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Studies on Properties of Certain Classes of Analytic Functions with Applications of Differential Subordination

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Summary

The purpose of this thesis is to define and study properties for certain classes of univalent and multivalent functions defined in the open unit disc $\mathrm{U} = \{z \in \mathbb{C} : |z| < 1\}$ and in the punctured unit disc $\mathrm{U}^{*} = \mathrm{U} \setminus \{0\}$ where $\mathbb{C}$ is the complex plane. These classes are defined by using some linear operators, integral operators, Hadamard product (or convolution) and $q-$ difference operators. Also, we define classes of uniformly convex and starlike functions. Further, let $\mathbf{P}_k$, $\mathbf{V}_k$ and $\mathbf{U}_k$, respectively, to denote the subclasses of A which have real parts bounded in the mean on U by k[illegible] bounded boundary rotation at most k[illegible] and bounded argument at most k[illegible] (see [119] and [129]). Also, we obtain subordination, superordination properties, sandwich results for classes associated with the operators $\mathrm{N}_{p,\ldots}^{m,\ldots}$ [illegible] and $\mathrm{H}_{p,\ldots}^{\ldots}$ [illegible] Distortion theorems for Classes of multivalent Non-Bazilevic analytic functions defined by linear operator are also obtained, we obtain also some preserving subordination results for classes of p-valent meromorphic functions associated with different operators. Furthermore, we study some inclusion relations for subclasses associated with a linear operator and we obtain necessary and sufficient conditions of Gaussian hypergeometric functions to be in various subclasses of univalent and uniformly classes. Finally, Fekete-Szegö inequalities for classes of non-Bazilevič functions with complex order defined by convolution, univalent and meromorphic functions involving $q-$ difference operator.

This thesis consists of seven Chapters.

Chapter 1

This chapter is considered as an introductory chapter and consists of basic concepts and previous definitions which are essential for completing the results in subsequent chapters.

Chapter 2

This chapter consists of three sections. The first section is an introductory section and contains the definition of the operator $\mathrm{H}_{p,\text{☼}\delta}^{\text{✞},\mu}$ defined by Tang et al. [158] (see also Seoudy and Aouf [143], Aouf et al. [23]). Also, we consider the linear operator $\mathrm{N}_{p,\text{✞},\delta,\text{☾}}^{m,\mu,\text{✈}}$ defined by Aouf et al. [23].

Section 2.2, contains preliminary results.

In Section 2.3, we study different properties of differential subordination, superordination and sandwich results of the classes associated to the operators $\mathrm{H}_{p,\text{☼}\delta}^{\text{✞},\mu}$ and $\mathrm{N}_{p,\text{✞},\delta,\text{☾}}^{m,\mu,\text{✈}}$.

Chapter 3

This chapter consists of five sections. The first section is an introductory section and contains the definition of the operator $\mathrm{H}_{p,\text{☺☹}}^{\text{☹}c}$ defined by Mostafa and Aouf [101] see also [26]. Also, we define the integral operator $I_{p,\delta}^{\mu}$ as follows:

For $f(z) \in \Sigma_{p,m}, 0 \leq \delta < 1, 0 \leq \mu \leq 1$ and $p \in \mathrm{N},$ we define the operator $I_{p,\delta}^{\mu}$

$$I_{p,\delta}^{\mu} f(z) = \frac{1}{(1-\delta)^{\mu+1}\Gamma(\mu+1)} \int_0^\infty t^{\mu+p} e^{-\left(\frac{t}{1-\delta}\right)} f(zt)\,dt$$

$$= z^{-p} + \sum_{k=m}^{\infty} \frac{\Gamma(\mu+k+1+p)}{\Gamma(\mu+1)} (1-\delta)^{k+p} a_k z^k.$$

Section 3.2, contains preliminary results.

In Section 3.3, by making use of the principle of subordination, we obtain inclusion results associated with $H_{p,\text{☺☹}}^{\text{☹}c}$

In Section 3.4, by making use of the principle of subordination, we obtain inclusion results associated with $I_{p,\delta}^{\mu}$.

In Section 3.5, some preserving sandwich results associated with $I_{p,\delta}^{\mu}$ are obtained.

Chapter 4

This chapter consists of three sections. The first section is an introductory section and contains the definition of the operator $D^m_{\lambda,p,l}$ defined by Aouf et al. [20]. Also, by using the operator $D^m_{\lambda,p,l}$, we define the subclasses $S^{*(m)}_{p,\lambda,l}(\phi)$, $K^m_{p,\lambda,l}(\phi)$, $C^m_{p,\lambda,l}(\phi,\psi)$ and $C^{*(m)}_{p,\lambda,l}(\phi,\psi)$.

