eta_\mathrm{I}(\mathrm{I}(\lambda,r),\lambda) > r$ for each $\varepsilon > 0$, then $\mathrm{I}_{\eta_\mathrm{I}}(\lambda, r-\varepsilon) \ge \mathrm{I}(\lambda, r)$.

(2) From (1) we only show η_I satisfies (ST6).

Suppose there exists $r \in (0,,1)$ such that

$$\eta_\mathrm{I}(\lambda, \mu_1 \wedge \mu_2) < r < \eta_\mathrm{I}(\lambda, \mu_2) \wedge \eta_\mathrm{I}(\lambda, \mu_2).$$

Since $\eta_I(\lambda,\mu_i) > r$ for $i \in \{0,1\}$, there exists r_i with $r_i > r$ and $\lambda \le I(\mu, r_i)$ such that

$$\eta_I(\lambda,\mu_i) \ge r_i > r.$$

Put $s = r_1 \wedge r_2$. Since I is a smooth interior operator,

$$\lambda \le I(\mu_1, r_1) \wedge I(\mu_2, r_2) \le I(\mu_1, s) \wedge I(\mu_2, s) \le I(\mu_1 \wedge \mu_2, s).$$

Thus, $\eta_I(\lambda, \mu_1 \wedge \mu_2) \ge s > r$. It is a contradiction.

Thus $\eta_I(\lambda, \mu_1 \wedge \mu_2) = \eta_I(\lambda, \mu_2) \wedge \eta_I(\lambda, \mu_2)$.

(3) Suppose there exists $r \in (0,1)$ such that

$$\eta_I \circ \eta_I(\lambda,\mu) < r < \eta_I(\lambda,\mu).$$

Since $\eta_I(\lambda,\mu) > r$, there exists r_1 with $r_1 > r$ and $\lambda \le I(\mu, r_1)$ such that

$$\eta_I(\lambda,\mu) \ge r_1 > r.$$

On the other hand, since $I(I(\mu, r_1), r_1) = I(\mu, r_1)$,

$$\eta_I \circ \eta_I(\lambda,\mu) \ge \eta_I(\lambda, I(\mu, r_1)) \wedge \eta_I(I(\mu, r_1), \mu) \ge r_1 > r.$$

It is a contradiction.

(4) It is trivial from 2) and 3).

(5) Suppose there exist $\lambda, \mu \in I^X$ and $r \in (0,1)$

$$\eta(\lambda,\mu) > r > \eta_{I_\eta}(\lambda,\mu).$$

Since $\eta(\lambda,\mu) > r$, we have $\lambda \le I(\mu, r)$. Thus

$$\eta_{I_\eta}(\lambda,\mu) \ge r.$$

It is a contradiction.

(6) Suppose there exist $\lambda, \mu \in I^X$ and $r \in (0,1)$ such that

$$\eta_{I_\eta}(\lambda,\mu) > r > \eta(\lambda,\mu).$$

Since $\eta_{I_\eta}(\lambda,\mu) > r$, there exists $r_\circ \in (0,1)$ with $\lambda \le I_\eta(\mu, r_0)$. Since

$I_\eta(\mu, r_0) = \sup\{\rho \setminus \eta(\rho,\mu) > r_\circ\}$ and η is perfect, $\eta(I_\eta(\mu, r_0), \mu) \ge r_\circ$.

From (ST3), it is a contradiction.

3.2.4 Theorem

Let τ be a smooth supra topology on X.

(1) $\eta_{I_\tau}(\lambda,\mu)=\sup\{\tau(\rho)\setminus\lambda\le\rho\le\mu\}$.

(2) $\tau_{\eta_{I_\tau}}=\tau$.

Proof

(1) Put $\eta_\tau(\lambda,\mu)=\sup\{\tau(\rho)\setminus\lambda\le\rho\le\mu\}$. We will show that $\eta_{I_\tau}=\eta_\tau$.

Suppose there exist $\lambda,\mu\in I^X$ and $r\in(0,1)$ such that

$$\eta_{I_\tau}(\lambda,\mu)<r<\eta_\tau(\lambda,\mu).$$

Since $\eta_\tau(\lambda,\mu)>r$, there exists $\rho\in I^X$ and $r_\circ$ with $\lambda\le\rho\le\mu$ such that

$$\eta_\tau(\lambda,\mu)\ge\tau(\rho)>r_\circ>r.$$

It implies $\lambda\le I_\tau(\rho,r_\circ)=\rho\le\mu$. Thus,

$$\eta_{I_\tau}(\lambda,\mu)\ge\eta_\tau(\lambda,\mu)\ge r_\circ>r.$$

It is a contradiction.

Suppose there exist $\lambda,\mu\in I^X$ and $r\in(0,1)$ such that

$$\eta_{I_\tau}(\lambda,\mu)>s>\eta_\tau(\lambda,\mu).$$

Since $\eta_{I_\tau}(\lambda,\mu)>s$, there exists $s_1<s$, $\lambda\le I_\tau(\mu,s_1)\le\mu$. Since

$$I_\tau(I_\tau(\mu,s_1),s_1)=I_\tau(\mu,s_1)$$

We have

$$\eta_\tau(\lambda,\mu)\ge\tau(I_\tau(\mu,s_1))\ge s_1>s$$

It is a contradiction.

(2) Since η_{I_τ} is perfect semi-topogenous order on X, by Theorem 3.2.2,

$$\begin{aligned}\tau_{\eta_{I_\tau}}(\lambda)&=\eta_{I_\tau}(\lambda,\lambda)\\&=\sup\{\tau(\rho)\setminus\lambda\le\rho\le\lambda\}\\&=\tau(\lambda)\end{aligned}$$

3.2.5 Example

Let $X=\{x,y,z\}$ be a set. Define smooth the topologies $\eta_i : I^X \times I^X \to I$ where $i=1,2,3,4$ as follows:

$$\eta_1(\lambda,\mu)=\begin{cases}1 & ,\text{if}\ \lambda=\underline{0}\ \text{or}\ \mu=\underline{1}\\ \frac{2}{3} & ,\text{if}\ \underline{0}\neq\lambda\leq\chi_{\{x\}},\underline{1}\neq\mu\geq\chi_{\{x,y\}}\\ \frac{1}{2} & ,\text{if}\ \underline{0}\neq\lambda\leq\chi_{\{y\}},\underline{1}\neq\mu\geq\chi_{\{x,y\}}\\ 0 & \text{otherwise}\end{cases}$$

$$\eta_2(\lambda,\mu)=\begin{cases}1 & ,\text{if}\ \lambda=\underline{0}\ \text{or}\ \mu=\underline{1}\\ \frac{2}{3} & ,\text{if}\ \underline{0}\neq\lambda\leq\chi_{\{y\}},\underline{1}\neq\mu\geq\chi_{\{x,y\}}\\ \frac{1}{2} & ,\text{if}\ \underline{0}\neq\lambda\leq\chi_{\{y\}},\underline{1}\neq\mu\geq\chi_{\{x,y\}}\\ 0 & \text{otherwise}\end{cases}$$

$$\eta_3(\lambda,\mu)=\begin{cases}1 & ,\text{if}\ \lambda=\underline{0}\ \text{or}\ \mu=\underline{1}\\ \frac{2}{3} & ,\text{if}\ \underline{0}\neq\lambda\leq\chi_{\{x\}},\underline{1}\neq\mu\geq\chi_{\{x,y\}}\\ 0 & \text{otherwise}\end{cases}$$

$$\eta_4(\lambda,\mu)=\begin{cases}1 & ,\text{if}\ \lambda=\underline{0}\ \text{or}\ \mu=\underline{1}\\ \frac{2}{3} & ,\text{if}\ \underline{0}\neq\lambda\leq\mu\neq1\\ 0 & \text{otherwise}\end{cases}$$

(1) η_1 is a smooth semi-topogenous order on X but not topogenous because:

$$
\begin{aligned}
0 &= \eta_1(\chi_{\{x\}} \vee \chi_{\{y\}}, \chi_{\{x,y\}}) \\
&\neq \eta_1(\chi_{\{x\}}, \chi_{\{x,y\}}) \wedge \eta_1(\chi_{\{y\}}, \chi_{\{x,y\}}) \\
&= \frac{1}{2}
\end{aligned}
$$

From Theorem 3.2.1, we can obtain smooth supra interior operator $\mathrm{I}_{\eta_1} : I^X \times I_1 \to I^X$ as follows:

$$
\mathrm{I}_{\eta_1}(\lambda, r) = \begin{cases} \underline{1} & \text{if } \lambda = \underline{1}, r \in I_1 \\ \chi_{\{x\}}, & \text{if } \chi_{\{x,y\}} \leq \lambda \neq 1, \frac{1}{2} \leq r < \frac{2}{3} \\ \chi_{\{x,y\}}, & \text{if } \chi_{\{x,y\}} \leq \lambda \neq 1, 0 \leq r < \frac{1}{2} \\ \underline{0}, & \text{otherewise} \end{cases}
$$

From Theorem 3.2.2, we can obtain smooth supra topology $\tau_{\eta_1} : I^X \to I$ as follows:

$$
\tau_{\eta_1}(\lambda) = \begin{cases} 1, & \text{if } \lambda \in \{\underline{0}, \underline{1}\} \\ \frac{1}{2}, & \text{if } \lambda = \chi_{\{x,y\}} \\ 0 & \text{otherwise} \end{cases}
$$

Since η_1 is not perfect, by Theorem 3.2.4 (3),

$$
\frac{1}{2} = \tau_{\eta_1}(\chi_{\{x,y\}}) \neq \eta_1(\chi_{\{x,y\}}, \chi_{\{x,y\}}) = 0 .
$$

From Theorem 3.2.3 (5), we have $\eta_1 \leq \eta_{1_{\eta_1}}$ but $\eta_1 \neq \eta_{1_{\eta_1}}$ as follows:

$$
\eta_{1_{\eta_1}}(\lambda, r) = \begin{cases} \underline{1}, & \text{if } \lambda = \underline{0}, \mu = \underline{1} \\ \frac{2}{3}, & \text{if } \chi_{\{x,y\}} \leq \lambda \neq 1, \frac{1}{2} \leq r < \frac{2}{3} \\ \chi_{\{x,y\}}, & \text{if } \chi_{\{x,y\}} \leq \lambda \neq 1, 0 \leq r < \frac{1}{2} \\ \underline{0}, & \text{otherewise} \end{cases}
$$

(2) η_2 is a perfect smooth semi-topogenous order on X but not topogenous because:

$$0 = \eta_2(\chi\{y\}, \chi\{x,y\} \wedge \chi\{y,z\}) \neq \eta_2(\chi\{y\}, \chi\{x,y\}) \wedge \eta_2(\chi\{y\}, \chi\{y,z\}) = \frac{1}{2}$$

From Theorem 3.1.1, we can obtain smooth supra interior operator

$$\mathrm{I}_{\eta_2}(\lambda, r) = \begin{cases} \underline{1}, & \text{if} \lambda = \underline{1}, r \in I_1 \\ \chi\{y\}, & \text{if } \chi\{\mathrm{x,y}\} \leq \underline{1}, 0 \leq r < \frac{2}{3} \\ \chi\{y\}, & \text{if } \chi\{\mathrm{x,y}\} \leq \underline{1}, 0 \leq r < \frac{1}{2} \\ \underline{0}, & \text{otherewise.} \end{cases}$$

But it is not a smooth interior operator because

$$\underline{0} = \mathrm{I}_{\eta_2}(\chi_{\{x,y\}}, \frac{1}{3}) \neq \mathrm{I}_{\eta_2}(\chi_{\{x,y\}}, \frac{1}{3}) \wedge \mathrm{I}_{\eta_2}(\chi_{\{x,y\}} \wedge \chi_{\{y,z\}}, \frac{1}{3}) = \chi_{\{y\}}.$$

From Theorem 3.2.3 (5), since η_2 is perfect, we have $\eta_2 = \eta_{\mathrm{I}_{\eta_2}}$ as follows:

$$\eta_{\mathrm{I}_{\eta_2}}(\lambda, \mu) = \begin{cases} 1, & \text{if } \lambda = \underline{0} \text{ or } \mu = \underline{1} \\ \frac{2}{3}, & \text{if } \underline{0} \neq \lambda \leq \chi_{\{y\}}, \underline{1} \neq \mu \geq \chi_{\{x,y\}} \\ \frac{1}{2}, & \text{if } \underline{0} \neq \lambda \leq \chi_{\{y\}}, \underline{1} \neq \mu \geq \chi_{\{y,z\}} \\ 0, & \text{otherewise} \end{cases}$$

Furthermore,

$$\tau_{\eta_2}(\lambda) = \eta_2(\lambda, \lambda) = \begin{cases} 1, & \text{if } \lambda \in \{\underline{0}, \underline{1}\} \\ 0, & \text{otherwise.} \end{cases}$$

(3) η_2 is a smooth topogenous order on X but not topogenous structure from the following statements: For any $\rho \in I^X$ with $\chi_{\{x\}} \leq \rho \leq \chi_{\{x,y\}}$, we have

$$\eta_3(\chi_{\{x\}}, \rho) \wedge \eta_2(\rho, \chi_{\{x,y\}}) = 0.$$

Thus,

$$0=\eta_3\circ\eta_3(\chi_{\{x\}},\chi_{\{x,y\}})<\eta_3(\chi_{\{x\}},\chi_{\{x,y\}})=\frac{2}{3}.$$

From Theorem 3.2.1, I_{η_3} is a smooth interior operator from:

$$I_{\eta_3}=\begin{cases}\underline{1}, & \text{if } \lambda=\underline{1},\\ \chi_{\{x\}}, & \text{if } \chi_{\{x,y\}}\le\lambda\ne\underline{1}, 0\le r<\dfrac{2}{3}\\ \underline{0}, & \text{otherwise.}\end{cases}$$

From Theorem 3.2.2(3), since $\eta_3\not\le\eta_3\circ\eta_3$, in general, we have

$$\chi_{\{x\}}=I_{\eta_3}(\chi_{\{x,y\}},\frac{1}{2})\ne I_{\eta_3}(I_{\eta_3}(\chi_{\{x,y\}},\frac{1}{2}),\frac{1}{2})=\underline{0}.$$

4) We easily show that η_1,η_2 and η_3 are not symmetric.

