INDUCED PROXIMITY IN FUZZY SPACES

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Abstract

The goal of this paper is to introduce and study the induced proximity on a fuzzy space due to the existence of a fuzzy proximity on another fuzzy space. Firstly, for every two complete lattices L, M, it is defined and studied the extension of the (L, M)-proximity on the fuzzy space L^Y to an (L, M)-proximity on the fuzzy space L^X; $Y \subset X$ and the restriction of the (L, M)-proximity on L^X to an (L, M)-proximity on L^Y. Then it is obtained the relations between their closure operators in each case (for general case and for special case when $M = 2$). Secondly, it is reformulated the definition of the fuzzy function on $P(\Lambda)^X$, where $P(\Lambda)$ is the lattice of the power set of a nonempty set Λ. Moreover, if $\mathcal{L}(\Lambda)$ denotes the family of all complete lattices defined on Λ, $i_X : L^X \to P(\Lambda)^X$ is the map $i_X(A)(x) = \{\alpha \in L : 0_L \leq \alpha \leq A(x)\}$, then it is shown that every (L, M)-basic proximity δ on L^X; $L \in \mathcal{L}(\Lambda)$ induces (L, M)-basic

proximity δ^* on $P(\Lambda)^X$ and the map i_X translates the family of δ-closed fuzzy subsets in L^X into the family of the δ^*-closed fuzzy subsets in $P(\Lambda)^X$. Thirdly, it is shown that the family of the categories of (L, M)-fuzzy basic proximity spaces on L^X; $L \in \mathcal{L}(\Lambda)$ is embedded in the category of $(P(\Lambda), M)$-fuzzy basic proximity spaces on $P(\Lambda)^X$.

1. Introduction

In his classical paper [34] in 1965, Zadeh introduced the fundamental concept of fuzzy set. Goguen [10] in 1967 replaced the unit interval of real numbers $I = [0, 1]$ by a complete lattice L. In 1968, Chang [5] initiated the theory of fuzzy topological spaces. Proximity has close relations with the concepts of topology, uniformity and metric. Efremovich has introduced the fundamental concept of the proximity space in [8]. In addition, Leader [22, 23] and Lodato [25, 26] have worked with weaker axioms than those of Efremovich proximity space enabling them to introduce an arbitrary topology on the underlying set. In the framework of L-topology, many authors generalized the crisp proximity to L-fuzzy setting; some different approaches to the concept of fuzzy proximity in the literature are introduced. The fuzzy proximity was introduced by Katsaras in 1979 [14-16]. Artico and Moresco in [1, 2] studied fuzzy proximity spaces, which are compatible with Lowen fuzzy topological spaces. Markin and Sostak in [27] introduced different concept of fuzzy proximity. They considered basic properties of these fuzzy proximities which described how a fuzzy proximity generates a fuzzy topology and studied the relations between their approaches and the approaches of the fuzzy proximity introduced by Katsaras and Artico. After then, the theory of proximity makes a massive progress (see [3, 4, 9, 11, 12, 17-19, 24, 29-33]). In [6], it was studied the $(P^*(L), 2)$-fuzzy topology on the fuzzy space $P^*(L)^X$, which is induced by an $(L, 2)$-fuzzy topological spaces on L^X, where the lattice $P^*(L)$ is defined by $P^*(L) = \{M \subset L :$

$0_L \in M\}$. It was obtained interesting relations between the category of $(P^*(L), 2)$-fuzzy topology on $P^*(L)^X$ and the category of the $(L, 2)$-fuzzy topology on L^X. This result has been a motivation to study the basic proximity spaces on $P(\Lambda)^X$ to find out its relation with the basic proximity spaces on L^X, where L is a lattice defined on a nonempty set Λ and $P(\Lambda)$ is the lattice of the power set of Λ. Moreover, we study the restriction of the (L, M)-fuzzy proximity δ on the fuzzy space L^X to an (L, M)-fuzzy proximity on the fuzzy space L^Y; $Y \subset X$ and conversely.

The outline of this paper is as follows: In Section 2, it is given basic concepts and useful results which will be used in the sequel. In Section 3, it was defined and studied the extension of the proximity on the fuzzy space L^Y to a proximity on the fuzzy space L^X; $Y \subset X$, the restriction of the fuzzy proximity on L^X to a fuzzy proximity on L^Y and it was obtained the relations between their closure operators. In Section 4, it was obtained some relations between the lattice of the power set $P(\Lambda)$, the family of complete lattices $\mathcal{L}(\Lambda)$ on Λ and reformulate a suitable definition of the fuzzy function on $P(\Lambda)$. In Section 5, it was defined the induced basic proximity on $P^*(\Lambda)^X$ for each given basic proximity on L^X; $L \in \mathcal{L}(\Lambda)$ and obtained a fundamental relation between their closure operators. In Section 6, it was introduced the definition of the (L, M)-proximity. In Section 7, it was studied the restriction and the extension of (L, M)-fuzzy proximities. In Section 8, it was defined and studied the induced (L, M)-proximity on $P^*(L)^X$ and on $P(\Lambda)^X$ corresponding to each (L, M)-proximity on L^X; $L \in \mathcal{L}(\Lambda)$. In Section 9, it is shown that the family of the categories of proximities on L^X; $L \in \mathcal{L}(\Lambda)$ is embedded in the category of proximities on $P(\Lambda)^X$.