In Section 4.2, some inclusion relations of the subclasses $S^{*(m)}_{p,\lambda,l}(\phi)$, $K^m_{p,\lambda,l}(\phi)$, $C^m_{p,\lambda,l}(\phi,\psi)$ and $C^{*(m)}_{p,\lambda,l}(\phi,\psi)$ are obtained.

In Section 4.3, some inclusion relations involving the operator

$$J_{c,p}f(z)=\frac{c+p}{z^c}\int_0^z t^{c-1}f(t)\,dt\quad(c>-p)$$

are obtained.

Chapter 5

This chapter consists of four sections. The first section is an introductory section and by using the linear operator $D^m_{\lambda,p,l}$ and the classes $R_k(p,\rho)$, $V_k(p,\rho)$, $T_k(p,\rho,\beta)$ and $T^*_k(p,\rho,\beta)$ of functions of bounded boundary rotations and bounded arguments defined by Seoudy [141] and for $p=1$ see (Noor [107]), we define the subclasses $R^m_{\lambda,p,l}(k,\rho)$, $V^m_{\lambda,p,l}(k,\rho)$, $T^m_{\lambda,p,l}(k,\rho,\beta)$ and $T^{*(m)}_{\lambda,p,l}(k,\rho,\beta)$ of analytic functions.

Also, we defined the operator $N_p^{\mu,\lambda}: A(p)\to A(p)$ by

$$N_p^{\mu,\lambda}f(z)=z^p+\sum_{n=1}^{\infty}\left(\frac{n+p+1}{p+1}\right)^{\mu}\frac{(\lambda+1)_n}{(1)_n}a_{n+p}z^{n+p}$$

and by using $N_p^{\mu,\lambda}$, we defined the subclasses $R_p^{\mu,\lambda}(k,\rho)$, $V_p^{\mu,\lambda}(k,\rho)$, $T_p^{\mu,\lambda}(k,\rho,\beta)$ and $T_p^{*(\mu,\lambda)}(k,\rho,\beta)$.

In Section 5.2, some inclusion relations of the subclasses $R^m_{\lambda,p,l}(k,\rho)$, $V^m_{\lambda,p,l}(k,\rho)$, $T^m_{\lambda,p,l}(k,\rho,\beta)$ and $T^{*(m)}_{\lambda,p,l}(k,\rho,\beta)$ are obtained.

In Section 5.3, some inclusion relations involving the operator

$$J_{c,p}f(z)=\frac{c+p}{z^c}\int_0^z t^{c-1}f(t)\,dt\quad(c>-p)$$

are obtained.

In Section 5.4, some inclusion relations of the subclasses $R_p^{\lambda}(\alpha,\beta)$, $V_p^{\lambda}(\alpha,\beta)$, $T_p^{\lambda}(\alpha,\beta)$ and $T_p^{*}(\alpha,\beta)$ are obtained.

Chapter 6

This chapter consists of three sections. The first section is an introductory section and contains the definitions of the classes $N^{*}(\lambda,\alpha,\beta)$ and $S_p(\lambda,\alpha,\beta)$ as follows:
For $0 \le \lambda, 0 \le \alpha < 1, 0 < \beta \le 1, z \in U$ and $f(z) \in S$ we defined the class $N^{*}(\lambda,\alpha,\beta)$, which consists of functions satisfying

$$\left| \frac{\dfrac{zf'(z)+\lambda z^2 f''(z)}{(1-\lambda)f(z)+\lambda zf'(z)} - 1}{\dfrac{zf'(z)+\lambda z^2 f''(z)}{(1-\lambda)f(z)+\lambda zf'(z)} + 1 - 2\alpha} \right| < \beta$$

and

$$T^{*}(\lambda,\alpha,\beta) = N^{*}(\lambda,\alpha,\beta) \cap T.$$

And for $-1 \le \alpha < 1, \beta \ge 0$, $0 \le \lambda < 1$, $f(z) \in S$ is said to be in the class $S_p(\lambda,\alpha,\beta)$ see [103] if

$$\operatorname{Re}\left\{ \frac{zf'(z)}{(1-\lambda)f(z)+\lambda zf'(z)} - \alpha \right\} > \beta \left| \frac{zf'(z)}{(1-\lambda)f(z)+\lambda zf'(z)} - 1 \right|$$

and

$$TS_p(\lambda,\alpha,\beta) = S_p(\lambda,\alpha,\beta) \cap T.$$

In Section 6.2, we determine necessary and sufficient conditions for Gaussian hypergeometric functions to be in the class $T^{*}(\lambda,\alpha,\beta)$ and sufficient conditions for Gaussian hypergeometric functions to be in the class $N^{*}(\lambda,\alpha,\beta)$.