5) η_4 is a symmetric smooth topogenous structure on X from:

$$\eta_4^s(\lambda,\mu)=\begin{cases}1, & \text{if } \lambda=\underline{0} \text{ or } \mu=\underline{1},\\ \dfrac{2}{3}, & \text{if } \underline{0}\ne\lambda\le\mu\ne\underline{1},\\ 0, & \text{otherwise.}\end{cases}$$

(6) We define a smooth interior operator $I: I^X\times I_1\to I^X$ as follows:

$$I(\lambda,r)=\begin{cases}\underline{1}, & \text{if } \lambda=\underline{1},\\ \chi_{\{x\}}, & \text{if } \chi_{\{x,y\}}\le\lambda\ne\underline{1}, 0\le r\le\dfrac{2}{3}\\ \underline{0}, & \text{otherwise.}\end{cases}$$

From Theorem 3.2.4 and Theorem 3.2.2, we obtain the followings:

$$\eta_I(\lambda,\mu)=\begin{cases}1, & \text{if } \lambda=\underline{0} \text{ or } \mu=\underline{1},\\ \dfrac{2}{3}, & \text{if } \underline{0}\ne\lambda\le\chi\{x\}, \underline{1}\ne\mu\ge\chi_{\{x,y\}},\\ 0, & \text{otherwise.}\end{cases}$$

$$1_{\eta_1}(\lambda,\mu)=\begin{cases}\underline{1}, & \text{if } \lambda=\underline{1},\\ \chi_{\{x\}}, & \text{if } \chi_{\{x,y\}}\leq\lambda\neq\underline{1}, 0\leq r<\frac{2}{3}\\ \underline{0}, & \text{otherwise.}\end{cases}$$

We have $1_{\eta_1}(\lambda,r)\leq I(\lambda,r)$ and $1_{\eta_1}(\lambda,r-\varepsilon)\geq I(\lambda,r)$ for $\varepsilon>0$.

3.2.6 Theorem [99]

If (X,η_1) and (X,η_2) are smooth semi-topogenuous spaces, then their supremum $\eta_1\cup\eta_2$ is also smooth semi-topogenuous on X.

(Notice, however, that If (X,η_1) and (X,η_2) are smooth topogenuous spaces, then it generally does not follow that $\eta_1\cup\eta_2$ is smooth topogenuous on X.)

3.2.7 Corollary

Let (X,η) be smooth semi-topogenuous spaces. We define, $\eta^*=\eta\cup\eta^s$.

Then from Proposition 3.1.3 and Theorem 3.2.6 we have η^* is smooth semi-topogenuous on X. Obvious η^* is symmetric.

3.2.8 Theorem

Let (S, X) be a smooth syntopogeneous space. Define a function $C_S: I^X\times I_1\to I^X$ by

$$C_S(\lambda,r)=\inf\{\mu/\eta(\lambda,\mu)>0, S(\eta)>r\}.$$

For each $\lambda,\lambda_1,\lambda_2\in I^X$ and $r,r_1,r_2\in I_1$, we have the following properties:

(1) $C_S(\underline{0},r)=\underline{0}$,

(2) $\lambda\leq C_S(\lambda,r)$,

(3) If $\lambda_1\leq\lambda_2$, then $C_S(\lambda_1,r)\leq C_S(\lambda_2,r)$,

(4) $C_S(\lambda_1\vee\lambda_2,r)=C_S(\lambda_1,r)\vee C_S(\lambda_2,r)$,

(5) If $r_1 \le r_2$, then $C_S(\lambda, r_1) \le C_S(\lambda, r_2)$,

(6) $C_S(C_S(\lambda, r), r) = C_S(\lambda, r)$.

Proof

(1) Since $\eta(\underline{0}, \underline{0}) = 1$ for $S(\eta) = 1$, then $C_S(\underline{0}, r) = \underline{0}$.

(2) Since $\lambda \le \mu$ for $\eta(\lambda, \mu) > 0$, implies $\lambda \le C_S(\lambda, r)$.

(3) and (5) are easily proved.

(4) From (3), we have

$$C_S(\lambda_1 \vee \lambda_2, r) \ge C_S(\lambda_1, r) \vee C_S(\lambda_2, r).$$

Conversely, suppose there exist $\lambda_1, \lambda_2 \in I^X$ and $r \in I$ such that

$$C_S(\lambda_1 \vee \lambda_2, r) \not\le C_S(\lambda_1, r) \vee C_S(\lambda_2, r).$$

There exists $x \in X$ and $t \in I_1$ such that

$$C_S(\lambda_1 \vee \lambda_2, r) > t > C_S(\lambda_1, r) \vee C_S(\lambda_2, r). \qquad \text{(F)}$$

Since $C_S(\lambda_i, r)(x) < t$, for each $i \in \{1,2\}$, there exists $\eta_i \in \Gamma_X$ with $S(\eta_i) > r$ and $\eta_i(\lambda_i, \mu_i) > 0$ such that

$$C_S(\lambda_i, r)(x) \le (\mu_i)(x) < t.$$

On the other hand, since $S(\eta_1) \wedge S(\eta_2) > r$, by (T2) of Definition 4.3.5, there exists η with $\eta \ge \eta_i$ and $S(\eta) > r$ such that

$$\begin{aligned} \eta(\lambda_1 \vee \lambda_2, \mu_1 \vee \mu_2)) &\ge \eta(\lambda_1, \mu_1 \vee \mu_2)) \wedge \eta(\lambda_2, \mu_1 \vee \mu_2)) \\ &\ge \eta(\lambda_1, \mu_1)) \wedge \eta(\lambda_2, \mu_2)) \\ &\ge \eta_1(\lambda_1, \mu_1)) \wedge \eta_2(\lambda_2, \mu_2)) \\ &> 0 \end{aligned}$$

Hence, $C_S(\lambda_1 \vee \lambda_2, r)(x) \le (\mu_1 \vee \mu_2)(x) < \mathrm{t}$

It is contradiction for the condition (F).

(6) Suppose there exists $\lambda \in I^X$ and $r \in I_1$ such that

$$C_S(C_S(\lambda, r), r) > C_S(\lambda, r).$$

There exists $x \in X$ and $t \in (0,1)$ such that

$$C_S(C_S(\lambda, r), r)(x) > t > C_S(\lambda, r)(x).$$

Since $C_S(\lambda,r)(x)<t$, there exists $\mu \in I^X$ with $S(\eta)>r$ and $\eta(\lambda,\mu)>0$ such that

$$C_S(\lambda,r)(x) \leq \mu(x) < t .$$

On the other hand, since $S(\eta)>r$, by (T3), of Definition 3.1.11, there exists $\zeta \in \Upsilon_X$ such that

$$\zeta \circ \zeta \geq \eta, \; S(\zeta) > r$$

Since $\zeta \circ \zeta(\lambda,\mu)>0$, there exists $\rho \in I^X$ such that

$$\zeta(\lambda,\rho) \wedge \zeta(\rho,\mu) > 0$$

It implies $C_S(\lambda,r) \leq \rho, C_S(\rho,r) \leq \mu$.

Hence,

$$C_S(C_S(\lambda,r),r) \leq \mu$$

Thus, $C_S(C_S(\lambda,r),r)(x) \leq \mu(x) < t$

It is contradiction.

3.2.9 Theorem

Let (X, S) be a smooth syntopogenous space. Define a function $\tau_S : I^X \to I$ by

$$\tau_S(\lambda) = \sup\{r \in I_1 \setminus C_S(\underline{1}-\lambda, r) = \underline{1}-\lambda\} .$$

Then τ_S is a smooth topology on X induced by S.

The proof is similar to the proof of Theorem 1.1.9

3.1.10 Theorem

Let (X,η_1) and (Y,η_2) be smooth topogenous spaces. Let $f : X \to Y$ be smooth topogenous continuous, then it satisfies the following statements:

(1) $f(C_{\eta_1}(\lambda,r)) \leq C_{\eta_2}(f(\lambda),r))$, for each $\lambda \in I^X$.

(2) $C_{\eta_1}(f^{-1}(\mu),r) \leq f^{-1}(C_{\eta_2}(f(\mu),r))$, for each $\mu \in I^Y$.

(3) $f:(X,\tau_{\eta_1}) \to (Y,\tau_{\eta_2})$ is smooth continuous.

Proof

(1) Suppose there exists $\lambda \in I^X$ and $r \in I_1$ such that

$$f(C_{\eta_1}(\lambda, r)) > C_{\eta_2}(f(\lambda), r)),$$

There exists $y \in Y$ and $t \in I_0$ such that

$$f(C_{\eta_1}(\lambda, r)) > t > C_{\eta_2}(f(\lambda), r)),$$

Since $f^{-1}(\{y\}) = \phi$, provides a contradiction that $f(C_{\eta_1}(\lambda, r))(y) = 0$, $f^{-1}(\{y\}) \neq \phi$, and there exists $x \in f^{-1}(\{y\})$ such that

$$f(C_{\eta_1}(\lambda, r))(y) \geq C_{\eta_1}(\lambda, r)(x) > t > C_{\eta_2}(f(\lambda), r))(f(x)). \qquad \text{(A)}$$

Since $C_{\eta_2}(f(\lambda), r))(f(x)) < t$, there exists $\nu \in I^Y$ with $\eta_2(\upsilon, \underline{1} - f(\lambda)) > r$ such that

$$C_{\eta_2}(f(\lambda), r))(f(x)) \leq (\underline{1} - \nu)(f(x)) = f^{-1}(\underline{1} - \nu)(x) < t.$$

On the other hand, since f is smooth topogenous continuous,

$$\eta_1(f^{-1}(\nu), f^{-1}(\underline{1} - f(\lambda))) \geq \eta_2(\nu, \underline{1} - f(\lambda)) > r.$$

Since $\eta_1(f^{-1}(\nu), \underline{1} - \lambda) \geq \eta_1(f^{-1}(\nu), f^{-1}(\underline{1} - f(\lambda)))$, we have

$$\begin{aligned} C_{\eta_1}(\lambda, r)(x) &= (\underline{1} - f^{-1}(\nu))(x) \\ &= f^{-1}(\underline{1} - \nu)(x) < t. \end{aligned}$$

Thus, $C_{\eta_1}(\lambda, r)(x) < t$, it is a contradiction for the equation (A).

(2) For each $\mu \in I^Y$ and $r \in I_1$, put $\lambda = f^{-1}(\mu)$. From (1),

$$f(C_{\eta_1}(f^{-1}(\mu), r)) \leq C_{\eta_2}(f(f^{-1}(\mu)), r)) \leq C\eta_2(\mu, r).$$

It implies

$$C_{\eta_1}(f^{-1}(\mu), r)) \leq f^{-1}(f(C_{\eta_1}(f^{-1}(\mu), r))) \leq f^{-1}(C\eta_2(\mu, r)).$$

(3) From (2), $C\eta_2(\mu, r) = \mu$ implies $C\eta_1(f^{-1}(\mu), r) = f^{-1}(\mu)$. It is easily proved from Theorem 3.2.8.

3.2.11 Theorem

Let (X,η_1) and (Y,η_2) be smooth semi-topogenuous spaces. Let $f:X\to Y$ be smooth topogenous continuous. Then it satisfies the following statements:

(1) $f:(X,\eta_1^s)\to(Y,\eta_2^s)$ is smooth topogenous continuous.

(2) $f:(X,\eta_1^*)\to(Y,\eta_2^*)$ is smooth topogenous continuous.

Proof

(1) $\eta_2^s(\lambda,\mu)=\eta_2^s(\underline{1}-\mu,\underline{1}-\lambda)$

$$\le\eta_1(f^{-1}(\underline{1}-\mu),f^{-1}(\underline{1}-\lambda))$$

$$=\eta_1^s(f^{-1}(\lambda),f^{-1}(\mu)).$$

(2) Suppose there exist $\lambda,\mu\in I^Y$ and $r\in I_1$ such that

$$\eta_2^s(\lambda,\mu)>r>\eta_1^s(f^{-1}(\lambda),f^{-1}(\mu))$$

Since $\eta_2^s(\lambda,\mu)>r$, then there are finite families $\{\lambda_j\setminus\lambda=\sup_{j=1}^{p}\lambda_j\}$ and $\{\mu_k\setminus\mu=\sup_{k=1}^{q}\mu_k\}$ such that

$$\eta_2^s(\lambda,\mu)\ge\inf_{j,k}(\eta_2(\lambda_j,\mu_k)\vee\eta_2^s(\lambda_j,\mu_k))$$

$$=\inf_{j,k}(\eta_2(\lambda_j,\mu_k)\vee\eta_2(\underline{1}-\mu_k,\underline{1}-\lambda_j))$$

$$>r$$

Since f is smooth topogenous continuous,

$$\eta_1^*(f^{-1}(\lambda),f^{-1}(\mu))\ge\inf_{j,k}(\eta_1(f^{-1}(\lambda_j),f^{-1}(\mu_k))\vee\eta_1^s(f^{-1}(\lambda_j),f^{-1}(\mu_k)))$$

$$\ge\inf_{j,k}(\eta_2(\lambda_j,\mu_k)\vee\eta_2^s(\lambda_j,\mu_k))$$

$$= \inf_{j,k} (\eta_2(\lambda_j, \mu_k) \vee \eta_2(\underline{1} - \mu_k, \underline{1} - \lambda_j))$$
$$> r$$

It is a contradiction.

3.2.12 Theorem

Let X be a set and (Y, η) be smooth topogenous space. Let $f : X \to Y$ be a function. We define $f^{-1}(\eta) : I^X \times I^X \to I$ as follows:

$$f^{-1}(\eta)(\lambda, \mu) = \eta(f(\lambda), \underline{1} - f(\underline{1} - \mu)).$$

Then:

(1) $(X, f^{-1}(\eta))$ is the coarsest topogenous spaces for which f is topogenous continuous.