2. Preliminaries

Let X be a given universal set and L be a given lattice. Denote by 0_L and 1_L the smallest element and the greatest element of L, respectively, and denote by $\mathbf{0}_X$ and $\mathbf{1}_X$ the smallest and the greatest fuzzy subset of L^X, respectively. In [6], it was used the lattice of the form $P^*(L) = \{M \subset L : 0_L \in M\}$. The algebraic structure $(P^*(L), \cup, \cap, ')$ forms a complemented, completely distributive and complete lattice with $0_{P^*(L)} = \{0_L\}$ which is the smallest element and $1_{P^*(L)} = L$ which is the greatest element. The complementary operation is defined by $' : P^*(L) \to P^*(L)$; where $M' = (L - M) \cup \{0_L\}$.

The empty $P^*(L)$-fuzzy subset of X is denoted by $\mathbf{0}_X$, where $\mathbf{0}_X(x) = \{0_L\}$; $\forall x \in X$. The greatest element of $P^*(L)$-fuzzy subset of X is denoted by $\mathbf{L}_X$, where $\mathbf{L}_X(x) = L$, $\forall x \in X$.

To study the relationship between $P^*(L)^X$ and L^X, we defined the mappings i_X and j_X as follows [6]:

(a) $i_X : L^X \to P^*(L)^X$ is defined as:

$$i_X(A)(x) = \{\lambda \in L : 0_L \leq \lambda \leq A(x)\}; \ \forall x \in X, A \in L^X.$$

(b) $j_X : P^*(L)^X \to L^X$ is defined as:

$$j_X(V)(x) = \sup(V(x)), \ \forall x \in X, V \in P^*(L)^X.$$

In [6], it is proved that the two operators i_X, j_X form adjunction functors and i_X is the unique write adjoint of j_X and j_X is the unique left adjoint of i_X.

The fuzzy function on L-fuzzy subsets is studied in [6]. The fuzzy

functions translate the fuzzy subsets L^X into the fuzzy subsets K^Y as follows:

Definition 2.1 [6]**.** Let X, Y be given nonempty sets and L, K be given lattices. The fuzzy function $\mathbf{F} = (F, \{f_x\}_{x \in X})$ from L^X into K^Y or simply the fuzzy function $\mathbf{F} = (F, f_X) : X \to Y$ is defined as an ordered pair (F, f_x), where $F : X \to Y$ is a function from X to Y and f_x; $x \in X$ is a family of onto comembership functions $f_x : L \to K$; $x \in X$ satisfying the following conditions:

(i) $f_x(0_L) = 0_K$ and $f_x(1_L) = 1_K$,

(ii) f_x is a nondecreasing function for all $x \in X$.

The action of the fuzzy function $\mathbf{F} = (F, f_x)$ on the L-fuzzy subsets A of X and the inverse image of the K-fuzzy subset B of Y are defined as follows:

$$\mathbf{F}^{\to}(A)(y) = \begin{cases} \vee_{y=F(x)} f_x(A(x)), & \text{if } F^{-1}(y) \neq \varnothing, \\ 0_K, & \text{if } F^{-1}(y) = \varnothing; \end{cases} \quad y \in Y \text{ and } A \in L^X,$$

$\mathbf{F}^{\leftarrow}(B)(x) = \vee f_x^{-1}(B(F(x)))$; $x \in X$ and $B \in K^Y$, where the supremum is taken over the set of values $f_x^{-1}(B(F(x)))$.

The fuzzy function $\mathbf{F} = (F, f_x)$ from L^X to L^Y is called a *uniform fuzzy function*, if $f_x = f$; for all $x \in X$. The ordinary functions are embedded in the family of fuzzy functions as uniform fuzzy functions in which $f_x = id_L$ is the identity function.

The concept of fuzzy topology on a set X was introduced by Chang in 1968 [5] as a collection of fuzzy subsets of I^X (where $I = [0, 1]$ is the closed unit interval of real numbers) satisfying the known axioms of the topology. This definition is extended to L-topology, where L is a complete lattice.

Kubiak in [20] generalized the L-topology by introducing the (L, M)-fuzzy topology.