In Section 6.3, we determine necessary and sufficient conditions for Gaussian hypergeometric functions to be in the class $TS_p(\lambda,\alpha,\beta)$ and sufficient conditions for Gaussian hypergeometric functions to be in the class $S_p(\lambda,\alpha,\beta)$.

Chapter 7

This chapter consists of three sections. The first section consists of two subsections **Subsection 7.1.1,** is an introductory of the first section and contains the lemmas and definition of the class $R_g^{\alpha,\mu}(b,\varphi)\ (b \in \mathbb{C}^*, \alpha \in \mathbb{C}$ and $0 \le \mu \le 1.)$ as follows:

A function $f(z) \in S$ is said to be in the class $R_g^{\alpha,\mu}(b,\varphi)\ (b \in \mathbb{C}^*, \alpha \in \mathbb{C}$ and $0 \le \mu \le 1)$ if it satisfies the following subordination condition:

$$1 + \frac{1}{b}\left\{(1-\alpha)\left(\frac{z}{(f*g)(z)}\right)^{\mu} + \alpha\frac{z(f*g)'(z)}{(f*g)(z)}\left(\frac{z}{(f*g)(z)}\right)^{\mu} - 1\right\} \prec \varphi(z)$$

In Subsection 7.1.2, we obtain Fekete-Szegö inequalities for the class $R_g^{\alpha,\mu}(b,\varphi)$.

Section 7.2, consists of two subsections.

Subsection 7.2.1, is an introductory of the second section and contains the following definition.

A function $f(z) \in S$ is said to be in the class $S_{\lambda,b}(q,\varphi)\ (0 \le \lambda \le 1, b \in \mathbb{C}^*, 0 < q < 1)$ if and only if

$$1 + \frac{1}{b}\left[\frac{zD_q f(z)}{(1-\lambda)f(z) + \lambda z D_q f(z)} - 1\right] \prec \varphi(z),$$

where $D_q f(z)$ is the $q-$ difference operator for a function $f(z)$ and is defined by ([58], [59] and [4]).

$$D_q f(z) = \frac{f(qz) - f(z)}{(q-1)z}$$

$$= 1 + \sum_{k=2}^{\infty} [k]_q a_k z^{k-1} \quad z \ne 0,$$

where

$$[k]_q = \frac{1-q^k}{1-q}.$$

As $q \to 1^-, [k]_q \to k,$ so $\lim_{q\to 1^-} D_q f(z) = f'(z)$

In Subsection 7.2.2, we obtain Fekete-Szegö inequalities for the class $S_{\lambda,b}(q,\varphi)$.

Section 7.3, consists of two subsections.

In Subsection 7.3.1, is an introductory of the third section and contains the following definitions.

A function $f(z) \in \mathcal{A}$ is said to be in the class $\Omega_b^{\lambda}(\mu, q, \varphi)\ (b \in \mathbb{C}^*, \mu \in \mathbb{C}\backslash(0,1], 0 < q < 1)$ if and only if

$$1+\frac{1}{b}\left[\frac{-\left(\left(1-\frac{\lambda}{q}\right)qzD_q^* f(z)+\lambda qzD_q^*\left(zD_q^* f(z)\right)\right)}{\left(1-\frac{\lambda}{q}\right)f(z)-\lambda zD_q^* f(z)}-1\right]\prec\varphi(z)$$

$$\left(b\in\mathbb{C}^*,\ \lambda\in\mathbb{C}\setminus(0,1],\ 0<q<1\right)$$

and in the class $\mathcal{S}^*_b(q,\varphi)\ (b\in\mathbb{C}^*, 0<q<1)$ if and only if

$$1-\frac{1}{b}\left[\frac{qzD_q^* f(z)}{f(z)}+1\right]\prec\varphi(z)\ \ (b\in\mathbb{C}^*; 0<q<1),$$

where $D_q^* f(z)$ is the $q-$ difference operator for the meromorphic function $f(z)$ and is defined by ([49], [102] and [159]).

$$D_q^* f(z)=\frac{f(qz)-f(z)}{(q-1)z},\ (z\in\mathbb{U}^*; 0<q<1),$$

$$=-\frac{1}{qz^2}+\sum_{k=0}^{\infty}[k]_q a_k z^{k-1},\ z\neq 0,$$

where

$$[k]_q=\frac{1-q^k}{1-q}\ (0<q<1).$$

As $q\to 1^-, [k]_q\to k,$ we have $\lim_{q\to 1^-} D_q^* f(z)=f'(z)$

In Subsection 7.3.2, we obtain Fekete-Szegö inequalities for the classes $\mathcal{S}^*_b(\lambda,q,\varphi)$ and $\mathcal{S}^*_b(q,\varphi)$.