(2) $C_{f^{-1}(\eta)}(\lambda, r) = f^{-1}\left(C_\eta(\lambda, r)\right)$ for each $\lambda \in I^X$ and $r \in I_1$

(3) $\tau_{f^{-1}(\eta)} = \tau_\eta$ where

Proof

(1) See the proof of Proposition 3.2 [99].

(2) Since $C_{f^{-1}(\eta)}(\lambda, r) = \inf\{\underline{1} - \rho \setminus f^{-1}(\eta)(\rho, \underline{1} - \lambda)) > r\}$

$$= \inf\{\underline{1} - \rho \setminus \eta(f(\rho), \underline{1} - f(\lambda)) > r\}$$

$$f^{-1}\left(C_\eta(\lambda, r)\right) = f^{-1}(\inf\{\underline{1} - \mu \setminus \eta(\mu, \underline{1} - f(\lambda)) > r\})$$

$$= \inf\{\underline{1} - f^{-1}(\mu) \setminus \eta(\mu, \underline{1} - f(\lambda)) > r\}$$

Let $\rho \in I^X$ such that $\eta(f(\rho), \underline{1} - f(\lambda)) > r$

Put $\mu = f(\rho)$. Then

$$\underline{1} - f^{-1}(\mu) = \underline{1} - f^{-1}(f(\rho)) \leq \underline{1} - \rho.$$

Thus, $C_{f^{-1}(\eta)}(\lambda, r) \geq f^{-1}\left(C_\eta(\lambda, r)\right)$.

Conversely, let $\mu \in I^Y$ such that $\eta(\mu, \underline{1} - f(\lambda)) > r$.

Since $f(f^{-1}(\mu) \leq \mu$ we have

$$\eta(f(f^{-1}(\mu),\underline{1}-f(\lambda)) \geq \eta(\mu,\underline{1}-f(\lambda)) > r$$

Thus, $C_{f^{-1}(\eta)}(\lambda,r) \leq f^{-1}\big(C_\eta(\lambda,r)\big)$.

(3) Suppose $\tau_{f^{-1}(\eta)} \not\leq \tau_\eta$. There exist $\lambda \in I^X$ and $r \in (0,1)$ such that

$$C_{f^{-1}(\eta)}(\lambda,r) > r > f^{-1}\big(C_\eta(\lambda,r)\big)$$

There exists $r_\circ \in I_\circ$ with $r_\circ > r$ such that

$$\underline{1}-\lambda = C_{f^{-1}(\eta)}(\lambda,r_\circ).$$

It implies

$$\lambda = \underline{1} - C_{f^{-1}(\eta)}(\lambda,r_\circ)$$

$$= \underline{1} - f^{-1}(C_\eta(\lambda,r_\circ))$$

$$= f^{-1}(\underline{1} - (C_\eta(\lambda,r_\circ))$$

Since $C_\eta(\lambda,r_\circ) = C_\eta(C_\eta(\lambda,r_\circ),r_\circ)$

$$\tau_\eta(\underline{1} - C_\eta(\lambda,r_\circ)) \geq r_\circ > r.$$

It is a contradiction.

Suppose $\tau_{f^{-1}(\eta)} \not\geq \tau_\eta$, There exist λ , $\in$ I^X and $r \in (0,1)$,such that

$$\tau_{f^{-1}(\eta)}(\lambda) < r < \tau_\eta(\lambda)$$

There exists $\mu \in I^Y$ with $f^{-1}(\mu) = \lambda$ such that

$$\tau_{f^{-1}(\eta)}(\lambda) < r < \tau_\eta(\mu) \leq \tau_f(\lambda)$$

From the definition of τ_η,

$$\underline{1} - \mu = C_\eta(\mu,r_\circ),\ r_\circ > r$$

$$C_{f^{-1}(\eta)}(f^{-1}(\underline{1}-\mu),r_\circ) = f^{-1}(C_\eta(f(f^{-1}(\underline{1}-\mu),r_\circ)))$$

$$\leq f^{-1}(C_\eta(\underline{1}-\mu,r_\circ))$$

$$= f^{-1}(\underline{1} - \mu)$$

Thus, $\tau_{f^{-1}(\eta)}(f^{-1}(\mu)) = \tau_{f^{-1}(\eta)}(\lambda) \geq r_\circ$

It is a contradiction.

3.2.13 Theorem

Let (X, τ) be a smooth supra topological space. We define a function $\tau' : I^X \to I$ as follow:

$$\tau'(\lambda) = \sup\{\inf_{j=1}^{m} \tau(\lambda_j) \setminus \lambda = \inf_{j=1}^{m} \lambda_j\}$$

where the supremum is taken for every finite family $\{\lambda_j \setminus \lambda = \inf_{j=1}^{m} \lambda_j\}$.

Then τ' is the coarsest smooth topology on X finer than τ.

Proof

First, we will show that τ' is a smooth topology on X.

(O1) It is easily proved from:

$$\tau'(\underline{0}) \geq \tau(\underline{0}) = 1,$$

$$\tau'(\underline{1}) \geq \tau(\underline{1}) = 1,$$

(O2) Suppose that there exists a family $\{\lambda_i \in I^X \setminus \lambda = \sup_{i \in \Gamma} \lambda_i\}$ and $r \in (0,1)$ such that

$$\tau'(\lambda) < r < \inf_{i \in \Gamma} \tau'(\lambda_i).$$

Since $\tau'(\lambda_i) > r$ for each $i \in \Gamma$, there is a finite family $\{\lambda_{i_j} \in I^X \setminus \lambda_i = \inf_{j \in J_i} \lambda_{i_j}\}$ such that

$$\tau'(\lambda_i) \geq \inf_{j \in J_i} \tau(\lambda_{i_j}) > r.$$

Since the unit interval I is complete distributive lattice (ref [10]), we have

$$\lambda = \sup_{i\in\Gamma}(\inf_{j\in J_i} \lambda_{i_j}) = \inf_{\psi\in\Pi J_i}(\sup_{i\in\Gamma} \lambda_{i_{\psi(i)}})$$

and

$$\inf_{i\in\Gamma}(\inf_{j\in J_i} \tau(\lambda_{i_j})) = \inf_{\psi\in\Pi J_i}(\inf_{i\in\Gamma} \tau(\lambda_{i_{\psi(i)}})).$$

Thus

$$\begin{aligned}\tau^t(\lambda) &\geq \inf_{\psi\in\Pi J_i}(\tau(\sup_{i\in\Gamma} \lambda_{i_{\psi(i)}}))\\ &\geq \inf_{\psi\in\Pi J_i}(\inf_{i\in\Gamma} \tau(\lambda_{i_{\psi(i)}}))\\ &= \inf_{i\in\Gamma}(\inf_{j\in J_i} \tau(\lambda_{i_j}))\\ &\geq r.\end{aligned}$$

It is a contradiction.

(O3) Suppose that there exist $\lambda, \mu \in I^X$ and $r \in (0,1)$ such that

$$\tau^t(\lambda \wedge \mu) < r < \tau^t(\lambda) \wedge \tau^t(\mu).$$

Since $\tau^t(\lambda) > r$ and $\tau^t(\mu) > r$, there are finite families $\{\lambda_i \setminus \lambda = \inf_{i=1}^{m} \lambda_i\}$, $\{\mu_j \setminus \mu = \inf_{j=1}^{m} \mu_j\}$ such that

$$\tau^t(\lambda) \geq \inf_{j=1}^{m} \tau(\lambda_i) > r, \tau^t(\mu) \geq \inf_{j=1}^{m} \tau(\mu_j) > r.$$

There exist a finite family $\{\lambda_i, \mu_j \setminus \lambda \wedge \mu = (\inf_{i=1}^{m} \lambda_i) \wedge (\inf_{j=1}^{n} \mu_j)\}$ such that

$$\begin{aligned}\tau^t(\lambda \wedge \mu) &\geq (\inf_{i=1}^{m} \tau(\lambda_i)) \wedge (\inf_{j=1}^{n} \tau(\mu_j))\\ &> r.\end{aligned}$$

It is a contradiction.

Second, it is proved that $\tau^t \geq \tau$ from the following: for a family $\{\lambda \setminus \lambda = \lambda\}$

$$\tau^{t}(\lambda) \geq \tau(\lambda).$$

Finally, if $\tau_1 \geq \tau$ and τ_1 is smooth topology on X, then we have

$$\tau^{t}(\lambda) = \sup\{\inf_{j=1}^{m} \tau(\lambda_j)\}$$

$$\leq \sup\{\inf_{j=1}^{m} \tau_1(\lambda_j)\}$$

$$\leq \tau_1(\lambda)$$

for every finite $\{\lambda_j \setminus \lambda = \inf_{j=1}^{m} \lambda_i\}$.

3.2.14 Theorem

Let η be a smooth semi-topogenous order on X. We define for all $\lambda, \mu \in I^X$,

$$\eta^{t}(\lambda, \mu) = \sup\{\inf_{j,k} \eta(\lambda_j, \mu_k)\}$$

where the supremum is taken for every finite families $\{\lambda_j \setminus \lambda = \sup \lambda_j\}$ and $\{\mu_k \setminus \mu = \sup \mu_k\}$. Then:

(1) η^t is a coarser smooth topogenous order on X finer than η.

(2) η is a coarser smooth topogenous order on X iff $\eta = \eta^t$.

(3) If $\eta \leq \eta \circ \eta$, then $\eta^t \leq \eta^t \circ \eta^t$.

(4) If $\eta \leq \eta \circ \eta$, then $\tau_{\eta_t} \leq (\tau_\eta)^t$

(5) If η is perfect, then $\tau_{\eta_t} \leq (\tau_\eta)^t$.

Proof

(1-3) See the proof of Theorem 2.2 [59] and Proposition 2.9 [59].

(4) Suppose $\tau_{\eta_t} \not\leq (\tau_\eta)^t$. There exist $\lambda \in I^X$ and $r \in (0,1)$ such that

$$\tau_{\eta_t}(\lambda) > r > (\tau_\eta)^t(\lambda).$$

There exists $r_\circ \in I_\circ$ with $r_\circ > r$ such that

$$\lambda = \mathrm{I}_{\eta^t}(\lambda, r_\circ).$$

It implies

$$\begin{aligned}\lambda &= \mathrm{I}_{\eta^t}(\lambda, r_\circ)\\ &= \sup\{\mu \in I^X \setminus \eta^t(\mu,\lambda) > r_\circ\}.\end{aligned}$$

Since $\eta^t(\mu,\lambda) > r_\circ$, there are finite families $\{\mu_j \setminus \mu = \sup_{j=1}^{n} \mu_j\}$ and $\{\lambda_k \setminus \lambda = \inf_{k=1}^{m} \lambda_k\}$ such that

$$\eta^t(\mu,\lambda) \geq \inf_{j,k} \eta(\mu_j, \lambda_k) > r_\circ$$

i.e. for all $j,k,\eta(\lambda_j,\mu_k) > r_\circ$. It implies

$$\mathrm{I}_\eta(\lambda_k, r_\circ) \geq \mu_j.$$

Thus,

$$\lambda \geq \inf_{k=1}^{m}\{\sup_{j=1}^{n} \mathrm{I}_\eta(\lambda_k, r_\circ)\} \geq \mu.$$

Put $\rho_k = \mathrm{I}_\eta(\lambda_k, r_\circ)$,

$$\begin{aligned}\lambda &= \mathrm{I}_{\eta^t}(\lambda, r_\circ)\\ &= \sup\{\mu \in I^X \setminus \eta^t(\mu,\lambda) > r_\circ\}\\ &= \sup\{\inf_{i=1}^{m}(\sup_{j=1}^{n} \rho_k)\}.\end{aligned}$$

Since $\eta \leq \eta \circ \eta$, by Theorem 3.1.5(4), we have

$$\mathrm{I}_\eta(\lambda_k, r_\circ) = \mathrm{I}_\eta(\mathrm{I}_\eta(\lambda_k, r_\circ), r_\circ),$$

$$\tau_\eta(\rho_k) \geq r_\circ > r.$$

It implies

$$\tau_\eta(\sup_{j=1}^{n} \rho_k) \geq r_\circ > r$$

From the definition of $(\tau_\eta)^t$,

$$(\tau_\eta)^t(\inf_{i=1}^{m}(\sup_{j=1}^{n}\rho_k)) \geq r_\circ$$

Thus

$$(\tau_\eta)^t(\lambda) \geq r_\circ > r.$$

It is a contradiction.

(5) Suppose $\tau_{\eta_t} \not\geq (\tau_\eta)^t$. There exist $\lambda \in I^X$ and $r \in (0,1)$ such that

$$\tau_{\eta_t}(\lambda) < r < (\tau_\eta)^t(\lambda).$$

Since $(\tau_\eta)^t(\lambda) > r$, there exists $\lambda_i \in I^X$ with $\inf_{j=1}^{m} \lambda_j = \lambda$ such that

$$\tau_{\eta_t}(\lambda) < r < \inf_{j=1}^{m} \tau_\eta(\lambda_j).$$

From (6) the definition of τ_η,

$$\lambda_i = \mathrm{I}_\eta(\lambda_i, r_\circ), r_\circ > r.$$

Since η is a perfect semi-topogenous order,

$$\eta(\lambda_i, \lambda_i) \geq r_\circ.$$

From the definition of η^t,

$$\eta^t(\lambda_i, \lambda) \geq r_\circ.$$

Thus, there exists r_1 with $r < r_1 < r_\circ$ such that

$$\eta^t(\lambda, \lambda) \geq \eta^t(\lambda_i, \lambda) \geq r_\circ > r_1 > r.$$

Hence,

$$\lambda = \mathrm{I}_{\eta^t}(\lambda, r_1).$$

Thus,

$$\tau_{\eta^t}(\lambda) \geq r_1 > r.$$

It is a contradiction.

3.2.15 Theorem

Let τ be a smooth supra topology on X. Then

$$(\eta_{\mathrm{I}_\tau})^t(\lambda,\mu) = \eta_{\mathrm{I}_{\tau^t}}(\lambda,\mu).$$

Proof

Suppose there exist $\lambda,\mu \in I^X$ and $r \in (0,1)$ such that

$$(\eta_{\mathrm{I}_\tau})^t(\lambda,\mu) < r < \eta_{\mathrm{I}_{\tau^t}}(\lambda,\mu).$$

Since $\eta_{\mathrm{I}_{\tau^t}}(\lambda,\mu) > r$, by Theorem 3.2.4, there exist $\rho \in I^X$ and $r_\circ \in (0,1)$ with $\lambda \le \rho \le \mu$ such that

$$\eta_{\mathrm{I}_{\tau^t}}(\lambda,\mu) \ge \tau^t(\rho) > r_\circ > r.$$

Since $\tau^t(\rho) > r_\circ$, there exists a finite family $\{\rho_j \setminus \rho = \inf_{j=1}^{m} \rho_j\}$ such that

$$\tau^t(\rho) \ge \inf_{j=1}^{m} \tau(\rho_j) > r_\circ > r.$$

Since $\tau(\rho_j) > r_\circ$ for each $j = 1,\ldots,m$, we have $\lambda \le \rho \le \mathrm{I}_\tau(\rho_i, r_\circ) = \rho_i$.

Thus,

$$\eta_{\mathrm{I}_\tau}(\lambda,\rho_i) \ge r_\circ.$$

From the definition of $(\eta_{\mathrm{I}_\tau})^t$,

$$(\eta_{\mathrm{I}_\tau})^t(\lambda,\rho) \ge \inf_{j=1}^{m} \eta_{\mathrm{I}_\tau}(\lambda,\rho_i) \ge r_\circ > r.$$

So, $(\eta_{\mathrm{I}_\tau})^t(\lambda,\mu) \ge (\eta_{\mathrm{I}_\tau})^t(\lambda,\rho) > r$. It is a contradiction.

Therefore, $(\eta_{\mathrm{I}_\tau})^t \ge \eta_{\mathrm{I}_{\tau^t}}$.

Conversely, suppose there exist $\lambda,\mu \in I^X$ and $r \in (0,1)$ such that

$$(\eta_{\mathrm{I}_\tau})^t(\lambda,\mu) > r > \eta_{\mathrm{I}_{\tau^t}}(\lambda,\mu).$$

Since $(\eta_{1_\tau})'(\lambda,\mu) > r$, by Theorem 3.2.7, there exist two finite families $\{\lambda_j \setminus \lambda = \sup_{j\in J} \lambda_j\}$ and $\{\mu_k \setminus \mu = \inf_{k\in K} \mu_k\}$ such that

$$(\eta_{1_\tau})'(\lambda,\mu) \geq \inf_{j,k} \eta_{1_\tau}(\lambda_j,\mu_k) > r.$$

For each $j \in J$ and $k \in K$, there exists $\rho_{jk} \in I^X$ with $\lambda_j \leq \rho_{jk} \leq \mu_k$ such that $\eta_{1_\tau}(\lambda_j,\mu_k) > \tau(\rho_{jk}) > r$.

For each $k \in K$, $\lambda = \sup_{j\in J} \lambda_j \leq \sup_{j\in J} \rho_{jk} \leq \mu_k$. Put $\omega_k = \sup_{j\in J} \rho_{jk}$

Thus, $\tau(\omega_k) = \sup_{j\in J} \tau(\rho_{jk}) > r$ because J is a finite index set. Furthermore,

$$\lambda = \sup_{j\in J} \lambda_j \leq \inf_{k\in K} \omega_k \leq \inf_{k\in K} \mu_k = \mu.$$

From the definition of τ',

$$\tau' \inf_{k\in K} \omega_k \geq \inf_{k\in K} \tau(\omega_k) > r.$$

It is a contradiction. Therefore, $(\eta_{1_\tau})' \leq (\eta_{1_\tau})'$.

3.2.16 Example

Let η_2 be defined as same as in example 3.1.11.

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From theorem 3.2.3, we can obtain smooth supra topology

$$\tau_\eta(\lambda) = \begin{cases} 1, & \text{if } \lambda \in \{\underline{0},\underline{1}\} \\ 0, & \text{otherwise.} \end{cases}$$

$$\tau_{\eta'}(\lambda) = \begin{cases} 1, & \text{if } \lambda \in \{\underline{0},\underline{1}\} \\ \frac{1}{2} & \text{if } \lambda = \chi_{\{y\}} \\ 0, & \text{otherwise.} \end{cases}$$

We can show $\tau_{\eta'} \geq (\tau_\eta)' = \tau_\eta$

3.2.17 Example

Let $X=\{a,b,c,d\}$ be a set. Define a function $\eta: I^X \times I^X \to I$ as follows:

$$\eta(\lambda,\mu)=\begin{cases} 1 & \text{if} \lambda=\underline{0} \text{ or } \mu=\underline{1}, \\ \frac{2}{3} & \text{if } \underline{0}\neq\lambda\leq\chi_{\{a,b\}}, \underline{1}\neq\mu\geq\chi_{\{a,b\}}, \\ \frac{1}{2} & \text{if } \underline{0}\neq\lambda\leq\chi_{\{b,c\}}, \underline{1}\neq\mu\geq\chi_{\{b,c\}}, \\ 0 & \text{otherwise.} \end{cases}$$

Then η is a smooth semi-topogenous order on X with $\eta=\eta\circ\eta$.

$$\eta'(\lambda,\mu)=\begin{cases} 1 & \text{if} \lambda=\underline{0} \text{ or } \mu=\underline{1}, \\ \frac{2}{3} & \text{if } \underline{0}\neq\lambda\leq\chi_{\{a,b\}}, \underline{1}\neq\mu\geq\chi_{\{a,b\}}, \\ \frac{1}{2} & \text{if } \underline{0}\neq\lambda\leq\chi_{\{b,c\}}, \underline{1}\neq\mu\geq\chi_{\{b,c\}}, \\ & , \underline{1}\neq\mu\geq\chi_{\{a,b,c\}} \\ \frac{1}{2} & \text{if } \underline{0}\neq\lambda\leq\chi_{\{b\}}, \underline{1}\neq\mu\geq\chi_{\{b\}} \\ & \mu\not\geq\chi_{\{a,b\}}, \mu\not\geq\chi_{\{b,c\}} \\ 0 & \text{otherwise.} \end{cases}$$

Then η' is a smooth topogenous structure on X. from Theorem 3.2.2, we can obtain smooth supra interior operator $\mathrm{I}_\eta : I^X \times I_1 \to I^X$ as follows:

$$\mathrm{I}_\eta(\lambda,r)=\begin{cases} \underline{1} & \text{if } \lambda=\underline{1}, r\in I_1 \\ \chi_{\{a,b\}}, & \text{if } \chi_{\{a,b\}}\le\lambda\ne 1, \frac{1}{2}\le r<\frac{2}{3} \\ & \text{or } \chi_{\{a,b\}}\le\lambda\not\ge\chi_{\{a,b,c\}}, 0\le r<\frac{1}{2} \\ \chi_{\{b,c\}}, & \text{if } \chi_{\{b,c\}}\le\lambda\ne\chi_{\{a,b,c\}}, 0\le r<\frac{1}{2} \\ \chi_{\{a,b,c\}}, & \text{if } \chi_{\{a,b,c\}}\le\lambda\ne\underline{1}, 0\le r<\frac{1}{2} \\ \underline{0}, & \text{otherewise} \end{cases}$$

From Theorem 3.2.2, we obtain smooth supra topology $\tau_\eta : I^X \to I$ as follows:

$$\tau_\eta(\lambda)=\begin{cases} 1, & \text{if } \lambda\in\{\underline{0},\underline{1}\} \\ \frac{3}{2}, & \text{if } \lambda=\chi_{\{a,b\}} \\ \frac{1}{2} & \text{if } \lambda\in\{\chi_{\{b,c\}},\chi_{\{a,b,c\}}\} \\ 0 & \text{otherwise.} \end{cases}$$

We can show $\tau_{\eta^t}=(\tau_\eta)^t$ as follows:

$$(\tau_\eta)^t(\lambda)=\begin{cases} 1 & \text{if } \lambda\in\{0,1\} \\ \frac{2}{3} & \text{if } \lambda=\chi_{\{a,b\}} \\ \frac{1}{2} & \text{if } \lambda\in\{\chi_{\{b\}},\chi_{\{b,c\}},\chi_{\{a,b,c\}}\} \\ 0 & \text{othewise} \end{cases}$$

But η is not perfect, that is, the converse of Theorem 3.2.3 (5) is not true because

$$0=\eta(\chi_{\{a,b\}}\vee\chi_{\{b,c\}},\chi_{\{a,b,c\}})$$
$$\ne\eta(\chi_{\{a,b\}}\},\chi_{\{a,b,c\}})\wedge\eta(\chi_{\{b,c\}},\chi_{\{a,b,c\}})=\frac{1}{2}.$$

3.2.18 Theorem

Let $(X_i, \eta_i)_{i\in\Gamma}$ be a family of smooth topogenous spaces. Let X be a set and, for each $i \in \Gamma, f_i : X \to X_i$ a functionping. Define the function $\eta : I^X \times I^X \to I$ on X by

$$\eta(\lambda,\mu) = \sup\{\inf_{j,k} \eta_i(f_i(\lambda_j), \underline{1} - f_i(\underline{1} - \mu_k))\}$$

where for every finite families $\{\lambda_j \setminus \lambda = \sup_{j=1}^{n} \lambda_j\}$ and $\{\mu_k \setminus \mu = \sup_{k=1}^{m} \mu_k\}$.

Then:

$$\tau_\eta = \Pi_{i\in\Gamma} \tau_{f_i^{-1}(\eta_i)}.$$

Proof

Suppose $\tau_\eta \not\leq \Pi_{i\in\Gamma} \tau_{f_i^{-1}(\eta_i)}$

There exist $\lambda \in I^X$ and $r \in (0,1)$ such that

$$\tau_\eta(\lambda) > r > \Pi_{i\in\Gamma} \tau_{f_i^{-1}(\eta_i)}(\lambda).$$

There exists $r_\circ \in I_\circ$ with $r_\circ < r$ such that

$$\lambda = I_\eta(\lambda, r_\circ).$$

It implies

$$\lambda = I_\eta(\lambda, r_\circ) = \sup\{\mu \in I^X \setminus \eta(\mu,\lambda) > r_\circ\}.$$

Since $\eta(\mu,\lambda) > r_\circ$ there are finite families $\{\mu_j \setminus \mu = \sup_{j=1}^{n} \mu_j\}$ and $\{\lambda_k \setminus \lambda = \sup_{k=1}^{m} \lambda_k\}$, such that

$$\eta(\mu,\lambda) \geq \inf_{j,k} \sup_{i\in\Gamma} \eta_i(f_i(\mu_j), \underline{1} - f_i(\underline{1} - \lambda_k)) > r$$

i.e. for all j, k,

$$\sup_{i\in\Gamma}\eta_i(f_i(\mu_j),\underline{1}-f_i(\underline{1}-\lambda_k))>r_\circ .$$

It follows that for any j,k, there exists an $i_{jk}\in\Gamma$ such that

$$f_{i_{jk}}^{-1}(\eta_{i_{jk}})(\mu_j,\lambda_k)=\eta_{i_{jk}}(f_{i_{jk}}(\mu_j),\underline{1}-f_{i_{jk}}(\underline{1}-\lambda_k))>r_\circ .$$

It implies

$$I_{f_{i_{jk}}^{-1}(\eta_{i_{jk}})}(\lambda_k,r_\circ)\geq\mu_j .$$

Thus,

$$\lambda\geq\inf_{k=1}^{m}\{\sup_{j=1}^{n}I_{f_{i_{jk}}^{-1}(\eta_{i_{jk}})}(\lambda_k,r_\circ)\}\geq\mu .$$

Put

$$\rho_{i_{jk}}=I_{f_{i_{jk}}^{-1}(\eta_{i_{jk}})}(\lambda_k,r_\circ)\geq\mu_j .$$

Since

$$\begin{aligned}\lambda&=I_\eta(\lambda,r_\circ)\\&=\sup\{\mu\in I^X\setminus\eta(\mu,\lambda)>r_\circ\}\\&=\sup\{\inf_{i=1}^{m}\{\sup_{j=1}^{n}\rho_{i_{jk}}\}\}.\end{aligned}$$

Since

$$I_{f_{i_{jk}}^{-1}(\eta_{i_{jk}})}(I_{f_{i_{jk}}^{-1}(\eta_{i_{jk}})}(\lambda_k,r_\circ),r_\circ)=I_{f_{i_{jk}}^{-1}(\eta_{i_{jk}})}(\lambda_k,r_\circ),$$

$$\tau_{f_{i_{jk}}^{-1}(\eta_{i_{jk}})}(\rho_{i_{jk}})\geq r_\circ>r .$$

It implies

$$\Pi_{i\in\Gamma}\tau_{f_i^{-1}(\eta_i)}(\lambda)\geq r_\circ>r .$$

It is a contradiction.

Suppose $\tau_\eta\ngeq\Pi_{i\in\Gamma}\tau_{f_i^{-1}(\eta_i)}$, There exist $\mu_i\in I^Y$ and $r\in(0,1)$ such that

$$\tau_\eta(\lambda) < r < \Pi_{i\in\Gamma}\tau_{f_i^{-1}(\eta_i)}(\lambda).$$

There exists $\mu_i \in I^Y$ with $\sup(\inf_{j=1}^{m} f_{i_j}^{-1}(\mu_{i_j})) = \lambda$ such that

$$\tau_\eta(\lambda) < r < \inf(\inf_{j=1}^{m} \tau_{\eta_{i_j}})(\mu_{i_j})).$$

From the definition of τ_η,

$$\mu_{i_j} = I_{\eta_{i_j}})(\mu_{i_j}, r_\circ), r_\circ > r.$$

Since

$$\eta(f_i^{-1}(\rho),\lambda) \geq \eta_i(f_i(f_i^{-1}(\rho), \underline{1} - f_i(\underline{1} - \lambda)) > r_\circ,$$

$$I_\eta(\lambda, r_\circ) \geq f_i^{-1}(I_{\eta_i}(\underline{1} - f_i(\underline{1} - f_i^{-1}(\mu_i)), r_\circ).$$

Thus,

$$I_\eta(f_{i_j}^{-1}(\mu_{i_j}), r_\circ) \geq f_{i_j}^{-1}(I_{\eta_{i_j}}(\underline{1} - f_{i_j}(\underline{1} - f_{i_j}^{-1}(\mu_{i_j})), r_\circ).$$

$$= f_{i_j}^{-1}(I_{\eta_{i_j}}(\underline{1} - f_{i_j}(f_{i_j}^{-1}(\underline{1} - \mu_{i_j})), r_\circ).$$

$$= f_{i_j}^{-1}(\mu_{i_j})$$

It implies

$$(I_\eta(\inf_{j=1}^{m} f_{i_j}^{-1}(\mu_{i_j}), r_\circ) = \inf_{j=1}^{m} f_{i_j}^{-1}(\mu_{i_j}).$$

Thus

$$I_\eta(\lambda) \geq \sup I_\eta(\inf_{j=1}^{m} f_{i_j}^{-1}(\mu_{i_j}), r_\circ)$$

$$= \sup(\inf_{j=1}^{m} f_{i_j}^{-1}(\mu_{i_j}))$$

$$= \lambda.$$

Thus

$$\tau_\eta(\lambda) \geq r_\circ > r.$$

It is a contradition.

CHAPTER IV

Chapter IV

Smooth topogenous spaces compatible with smooth uniform spaces.

In this chapter we study a natural relationship between smooth topogenous structures and smooth quasi-uniformities. The family $\Pi(\eta)$ of all smooth quasi-uniformities U compatible with smooth topogenous structure η on X is neverempty and it contains smooth quasi-uniformity U_η which is the coarsest member of $\Pi(\eta)$.

4.1 Smooth topogenous spaces induced by smooth uniform spaces.

4.1.1 Lemma

To every $\alpha \in \Omega_X$, we define $\eta_\alpha : I^X \times I^X \to I$ as

$$\eta_\alpha(\mu, \lambda) = \begin{cases} 1 & \text{if } \lambda \geq \alpha(\mu), \\ 0 & \text{otherwise.} \end{cases}$$

Then it satisfies the following properties:

(1) The function $\eta_\alpha \in \Upsilon_X$ is a biperfect smooth topogenous order.

(2) If $\alpha \leq \beta$, then $\eta_\beta \leq \eta_\alpha$.

(3) If $\beta \leq \alpha_1 \wedge \alpha_2$, then $\eta_{\alpha_1}, \eta_{\alpha_2} \leq \eta_\beta$.

(4) For each $\alpha \in \Omega_X$, we have $\eta_\alpha^s = \eta_{\alpha^{-1}}$.

(5) If $\beta \circ \beta \leq \alpha$, then $\eta_\beta \circ \eta_\beta \geq \eta_\alpha$.

Proof

(1) Since $\alpha(\underline{1}) = \underline{1}$ and $\alpha(\underline{0}) = \underline{0}$, then $\eta_\alpha(\underline{1}, \underline{1}) = \eta_\alpha(\underline{0}, \underline{0}) = 1$. Let

$$\eta_\alpha(\mu, \lambda) \neq 0.$$

Then $\eta_\alpha(\mu,\lambda)=1$ implies $\mu \le \alpha(\mu) \le \lambda$. Since $\mu \le \mu_1$ and $\lambda \le \lambda_1$ implies $\alpha(\mu) \le \alpha(\mu_1)$, then

$$\eta_\alpha(\mu_1,\lambda_1) \le \eta_\alpha(\mu,\lambda).$$

To prove the biperfect condition, since $\alpha(\sup_{i\in\Gamma} \mu_i) = \sup_{i\in\Gamma} \alpha(\mu_i) \le \lambda$ iff

$\mu_i \le \lambda$ for all $i \in \Gamma$,

$$\eta_\alpha(\sup_{i\in\Gamma} \mu_i, \lambda) = \inf_{i\in\Gamma} \eta_\alpha(\mu_i,\lambda).$$

Since $\mu \le \inf_{j\in\Lambda} \lambda_j$ iff $\mu \le \lambda_j$ for all $j \in \Lambda$,

$$\eta_\alpha(\mu, \inf_{j\in\Lambda} \lambda_j) = \inf_{j\in\Lambda} \eta_\alpha(\mu,\lambda_j).$$

Others are similarly proved.

(2) Since $\alpha(\mu) \le \beta(\mu)$, $\eta_\beta \le \eta_\alpha$.

(3) Since $\alpha_1 \wedge \alpha_2(\mu) \le \alpha_1(\mu) \wedge \alpha_2(\underline{0})$, we have $\alpha_1 \wedge \alpha_2 \le \alpha_1$. From (2), $\eta_{\alpha_1} \le \eta_\beta$. Similarly $\eta_{\alpha_2} \le \eta_\beta$.

(4) It is easily proved from $\alpha^{-1}(\mu) \le \lambda$ iff $\alpha(\underline{1}-\lambda) \le \underline{1}-\mu$.

(5) From (2), we only show that $\eta_\beta \circ \eta_\beta = \eta_{\beta\circ\beta}$. Since

$\eta_\beta \circ \eta_\beta(\mu,\lambda) = \sup_{\rho \in I^X} \{\eta_\beta(\mu,\rho) \wedge \eta_\beta(\rho,\lambda)\}$, we have

$$\eta_\beta \circ \eta_\beta(\mu,\lambda) = \begin{cases} 1 & \text{if } \lambda \ge \beta(\beta(\mu)) \\ 0 & \text{otherwise.} \end{cases}$$

From Lemma 4.1.1, we easily prove the following theorem.

4.1.2 Theorem

Let $B:\Omega_X \to I$ be a smooth quasi-uniform (resp. smooth uniform) base on X. Define $S_B:\Upsilon_X \to I$ as

$$S_B(\eta_\alpha)=B(\alpha).$$

Then S_B is a smooth (resp. smooth symmetric) syntopogenous structure on X.

4.1.3 Theorem

Let (X,B_1) and (Y,B_2) be smooth quasi-uniform bases. Then $B_2(\alpha)\le B_1(f^{\leftarrow}(\alpha))$ for each $\alpha\in\Omega_Y$ iff $f:(X,S_{B_1})\to(Y,S_{B_2})$ is syntopogenous continuous.

Proof

Let $\eta_\alpha \in \Upsilon_Y$ be given. Since $f^{\leftarrow}(\alpha)(\rho)=f^{-1}(\alpha(f(\rho)))$

$$\eta_{f^{\leftarrow}(\alpha)}(f^{-1}(\mu),f^{-1}(\lambda))=\begin{cases}1 & \text{if } \lambda\ge f^{-1}(\alpha(f(f^{-1}(\mu))))\\ 0 & \text{otherwise.}\end{cases}$$

Since $\lambda\ge\alpha(\mu)$ implies

$$f^{-1}(\lambda)\ge f^{-1}(\alpha(\mu))\ge f^{-1}(\alpha(f(f^{-1}(\mu)))).$$

We have

$$\eta_{f^{\leftarrow}(\alpha)}(f^{-1}(\mu),f^{-1}(\lambda))\ge\eta_\alpha(\mu,\lambda).$$

Since $B_2(\alpha)\le B_1(f^{\leftarrow}(\alpha))$, $S_{B_1}(\eta_\alpha)\le S_{B_2}(\eta_{f^{\leftarrow}(\alpha)})$.

Thus, f is smooth syntopogenous continuous.

Conversely, since $B_{S_{B_i}}=B_i$ for $i\in\{1,2\}$, it is easily proved.

4.1.4 Theorem

Let (X,U) be smooth quasi-uniform space. Define

$$\eta_U(\mu,\rho)=\sup\{U(\alpha)\setminus\alpha(\mu)\leq\rho\}.$$

Then (X,η_U) is smooth topogenous space. If (X,U) is smooth uniform space,

Then (X,η_U) is a symmetric smooth topogenous space.

Proof

(ST1) From (FQU4), there exists $\alpha\in\Omega_X$ such that $U(\alpha)=1$. Since $\alpha(\underline{1})=\underline{1}$ and $\alpha(\underline{0})=\underline{0}$, then $\eta_U(\underline{1},\underline{1})=\eta_U(\underline{0},\underline{0})=1$.

(ST2) If $\eta_U(\mu,\rho)\neq 0$, then there exists $\alpha\in\Omega_X$ such that $U(\alpha)\neq 0$ and $\alpha(\mu)\leq\rho$. It implies $\mu\leq\rho$.

(ST3) If $\lambda\leq\lambda_1$ and $\mu_1\leq\mu$, then for each $\alpha\in\Omega_X$ with $\alpha(\lambda_1)\leq\mu_1$, we have $\alpha(\lambda)\leq\alpha(\lambda_1)\leq\mu_1\leq\mu$. Thus, $\eta_U(\lambda_1,\mu_1)\leq\eta_U(\lambda,\mu)$.

(ST4) Suppose there exist $\lambda_1,\lambda_2,\mu\in I^X$ such that

$$\eta_U(\lambda_1\vee\lambda_2,\mu)\not\geq\eta_U(\lambda_1,\mu)\wedge\eta_U(\lambda_2,\mu).$$

From the definition of $\eta_U(\lambda_i,\mu)$ for $i\in\{0,1\}$, there exists $\alpha_i\in\Omega_X$ with $\alpha_i(\lambda_i)\leq\mu$ such that

$$\eta_U(\lambda_1\vee\lambda_2,\mu)\not\geq U(\alpha_1)\wedge U(\alpha_2).$$

On the other hand, since $(\alpha_1\wedge\alpha_2)(\lambda_1\vee\lambda_2)\leq\alpha_1(\lambda_1)\vee\alpha_2(\lambda_2)\leq\mu$, then

$$\eta_U(\lambda_1\vee\lambda_2,\mu)\geq U(\alpha_1\wedge\alpha_2).$$

Since $U(\alpha_1\wedge\alpha_2)\geq U(\alpha_1)\wedge U(\alpha_2)$, it is a contradiction.

Thus, $\eta_U(\lambda_1\vee\lambda_2,\mu)\geq\eta_U(\lambda_1,\mu)\wedge\eta_U(\lambda_2,\mu)$.

(ST5) Suppose there exist $\lambda,\mu_1,\mu_2\in I^X$ such that

$$\eta_U(\lambda,\mu_1\wedge\mu_2)\not\geq\eta_U(\lambda,\mu_1)\wedge\eta_U(\lambda,\mu_2).$$

From the definition of $\eta_U(\lambda,\mu_i)$ for $i\in\{0,1\}$, there exists $\alpha_i\in\Omega_X$ with $\alpha_i(\lambda)\leq\mu_i$ such that

$$\eta_U(\lambda,\mu_1 \wedge \mu_2) \not\geq U(\alpha_1) \wedge U(\alpha_2).$$

Since $(\alpha_1 \wedge \alpha_2)(\lambda) \leq \alpha_1(\lambda) \wedge \alpha_2(\lambda) \leq \mu_1 \wedge \mu_2$, $\eta_U(\lambda,\mu_1 \wedge \mu_2) \geq U(\alpha_1 \wedge \alpha_2)$. Since $U(\alpha_1 \wedge \alpha_2) \geq U(\alpha_1) \wedge U(\alpha_2)$, it is a contradiction.

(ST6) Suppose there exist $\mu, \rho \in I^X$ such that

$$\eta_U \circ \eta_U(\mu,\rho) \not\geq \eta_U(\mu,\rho).$$

From the definition of $\eta_U(\mu,\rho)$, there exists $\alpha \in \Omega_X$ with $\alpha(\mu) \leq \rho$ such that

Since $\sup\{U(\beta) \setminus \beta \circ \beta \leq \alpha\}$, there exists $\beta \in \Omega_X$ with $\beta \circ \beta(\mu) \leq \alpha(\mu) \leq \rho$ such that

$$\eta_U \circ \eta_U(\mu,\rho) \not\geq U(\beta).$$

On the other hand, since $\beta(\mu) = \beta(\mu)$ and $\beta \circ \beta(\mu) \leq \rho$, we have

$$\eta_U \circ \eta_U(\mu,\alpha(\mu)) \not\geq U(\beta).$$

Hence, $\eta_U \circ \eta_U(\mu,\rho) \not\geq U(\beta)$. It is a contradiction.

Let (X,U) be smooth uniform space. From Lemma 4.1.1, since $\alpha(\mu) \leq \rho$ iff $\alpha^{-1}(\rho') \leq \mu'$ and $U(\alpha) = U(\alpha^{-1})$, we have $\eta_U = \eta_U^s$. Hence, (X,η_U) is a symmetric smooth topogenous space.

4.2 Smooth uniform spaces induced by smooth topogenous spaces.

4.2.1 Lemma

Let Υ_X be asmooth biperfect syntopogenous structure on X where for each $\eta \in \Upsilon_X$ the range of η is finite. We define as

$$\alpha_\eta(\mu) = \inf\{\lambda \in I^X \setminus \eta(\mu,\lambda) > 0\}.$$

Then it satisfies the following conditions:

(1) $\alpha \in \Omega_X$,

(2) If $\eta \le \zeta$ and $\alpha_\eta \in \Omega_X$, then $\alpha_\zeta \le \alpha_\eta$,

(3) If $\gamma, \zeta \le \eta$ and $\alpha_\eta, \alpha_\gamma \in \Omega_X$, then $\alpha_\eta \le \alpha_\zeta \wedge \alpha_\gamma$,

(4) $\alpha_{\eta^s} = (\alpha_\eta)^{-1}.$,

(5) For each $\alpha_\eta \in \Omega_X$, there exists $\alpha_\eta \in \Omega_X$ such that $\alpha_\zeta \circ \alpha_\zeta \le \alpha_\eta$

(6) $\alpha_{\eta_\alpha} = \alpha$.

Proof

Since $\eta(\underline{0},\underline{0}) = 1, \alpha_\eta(\underline{0}) = 0$. Since $\eta(\mu,\lambda) > 0$, then $\mu \le \lambda$.

Thus, $\mu \le \alpha_\eta(\mu)$.

Since the range of η is finite,

$$\inf_{i\in\Gamma} \eta(\mu_i,\lambda) > 0 \text{ iff } \eta(\mu_i,\lambda) > 0, \forall i \in \Gamma.$$

It implies

$$\begin{aligned}\alpha_\eta(\sup_{i\in\Gamma}\mu_i) &= \inf\{\lambda / \eta(\sup_{i\in\Gamma}\mu_i,\lambda) > 0\}\\ &= \inf\{\lambda / \inf_{i\in\Gamma}\eta(\mu_i,\lambda) > 0\}\\ &= \sup_{i\in\Gamma}(\inf\{\lambda / \eta(\mu_i,\lambda) > 0\})\\ &= \sup_{i\in\Gamma}\alpha_\eta(\mu_i).\end{aligned}$$

(2) Since $\zeta(\mu,\lambda) \ge \eta(\mu,\lambda) > 0$, we have $\alpha_\zeta \le \alpha_\eta$.

(3) From (2), we only show that $\alpha_1, \alpha_2 \ge \beta$ implies $\alpha_1 \wedge \alpha_2 \ge \beta$ for each $\alpha_1, \alpha_2, \beta \in \Omega_X$.

Suppose there exists $\mu \in I^X$ and $t \in (0,1)$ such that

$$(\alpha_1 \wedge \alpha_2)(\mu)(x) < t < \beta(\mu)(x). \tag{E}$$

Since $(\alpha_1 \wedge \alpha_2)(\mu)(x) < t$, there exist $\mu_1, \mu_2 \in I^X$ with $\mu = \mu_1 \vee \mu_2$ such that

$$(\alpha_1 \wedge \alpha_2)(\mu)(x) \le \alpha_1(\mu_1)(x) \vee \alpha_2(\mu_2)(x) < t.$$

On the other hand,

$$\beta(\mu) = \beta(\mu_1) \vee \beta(\mu_2) \le \alpha_1(\mu_1) \vee \alpha_2(\mu_2).$$

Thus, $\beta(\mu)(x) < t$. It is a contradiction for the equation (E).

(4) Since $\alpha_\eta{}^{-1}(\mu) = \inf\{\lambda \in I^X \setminus \alpha_\eta(\underline{1} - \lambda) \le \underline{1} - \mu\}$, and

$\alpha_{\eta^s}(\mu) = \inf\{\lambda \in I^X \setminus \eta(\underline{1} - \lambda, \underline{1} - \mu) > 0\}$, we only show that

$$\alpha_\eta(\underline{1} - \lambda) \le \underline{1} - \mu \text{ iff } \eta(\underline{1} - \lambda, \underline{1} - \mu) > 0.$$

($\Leftarrow$) It is trivial.

($\Rightarrow$) Since $\alpha_\eta(\underline{1} - \lambda) = \inf\{\rho_i \in I^X \setminus \eta(\underline{1} - \lambda, \rho_i) > 0\}$, we have the following:

$$\begin{aligned} \eta(\underline{1} - \lambda, \underline{1} - \mu) &\ge \eta(\underline{1} - \lambda, \alpha_\eta(\underline{1} - \lambda)) \\ &= \eta(\underline{1} - \mu, \inf \rho_i) \\ &= \inf \eta(\underline{1} - \mu, \rho_i) > 0, \end{aligned}$$

because the range of η is finite.

(5) Let $\alpha_\eta \in \Omega_X$. For $\eta \in \Upsilon_X$ and $\eta(\lambda, \mu) > 0$, there exists $\zeta \in \Upsilon_X$ such that $\zeta \circ \zeta \ge \eta$. Since $\zeta \circ \zeta(\mu, \lambda) > 0$, there exists $\rho \in I^X$ such that

$$\zeta(\mu, \rho) \wedge \zeta(\mu, \rho) > 0.$$

It implies

$$\alpha_\zeta(\mu) \le \rho, \alpha_\zeta(\rho) \le \lambda.$$

Thus, $\alpha_\zeta(\alpha_\zeta(\mu)) \le \lambda$. Hence, $\alpha_\zeta(\alpha_\zeta(\mu)) \le \alpha_\zeta(\mu)$. Thus, $\alpha_\zeta \circ \alpha_\zeta \le \alpha_\eta$.

(6) $$\alpha_{\eta_\alpha}(\mu) = \inf\{\lambda / \eta_\alpha(\mu,\lambda) > 0\}$$
$$= \inf\{\lambda / \eta_\alpha(\mu,\lambda) = 1\}$$
$$= \inf\{\lambda / \lambda) \geq \alpha(\mu)\}$$
$$= \alpha(\mu).$$

From Lemma 4.3.13, we easily prove the following theorem.

4.2.2 Theorem

Let $S : \Upsilon_X \to I$ be a smooth (resp. symmetric) syntopogenous structure on X where for each $\eta \in \Upsilon_X$ the range of η is finite. Define $B_S : \Omega_X \to I$ as

$$B_S(\alpha_\eta) = \sup\{S(\eta) \setminus \eta \text{ induces } \alpha_\eta\}.$$

Then:

(1) B_S is a smooth quasi-uniform (resp. smooth uniform) base on X.

(2) If $B : \Omega_X \to I$ is a smooth quasi-uniform base on X, then $B_{S_B} = B$.

4.2.3 Theorem

Let (X, S_1) and (Y, S_2) be smooth syntopogenous spaces. Let $f : (X, S_1) \to (Y, S_2)$ be smooth syntopogenous continuous then we have the following properties.

(1) If the ranges of η and ζ are finite sets for each $\eta \in \Upsilon_X$ and $\zeta \in \Upsilon_Y$, then $f : (X, U_{S_1}) \to (Y, U_{S_2})$ is smooth quasi-uniform continuous where U_{S_i} is generated by B_{S_i} for $i \in \{0,1\}$.

(2) $f(C_{S_1}(\lambda, r)) \leq C_{S_2}(f(\lambda), r)$, for each $\lambda \in I^X$.

(3) $C_{S_1}(f^{-1}(\mu), r) \leq f^{-1}(C_{S_2}(f(\mu), r))$, for each $\mu \in I^Y$.

(4) $f : (X, \tau_{S_1}) \to (Y, \tau_{S_2})$ is smooth continuous.

Proof

(1) From Theorem 4.3.10, we show that

$$B_{S_2}(\alpha_\zeta) \le B_{S_1}(f^{\leftarrow}(\alpha_\zeta)).$$

Since $f^{\leftarrow}(\alpha_\zeta)(\lambda) = f^{-1}(\alpha_\zeta(f(\lambda)))$,

$$f^{-1}(\alpha_\zeta(f(\lambda))) = f^{-1}(\inf\{\rho / \zeta(f(\lambda),\rho) > 0\})$$
$$= \inf\{f^{-1}(\rho) / \zeta(f(\lambda),\rho) > 0\}).$$

Since f is smooth syntopogenous continuous, for each $\zeta \in \Upsilon_Y$, there exists $\eta \in \Upsilon_X$ with $\eta(f^{-1}(f(\lambda)), f^{-1}(\rho)) \ge \zeta(f(\lambda),\rho)$ such that $S_2(\zeta) \le S_1(\eta)$.

Since $\eta(\lambda, f^{-1}(\rho)) \ge \eta(f^{-1}(f(\lambda)), f^{-1}(\rho))$, $f^{\leftarrow}(\alpha_\zeta)(\lambda) = \alpha_\zeta(\lambda)$. It implies

$$B_{S_1}(f^{\leftarrow}(\alpha_\zeta)) \ge B_{S_1}(\alpha_\eta)$$
$$\ge B_{S_2}(\alpha_\zeta).$$

(2) Suppose there exists $\lambda \in I^X$ and $r \in I_1$ such that

$$f(C_{S_1}(\lambda, r)) > C_{S_2}(f(\lambda), r)).$$

There exists $y \in Y$ and $t \in I_\circ$ such that

$$f(C_{S_1}(\lambda, r))(y) > t > C_{S_2}(f(\lambda), r))(y).$$

Since $f^{-1}(\{y\}) = \phi$, provides a contradiction that $f(C_{S_1}(\lambda, r))(y) = 0$,

$f^{-1}(\{y\}) \ne \phi$, and there exists $x \in f^{-1}(\{y\})$ such that

$$f(C_{S_1}(\lambda, r))(y) \ge C_{S_1}(\lambda, r)(x) > t > C_{S_2}(f(\lambda), r))(f(x)).$$

Since $C_{S_2}(f(\lambda), r))(f(x)) < t$, there exists $\zeta \in \Upsilon_Y$ with $S_2(\zeta) > r$ and $\zeta(f(\lambda), \mu) > 0$ such that

$$C_{S_2}(f(\lambda), r))(f(x)) \le \mu(f(x)) < t.$$

On the other hand, since f is syntopogenous continuous, for each $\zeta \in \Upsilon_Y$.

There exists $\eta \in \Upsilon_X$ with $\eta(f^{-1}(f(\lambda), f^{-1}(\mu)) \geq \zeta(f(\lambda)\mu)$ such that

$$S_1(\eta) \geq S_2(\zeta) > r.$$

It implies, $C_{S_1}(\lambda, r)(x) \leq f^{-1}(\mu)(x) < t$. It is a contradiction.

4.2.4 Lemma

Let (X, η) be smooth topogenous space. Let

$$\eta^\circ = \{(\mu, \rho) \in I^X \times I^X \setminus \eta(\mu, \rho) \neq 0\}.$$

For every $(\mu, \rho) \in \eta^\circ$ we define $\alpha_{\mu,\rho} : I^X \to I^X$ as follows:

$$\alpha_{\mu,\rho}(\lambda) = \begin{cases} \underline{0} & \text{if } \lambda = \underline{0}, \\ \rho & \text{if } \underline{0} \neq \lambda \leq \rho, \\ \underline{1} & \text{otherwise.} \end{cases}$$

We have the following statements.

(1) $\alpha_{\mu,\rho} \in \Omega_X$.

(2) If $\lambda \leq \mu, \upsilon \leq \rho$ and $\alpha_{\mu,\upsilon} \in \Omega_X$, then $\alpha_{\mu,\upsilon} \leq \alpha_{\lambda,\rho}$.

(3) For each $\alpha_{\mu,\rho}$ there exists $\upsilon \in I^X$ such that $\alpha_{\mu,\rho} \circ \alpha_{\mu,\rho} = \alpha_{\mu,\rho}$.

(4) If (X, η) is a symmetric smooth topogenous space and $\alpha_{\mu,\rho} \in \Omega_X$, then

$$(\alpha_{\mu,\rho})^{-1} = \alpha_{\mu',\rho'}.$$

(5) For each $i = 1, \ldots, n, \alpha_{\mu_i,\rho_i}$ with $(\mu_i, \rho_i) \in \eta^\circ$, denote

$$\Gamma = \{J \subset \{1, \ldots, n\} \setminus \lambda \leq \sup_{j \in J} \mu_j\}$$

and put $\tau_J = \sup_{j \in J} \rho_j$ for any nonempty subset J of $\{1, \ldots, n\}$. Then

$$\inf_{i=1}^{n} \alpha_{\mu_i,\rho_i}(\lambda) = \begin{cases} \underline{0} & \text{if } \lambda = \underline{0}, \\ \inf_{j\in\Gamma} \tau_J & \text{if } \Gamma \neq \phi, \\ \underline{1} & \text{if } \Gamma = \phi. \end{cases}$$

Proof

(1) From the definition of $\alpha_{\mu,\rho}$, we have $\alpha_{\mu,\rho}(\underline{0}) = \underline{0}$. If $0 \neq \lambda \leq \mu$, then $\alpha_{\mu,\rho}(\lambda) = \rho$. Since $(\mu,\rho) \in \eta^\circ$, that is, $\eta(\mu,\rho) \neq 0$, by (T2), $\mu \leq \rho$. Hence, $\lambda \leq \alpha_{\mu,\rho}(\lambda)$. If $\lambda \not\leq \mu$, then $\lambda \leq \alpha_{\mu,\rho}(\lambda) = \underline{1}$. It follows that $\lambda \leq \alpha_{\mu,\rho}(\lambda)$.

Finally, we easily show that $\alpha_{\mu,\rho}(\sup_{i\in\Gamma} \upsilon_i) = \sup_{i\in\Gamma} \alpha_{\mu,\rho}(\upsilon_i)$ from the following conditions (a) and (b):

(a) $\sup_{i\in\Gamma} \upsilon_i \leq \mu$ iff for all $i \in \Gamma$, $\upsilon_i \leq \mu$,

(b) $\sup_{i\in\Gamma} \upsilon_i \not\leq \mu$ iff for some $i \in \Gamma$, $\upsilon_i \not\leq \mu$. Hence, $\alpha_{\mu,\rho} \in \Omega_X$.

(2) From definitions of $\alpha_{\mu,\rho}$ and $\alpha_{\lambda,\rho}$ it is trivial.

(3) From (T6), since $\eta \circ \eta(\mu,\rho) = \sup_{\upsilon \in I^X} (\eta(\mu,\upsilon) \wedge \eta(\upsilon,\rho)) \geq \eta(\mu,\rho) \neq 0$, there exists $\upsilon \in I^X$ such that $\eta(\mu,\upsilon) \neq 0$ and $\eta(\upsilon,\rho) \neq 0$. Hence, $\alpha_{\mu,\upsilon}, \alpha_{\upsilon,\rho} \in \Omega_X$.

Moreover, it is easily proved $\alpha_{\upsilon,\rho} \circ \alpha_{\mu,\upsilon}(\lambda) = \alpha_{\mu,\rho}(\lambda)$ for any $\lambda \in I^X$.

(4) Since (X,η) is a symmetric smooth topogenous space and $\alpha_{\mu,\rho} \in \Omega_X$, then $\eta(\mu,\rho) = \eta(\rho',\mu') \neq 0$. It follows that $\alpha_{\rho',\mu'} \in \Omega_X$. We show that

$$\alpha_{\mu,\rho}(\lambda) = \alpha_{\rho',\mu'}(\lambda).$$

From the following statements (a),(b) and (c):

(a) If $\lambda = \underline{0}$, then $(\alpha_{\mu,\rho})^{-1}(\underline{0}) = \inf\{\upsilon \setminus \alpha_{\mu,\rho}(\upsilon') \leq \underline{1}\} = \underline{0} = \alpha_{\rho',\mu'}(\underline{0})$

(b) If $\underline{0} \neq \lambda \leq \rho'$, then, by the definition of $\alpha_{\mu,\rho}$, we have

$$\alpha_{\mu,\rho}(\upsilon') \leq \lambda' \text{ iff } \alpha_{\mu,\rho}(\upsilon') \leq \rho \text{ iff } \upsilon' \leq \mu.$$

Hence,

$$\begin{aligned}(\alpha_{\mu,\rho})^{-1}(\lambda) &= \inf\{\upsilon \in I^X \setminus \alpha_{\mu,\rho}(\upsilon') \leq \lambda'\} \\ &= \inf\{\upsilon \in I^X \setminus \upsilon \geq \mu'\} \\ &= \mu' \\ &= \alpha_{\rho',\mu'}(\lambda).\end{aligned}$$

(c) If $\lambda = \underline{0}$ and $\alpha_{\mu,\rho}(\upsilon') \leq \lambda'$, then, by the definition of $\alpha_{\mu,\rho}$, we only have

$\alpha_{\mu,\rho}(\upsilon') = \underline{0}$. It implies that $\upsilon = \underline{1}$. Hence, $(\alpha_{\mu,\rho})^{-1}(\lambda) = \alpha_{\rho',\mu'}(\lambda) = 1$.

(5) If $\lambda = \underline{0}$ or $\Gamma = \phi$, then it is trivial.

We only show that for $\Gamma \neq \phi$, $\inf_{i=1}^{n} \alpha_{\mu_i,\rho_i}(\lambda) = \inf_{J \in \Gamma} \tau_J$.

Suppose $\inf_{i=1}^{n} \alpha_{\mu_i,\rho_i}(\lambda) \not\leq \inf_{J \in \Gamma} \tau_J$ Since $\Gamma \neq \phi$, there exists $J \in \Gamma$ with $\lambda \leq \sup_{j \in J} \mu_j$ such that

$$\inf_{i=1}^{n} \alpha_{\mu_i,\rho_i}(\lambda) \not\leq \tau_J.$$

Put for $i \in \{1,...,n\}$,

$$\lambda_i = \begin{cases} \lambda \wedge \mu_i & \text{if } i \in J, \\ \underline{0} & \text{otherwise.} \end{cases}$$

Since $\lambda = \sup_{i \in J} \lambda_i$ and $\lambda_i \leq \mu_i$ for all $i \in J$, we obtain

$$\inf_{i=1}^{n} \alpha_{\mu_i,\rho_i}(\lambda) \leq \sup_{i=1}^{n} \alpha_{\mu_i,\rho_i}(\lambda_i) \leq \sup_{i\in J} \rho_i = \tau_J .$$

It is a contradiction. Hence, $\inf_{i=1}^{n} \alpha_{\mu_i,\rho_i}(\lambda) \leq \inf_{J\in\Gamma} \tau_J$.

Suppose $\inf_{i=1}^{n} \alpha_{\mu_i,\rho_i}(\lambda) \not\geq \inf_{J\in\Gamma} \tau_J$. There exist $\lambda_i \in I^X$ with $\lambda = \sup_{i=1}^{n} \lambda_i$ such that

$$\sup_{i=1}^{n} \alpha_{\mu_i,\rho_i}(\lambda_i) \not\geq \inf_{J\in\Gamma} \tau_J .$$

Put $\upsilon = \sup_{i=1}^{n} \alpha_{\mu_i,\rho_i}(\lambda_i)$ and $K = \{k \in \{1,\ldots,n\} \setminus \rho_k \leq \upsilon\}$. We obtain $\tau_K \leq \upsilon$.

If $i \notin K$, then $\rho_i \not\leq \upsilon$. Hence, $\alpha_{\mu_i,\rho_i}(\lambda_i) = \underline{0}$, which implies $\lambda_i = \underline{0}$.

If $k \in K$, then $\lambda_k \leq \mu_k$ because $\alpha_{\mu_k,\rho_k}(\lambda_k) \neq \underline{1}$.

It implies that

$$\lambda = \sup_{i=1}^{n} \lambda_i = \sup_{k\in K} \lambda_k \leq \sup_{k\in K} \mu_k .$$

Then there exists $K \in \Gamma$ such that

$$\sup_{i=1}^{n} \alpha_{\mu_i,\rho_i}(\lambda_i) = \upsilon \geq \tau_K \geq \inf_{K\in\Gamma} \tau_K .$$

It is a contradiction. Hence, $\inf_{i=1}^{n} \alpha_{\mu_i,\rho_i}(\lambda) \geq \inf_{J\in\Gamma} \tau_J$.

4.2.5 Example

For each $i = 1,2$, α_{μ_i,ρ_i} with $(\mu_i,\rho_i) \in \eta^\circ$, we have

$$\alpha_{\mu_1,\rho_1} \wedge \alpha_{\mu_2,\rho_2} = \begin{cases} \underline{0} & \text{if } \lambda = \underline{0}, \\ \rho_1 \wedge \rho_2 & \text{if } 0 \neq \lambda \leq \mu_1 \wedge \mu_2, \\ \rho_1 & \text{if } \lambda \leq \mu_1, \lambda \not\leq \mu_2, \\ \rho_2 & \text{if } \lambda \not\leq \mu_1, \lambda \leq \mu_2, \\ \rho_1 \vee \rho_2 & \text{if} \lambda \leq \mu_1 \vee \mu_2, \lambda \not\leq \mu_1, \lambda \not\leq \mu_2, \\ \underline{1} & \text{otherwise.} \end{cases}$$

4.2.6 Theorem

Let (X,η) be smooth topogenous space. Define a function $U_\eta : \Omega_X \to I$ by

$$U_\eta(\alpha) = \sup\{\inf_{i=1}^{n} \eta(\mu_i,\rho_i) \setminus \inf_{i=1}^{n} \alpha_{\mu_i,\rho_i} \leq \alpha\}.$$

Where the supremum is taken over every finite family $\{\alpha_{\mu_i,\rho_i} \setminus i = 1,\ldots,n\}$.

Then U_η is smooth quasi-uniformity on X. If (X,η) is a symmetric smooth topogenous space, U_η is smooth uniformity on X.

Proof

(SQU1) It is trivial from the definition of U_η.

(SQU2) Suppose there exist $\alpha,\beta \in \Omega_X$ such that

$$U_\eta(\alpha \wedge \beta) \not\geq U_\eta(\alpha) \wedge U_\eta(\beta).$$

There exist finite families $\{\inf_{i=1}^{m} \alpha_{\mu_i,\rho_i} \leq \alpha\}$ and $\{\inf_{j=1}^{n} \beta_{\upsilon_j,w_j} \leq \beta\}$ such that

$$U_\eta(\alpha \wedge \beta) \not\geq (\inf_{i=1}^{m} \eta(\mu_i,\rho_i)) \wedge (\inf_{j=1}^{n} \eta(\upsilon_j,w_j)).$$

Since $\alpha \wedge \beta \geq (\inf_{i=1}^{m} \alpha_{\mu_i,\rho_i}) \wedge (\inf_{j=1}^{n} \beta_{\upsilon_j,w_j})$, we have

$$U_\eta(\alpha \wedge \beta) \geq U_\eta(\alpha) \wedge U_\eta(\beta).$$

It is a contradiction.

(SQU3) Suppose there exists $\alpha \in \Omega_X$ such that

$\sup\{U_\eta(\beta) \setminus \beta \circ \beta \leq \alpha\} \ngeq U_\eta(\alpha)$

Put $t = \sup\{U_\eta(\beta) \setminus \beta \circ \beta \leq \alpha\}$. From the definition of $U_\eta(\alpha)$, there exists a family $\{\alpha_{\mu_i,\rho_i} \setminus \inf_{i=1}^{m} \alpha_{\mu_i,\rho_i} \leq \alpha\}$ such that

$$t \ngeq \inf_{i=1}^{m} \eta(\mu_i, \rho_i).$$

Since $t \ngeq \eta(\mu_i, \rho_i)$, for all $i = 1,\dots,m$ by (T6), we have

$$t \ngeq (\eta \circ \eta)(\mu_i, \rho_i) = \sup_{\upsilon \in I^X} (\eta(\mu_i, \upsilon) \wedge \eta(\upsilon, \rho_i)).$$

There exists $\upsilon_i \in I^X$ such that

$$t \ngeq \inf_{i=1}^{m} (\eta(\mu_i, \upsilon_i) \wedge \eta(\upsilon_i, \rho_i)).$$

On the other hand, let $\beta_i = \alpha_{\upsilon_i,\rho_i} \wedge \alpha_{\mu_i,\upsilon_i}$, be given. It satisfies

$$\beta_i \circ \beta_i \leq \alpha_{\upsilon_i,\rho_i} \wedge \alpha_{\mu_i,\upsilon_i} = \alpha_{\mu_i,\rho_i},\ U_\eta(\beta) = \eta(\mu_i, \upsilon_i) \wedge \eta(\upsilon_i, \rho_i).$$

Let $\beta = \inf_{i=1}^{n} \beta_i$ be given. Since $\beta_i \circ \beta_i \leq \alpha_{\mu_i,\rho_i}$, for each $i = 1,\dots,m$, we have

$$(\inf_{i=1}^{n} \beta_i) \circ (\inf_{i=1}^{n} \beta_i) \leq \beta_i \circ \beta_i \leq \alpha_{\mu_i,\rho_i}.$$

Hence, by Lemma 1.1.5, we have

$$(\inf_{i=1}^{n} \beta_i) \circ (\inf_{i=1}^{n} \beta_i) \leq \inf_{i=1}^{n} \alpha_{\mu_i,\rho_i} \leq \alpha.$$

Then we have $\beta \circ \beta \leq \alpha$ and

$$U_\eta(\beta) \geq \inf_{i=1}^{m} U_\eta(\beta_i).$$

Thus,

$$U_\eta(\beta) \geq \inf_{i=1}^{m} (\eta(\mu_i, \upsilon_i) \wedge \eta(\upsilon_i, \rho_i)).$$

It implies, $\sup\{U_\eta(\beta) \setminus \beta \circ \beta \leq \alpha\} \geq t$. It is a contradiction.

(SQU4) Since $\eta(\underline{1},\underline{1}) = 1$, there exists $\alpha_{\underline{1},\underline{1}} \in \Omega_X$ such that $U_\eta(\alpha_{\underline{1},\underline{1}}) = 1$.

Then U_η is smooth quasi-uniformity on X.

Let (X, η) be a symmetric smooth topogenous space.

(SU) Suppose there exists $\alpha \in \Omega_X$ such that

$$\sup\{U_\eta(\beta) \setminus \beta \leq \alpha^{-1}\} \not\geq U_\eta(\alpha).$$

Put. $s = \sup\{U_\eta(\beta) \setminus \beta \leq \alpha^{-1}\}$. Since $U_\eta(\alpha) \not\leq s$, there exists a finite family $\{\alpha_{\mu_i,\rho_i} \setminus \inf_{i=1}^{n} \alpha_{\mu_i,\rho_i} \leq \alpha\}$ such that

$$\inf_{i=1}^{n} \eta(\mu_i, \rho_i) \not\leq s.$$

Since $\eta(\mu_i, \rho_i) \not\leq s$ for all $1 \leq i \leq n$ and (X, η) is a symmetric smooth topogenous space, we have $\eta(\mu_i, \rho_i) = \eta(\rho_i', \mu_i')$.

Since $(\alpha_{\mu_i,\rho_i})^{-1} = \alpha_{\rho_i',\mu'_i}$ from Lemma 4.2.4, by Lemma 1.1.5, we have

$$\alpha^{-1} = (\inf_{i=1}^{n} \alpha_{\mu_i,\rho_i})^{-1}$$

$$= \inf_{i=1}^{n} (\alpha_{\mu_i,\rho_i})^{-1}$$

$$= \inf_{i=1}^{n} \alpha_{\rho_i',\mu'}.$$

Hence,

$$U_\eta(\alpha^{-1}) = \inf_{i=1}^{n} \eta(\rho_i', \mu_i')$$

$$= \inf_{i=1}^{n} \eta(\mu_i, \rho_i).$$

It implies $s \geq \inf_{i=1}^{n} \eta(\mu_i, \rho_i)$. It is a contradiction. Then U_η is a smooth uniformity on X.

4.2.8 Definition

The smooth quasi-uniform space (X, U) is said to be compatible with smooth topogenous space (X, η) if $\eta_U = \eta$.

The class $\Pi(\eta)$ denotes the family of all smooth quasi-uniformities which are compatible with a given smooth topogenous structure η.

4.2.9 Theorem

Let (X, η) be smooth topogenous space and the smooth topogenous structure η_{U_η} induced by U_η. Then we have:

(1) $\eta_{U_\eta} = \eta$, that is, $U_\eta \in \Pi(\eta)$.

(2) U_η is the coarsest member of $\Pi(\eta)$.

Proof

(1) First, we will show that $\eta_{U_\eta} \geq \eta$. If $\eta(\mu, \rho) = 0$, then it is trivial. If $\eta(\mu, \rho) \neq 0$ then by Lemma 4.2.4 (1), there exists $\alpha_{\mu,\rho} \in \Omega_X$ such that $U_\eta(\alpha_{\mu,\rho}) \geq \eta(\mu, \rho)$ from Theorem 4.2.6, It follows that $\alpha_{\mu,\rho}(\mu) = \rho$ from Theorem 4.1.4, $\eta_{U_\eta}(\mu, \rho) \geq U_\eta(\alpha_{\mu,\rho})$. Hence, $\eta_{U_\eta} \geq \eta$.

Suppose that $\eta_{U_\eta} \not\leq \eta$. Then there exist $\mu, \rho \in I^X$ such that

$$\eta_{U_\eta}(\mu,\rho) \not\leq \eta(\mu,\rho). \tag{I}$$

From the definition of $\eta_{U_\eta}(\mu,\rho)$ there exists $\alpha \in \Omega_X$ with $\alpha(\mu) \leq \rho$ such that

$$U_\eta(\alpha) \not\leq \eta(\mu,\rho).$$

From the definition of U_η, there exists a finite family

$\{\alpha_{\mu_i,\rho_i} \setminus \inf_{i=1}^{n} \alpha_{\mu_i,\rho_i} \leq \alpha\}$ such that

$$\inf_{i=1}^{n} \eta(\mu_i,\rho_i) \not\leq \eta(\mu,\rho). \tag{II}$$

On the other hand, put $\Gamma = \{J \subset \{1,\ldots,m\} \setminus \mu \leq \sup_{j\in J} \mu_j\}$.

If $\Gamma = \phi$, $\inf_{i=1}^{n} \alpha_{\mu_i,\rho_i}(\mu) = \underline{1} \leq \rho$. Thus, $\rho = \underline{1}$, and $\eta(\mu,\rho) \geq \eta(\underline{1},\underline{1}) = 1$.

It is a contradiction for the equation (I).

If $\rho = \underline{0}$ by $\eta_{U_\eta}(\mu,\rho) \neq 0$ and (ST2), $\mu = \underline{0}$. Hence, $\eta(\underline{0},\underline{0}) = 1$. It is a contradiction for the equation (I).

If $\Gamma \neq \phi$ and $\rho \neq \underline{0}$, by Lemma 4.2.4 (5), then there exists

$\Gamma = \{J \subset \{1,\ldots,m\} \setminus \mu \leq \sup_{j\in J} \mu_j\}$ such that

$$\inf_{i=1}^{n} \alpha_{\mu_i,\rho_i}(\mu) = \inf_{J\in\Gamma} \tau_J \leq \rho.$$

Hence, $\rho \geq \inf_{J\in\Gamma}(\sup_{j\in J} \rho_j)$. Moreover, we have $\mu \leq \inf_{J\in\Gamma}(\sup_{j\in J} \mu_j)$. Since

$$\eta(\sup_{j\in J} \mu_j, \sup_{j\in J} \rho_j) \geq \inf_{i=1}^{m} \eta(\mu_i,\rho_i).$$

we have

$$\eta(\mu,\rho) \geq \eta(\inf_{J\in\Gamma}(\sup_{j\in J}\mu_j), \inf_{J\in\Gamma}(\sup_{j\in J}\rho_j)) \geq \inf_{i=1}^{m}\eta(\mu_i,\rho_i).$$

It is a contradiction for the equation (II).

Therefore $\eta \geq \eta_{U_\eta}$.

(2) By (1), we have that U_η is compatible with η. Let U be an arbitrary member of $\Pi(\eta)$.

We will show that $U_\eta(\alpha) \leq U(\alpha)$, for all $\alpha \in \Omega_X$.

Suppose that there exists $\alpha \in \Omega_X$ such that

$$U_\eta(\alpha) \not\leq U(\alpha).$$

There exists a finite family $\{\alpha_{\mu_i,\rho_i} \setminus \inf_{i=1}^{m}\alpha_{\mu_i,\rho_i} \leq \alpha\}$ such that

$$\inf_{i=1}^{m}\eta(\mu_i,\rho_i) \not\leq U(\alpha).$$

Since $U \in \Pi(\eta)$, that is, $\eta(\mu_i,\rho_i) = \eta_U(\mu_i,\rho_i)$ for $i = 1,\ldots,m$, there exists $\beta_i \in \Omega_X$ with $\beta_i(\mu_i) \leq \rho_i$ such that

$$\inf_{i=1}^{m}U(\beta_i) \not\leq U(\alpha). \qquad \text{(III)}$$

On the other hand, put $\beta = \inf_{i=1}^{m}\beta_i$. Since $\beta_i(\mu_i) \leq \rho_i$, by the definition of α_{μ_i,ρ_i}, we have $\beta_i \leq \alpha_{\mu_i,\rho_i}$. It follows that

$$\beta = \inf_{i=1}^{m}\beta_i \leq \inf_{i=1}^{m}\alpha_{\mu_i,\rho_i} \leq \alpha.$$

Hence, $U(\alpha) \geq U(\beta) \geq \inf_{i=1}^{m}U(\beta_i)$. It is a contradiction for the equation (III).

4.1.14 Example

Define a function $\eta : I^X \times I^X \to I$ as follows:

$$\eta(\lambda,\mu)=\begin{cases}1 & \text{if } \lambda=\underline{0} \text{ or } \mu=\underline{1},\\ \dfrac{2}{3} & \text{if } \underline{0}\neq\lambda\leq\chi_{\{x\}}, \underline{1}\neq\mu\leq\chi_{\{x\}},\\ 0 & \text{otherwise.}\end{cases}$$

Where χ_A is a characteristic function of A. Then (X,η) is smooth topogenous space.

From Theorem 4.2.6, we can obtain a smooth quasi-uniformity $U_\eta : \Omega_X \to I$ on X as follows:

$$U_\eta(\alpha)=\begin{cases}1 & \text{if } \alpha=\alpha_{\underline{1},\underline{1}}\\ \dfrac{2}{3} & \text{if } \alpha_{\chi_{\{x\}},\chi_{\{x\}}}\leq\alpha=\alpha_{\underline{1},\underline{1}},\\ 0 & \text{otherwise.}\end{cases}$$

If $\underline{0}\neq\lambda\leq\chi_{\{x\}}, \underline{1}\neq\mu\leq\chi_{\{x\}}$, then, by Lemma 4.2.4 (2), $\alpha_{\chi_{\{x\}},\chi_{\{x\}}}\leq\alpha_{\lambda,\mu}$.

Hence, $\eta_{U_\eta}(\lambda,\mu)=\dfrac{2}{3}$. By a similar method, we have $\eta_{U_\eta}=\eta$.

4.2.11 Theorem

Let (X,η) be smooth (quasi-)uniform space. The smooth (quasi-)uniformity U_{η_U} induced by η_U is coarser than U.

Proof

Suppose that $U_{\eta_U}\not\leq U$. There exists $\alpha\in\Omega_X$ such that

$$U_{\eta_U}(\alpha)\not\leq U(\alpha).$$

From the definition of $U_{\eta_U}(\alpha)$, there exists a finite family

$\{\alpha_{\mu_i,\rho_i} \setminus \inf_{i=1}^{n} \alpha_{\mu_i,\rho_i} \leq \alpha\}$ such that

$$\inf_{i=1}^{n} \eta_U(\mu_i,\rho_i) \not\leq U(\alpha).$$

From the definition of η_U for each $i \in \{1,\ldots,n\}$, there exists $\beta_i \in \Omega_X$ with $\beta_i(\mu_i) \leq \rho_i$ such that

$$\inf_{i=1}^{n} U(\beta_i) \not\leq U(\alpha).$$

Let $\beta = \inf_{i=1}^{n} \beta_i$ be given. Since $\beta_i(\mu_i) \leq \rho_i$, by the definition of α_{μ_i,ρ_i} we have

$$\beta_i \leq \alpha_{\mu_i,\rho_i}.$$

Hence, $\beta \leq \alpha$ and

$$U(\alpha) \geq U(\beta) \geq \inf_{i=1}^{n} U(\beta_i),$$

It is a contradiction. Therefore, $U_{\eta_U} \leq U$.

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ARABIC SUMMARY

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مقدمة إلى كلية العلوم - فرع الفيوم
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من

مصطفى الدرديرى أحمد حسين
ماجستير رياضيات
جامعة القاهرة

للحصول على درجة

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(رياضيات)

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الملخص العربى

يعتمد جزء كبير من الرّياضيات على مفهوم الفئة وعلى المنطق الثنائى. فكما أن الجمل الخبرية أمّا أن تكون صحيحة وأمّا أن تكون خاطئة فان العنصر أيضا أمّا أن ينتمى أو لاينتمى إلى الفئة.

لقد عرّف زاده [111] فى عام (1965) مفهوم الفئة الفازية Fuzzy set فالتعبير $\mu \in I^X$ نقصدُ به الفئة الفازية μ من X حيث أن $\mu(x)$ تمثل درجة انتماء العنصر x بالنسبة للفئة الفازية μ . وقال أنّ هذا المفهوم يمتد الى حساب التفاضل والتكامل ومن ناحية أخرى الى كثير من العلوم منها علوم الحاسب والتحكم ولقد وجدت تطبيقاتٍ كثيرة للفئة الفازية وكمثال لذلك ياجير [108] في عام (1982).

ومنذ ذلك الحين وما زال الرياضيون يُحاولون أدخال مفهوم الفئة الفازية الى كثير من فروع الرياضيات المختلفة مثل الجبر والتوبولوجى مع العلم بان بعض هذه المحاولات لم توضح فروقا ملموسة بين المفهومين الكلاسيكى والفازى.

وقد دخلت الرياضيات الفازية الى علم التوبولوجى والموضوعات المرتبطة به كالتقارب Proximity والأنتظام Uniformity والتجانس Topogenous ففى عام (1968) عرف تشانج [14] لأول مرة مفهوم التوبولوجى الفازى. ومنذ ذلك الحين والعديد من الباحثين مثل ونج [105] وهوتون [45] و لوين [63] وغووجين [36] وباو وينج [76,77] وأخرين يحاولون إدخال المفاهيم المختلفة للتوبولوجى الكلاسيكى الى التوبولوجى الفازى.

ففى عام (1979) عرف كاتاسارس [50] مفهوم التقارب الفازى Fuzzy proximity على أنه علاقة ثنائية على عائلة من الفئات الفازية و التى تحقق مجموعة معينة من المسلمات وهى نفس المسلمات للتقارب الكلاسيكى وبالتالى فهى معالجةً ليست مجدية وذلك لان التوبولوجى المولد بها توبولوجى كلاسيكى.

وفى عام (1989) قدم مرسى [49] مفهوما جديدا للتقارب الفازى متوائما مع أرتيكو وموريسكو [2] ولكنه ايضا تتبع اسلوب تشانج فى تعريف التوبولوجى الفازى. نود ان نشير الى أن التوبولوجى الفازى للوين [64] حالة خاصةٍ لتشانج.

وللانتظام الفازى Fuzzy uniformity جذران يرجع احدهما إلى لوين [66] والآخر إلى هوتون [46]. اما لوين فقد عرف الانتظام الفازى معتمدا على مفهوم الحاشية Entourage ، بينما هوتون اعتمد على تغطية التقارب Covering approach.

وفى [71] قدم مينجشينج مفهوم الانتظام الزغبى Fuzzifying uniformity والذى اعتمد فيه على مفهومه للتوبولوجى الزغبى [70 – 68].

كما قدم كاتساراس [19 – 18] نظرية البناءات المتجانسة الفازية بشكلين مختلفين احدهما اعتمد فيه على التوبولوجى الفازى لتشانج [14] و الانتظام الفازى لهوتون [46] والتقارب الفازى له والثاني اعتمد فيه على التوبولوجى و الانتظام الفازى لوين [63,66] والتقارب الفازى لآرتيكو موريسكو [3] وكلا من الطريقتين اعتمد على التوبولوجى الفازى لتشانج.

إن بعض الدراسات السابقة لا تعنى تغيرا واضحا بين الدراستين الكلاسيكية والفازية للبناءات التوبولوجية المختلفة حيث ان الشكل العام لا يتغير فيه سوى استبدال الفئات الكلاسيكية بفئات فازية مع الاحتفاظ بشكل المسلمات التى يقوم عليها البناء كما هى على الرغم من انه يمكن ادخال مفهوم الفازى الى المسلمات نفسها و ذلك باستخدام المنطق المتعدد القيم على ان تكون قيم الصدق والكذب داخل فترة الوحدة المغلقة [0,1] .

مع نهاية الثمانيناتِ وبداية التسعينات أدرك كثيرا من الباحثين هذا الغرض و اهتم كلا منهم بمحاولة دخول الفازى الى البناء نفسه. وبذلك قدم سوستاك [93] فى عام (1985) شكلا جديدا للتوبولوجى الفازى والذى نسميه فى دراستنا الحالية بالتوبولوجى الاملس Smooth topology

ولقد اهتم رمضان بهذا الاتجاه ففى عام (1992) قدم مفهوما للتوبولوجى الاملس [80] وتعريفا لقبل الانتظام وقبل التقارب الاملس وذلك مع بادارد و مشهور [8] وتعريفا لقبل التجانس الاملس [80]

كاستمرار لدراسة البناءات التوبولوجية الملساء فان هذه الدراسة تهدف الى بحث و استقصاء اكثر لكلا من التقارب والتجانس والانتظام والتوبولوجى الاملس.

تتكون هذه الرسالة من فصل تمهيدى واربعة فصول رئيسية و قائمة من المراجع.

الفصل التمهيدى: يحتوى على المفاهيم الاولية والتعريفات والنظريات الاساسية التى استخدمت فى هذه الرسالة.

الفصل الاول: يهدف الى تقديم مفهوم الفراغات المنتظمة الملساء مع دراسة للفراغ الجزئى منه ودراسة لحاصل الضرب مع توضيح العلاقة بينه وبين التوبولوجى الاملس.

الفصل الثانى: نقدم مفهومى الجريل الاملس Smooth grill والفراغ التقاربى الاملس مع دراسة لخواص كلا منهما. اعطيت ايضا العلاقة بين الفراغ التوبولوجى الاملس والفراغ المنتظم الاملس والفراغ التقاربى الاملس.

الفصل الثالث: نعتنى فى هذا الفصل بنظرية البناءات المتجانسة الملساء ففى الجزء 3.1 اعطيت المفاهيم والخواص مع دراسة لحاصل الضرب والفراغ الجزئى للبناءات المتجانسة الملساء. قدمنا ايضا فى الجزء 3.2 العلاقة بينه و بين الفراغ التوبولوجى الاملس الفوقى (Smooth supra topology) والفراغ المنتظم الاملس.

الفصل الرابع: قدمنا الفراغات المتجانسة الملساء المتوائمة مع الفراغ المنتظم الاملس ففى الجزء 4.1 انشانا فراغا متجانسا املسا من فراغ منتظم املس والعكس فى الجزء 4.2.

نود ان نشير الى ان معظم نتائج هذه الرسالة بعضها قبل للنشر وبعضها أرسل للنشر.