Definition 2.2 [20]**.** Let L, M be complete lattices. A mapping $\tau : L^X \to M$ is called an (L, M)-*fuzzy topology* on X if it satisfies the following conditions:

(a) $\tau(\mathbf{0}_X) = \tau(\mathbf{1}_X) = 1_M$,

(b) $\tau(A \wedge B) \geq \tau(A) \wedge \tau(B)$ if $A, B \in L^X$,

(c) $\tau(\vee_i A_i) \geq \wedge_i \tau(A_i)$ for every $\{A_i; i \in \alpha\} \subset L^X$.

τ is called an (L, M)-*fuzzy topology*, (X, L, M, τ) is called *fuzzy topological space* and $\tau(A)$ is called the *degree of openness of A*, for each $A \in L^X$.

Definition 2.3 [6]**.** Let $\boldsymbol{F} = (F, \{f_x\}_{x \in X})$ be a fuzzy function from (X, L_1, M, τ_1) to (Y, L_2, M, τ_2). The fuzzy function $\boldsymbol{F} = (F, \{f_x\}_{x \in X})$ is called τ_1-τ_2 *continuous fuzzy function* if $\tau_1(\boldsymbol{F}^{\leftarrow}(B)) \geq \tau_2(B)$, for all $B \in L_2^Y$.

Definition 2.4 [6]**.** The composition of the two fuzzy functions $\boldsymbol{F} = (F, f_x) : X \to Y$ from (X, L_1, M, τ_1) to (X, L_2, M, τ_2) and $\boldsymbol{G} = (G, g_x) : Y \to Z$ from (Y, L_2, M, τ_2) to (Z, L_3, M, τ_3) is a fuzzy function, which is denoted by $\boldsymbol{G} \circ \boldsymbol{F}$ and it is defined as follows: $\boldsymbol{G} \circ \boldsymbol{F} : (X, L_1, M, \tau_1) \to (X, L_3, M, \tau_3)$, where

$$(\boldsymbol{G} \circ \boldsymbol{F})^{\to}(A) = (G \circ F, g_{F(x)} \circ f_x)^{\to}(A), \text{ for every } A \in L_1^X.$$

Theorem 2.1 [6]**.** *The composition $\boldsymbol{G} \circ \boldsymbol{F}$ of the two continuous fuzzy functions*

$$\boldsymbol{F} = (F, f_x) : (X, L_1, M, \tau_1) \to (X, L_2, M, \tau_2),$$

$$\boldsymbol{G} = (G, g_x) : (X, L_2, M, \tau_2) \to (X, L_3, M, \tau_3)$$

is continuous fuzzy function.

Katsaras in [14, 15] defined the proximity on I^X as a binary relation Σ on I^X, satisfying the usual conditions of proximity. This concept was extended into L^X, where L is a complete lattice.

Definition 2.5 [14, 15]**.** A binary relation Σ on L^X is called *proximity* on L^X if Σ satisfies the following conditions:

(Fp1) $A\Sigma B$ implies $B\Sigma A$,

(Fp2) $(A \vee B)\Sigma C$ if and only if $A\Sigma C$ or $B\Sigma C$,

(Fp3) $A\Sigma B$ implies $A \neq \mathbf{0}_X$ and $B \neq \mathbf{0}_X$,

(Fp4) $A \wedge B \neq \mathbf{0}_X$ implies $A\Sigma B$,

(Fp5) $A\overline{\Sigma}B((A, B) \notin \Sigma)$ implies that there exists $C \in L^X$ such that $A\overline{\Sigma}C$ and $C^c\overline{\Sigma}B$.

The ordered pair (X, Σ) is called a *fuzzy proximity space*.

Definition 2.6 [24, 27]**.** A binary relation Σ on L^X which satisfies the axioms (Fp1), (Fp2), (Fp3) and (Fp4) is called *basic proximity* on L^X.

Definition 2.7 [14, 15]**.** Let (X, Σ_1) and (Y, Σ_2) be two (basic) proximity spaces on L^X, L^Y, respectively. A function $F : X \to Y$ is said to be a *proximity mapping* if for all $A, B \in L^X$, $A\Sigma_1 B$ implies $F^{\to}(A)\Sigma_2 F^{\to}(B)$. Equivalently, $H\overline{\Sigma}_2 K$ implies $F^{\leftarrow}(H)\overline{\Sigma}_1 F^{\leftarrow}(K)$, for all $H, K \in L^X$.

Definition 2.8 [6]**.** Let (X, Σ_1) and (Y, Σ_2) be two L-(basic) proximity spaces. A fuzzy function $\mathbf{F} = (F, f_x) : X \to Y$ is called a *fuzzy proximity mapping* with respect to Σ_1 and Σ_2 or simply a *fuzzy proximity mapping* if $A\Sigma_1 B$, then $\mathbf{F}^{\to}(A)\Sigma_2 \mathbf{F}^{\to}(F)$, for all $A, B \in L^X$.

Remark 2.1. Definition 2.7 is obtained from Definition 2.8, if the fuzzy function is uniform and all its comembership functions are equal to the identity function.

The family of (basic) proximities as objects and proximity mappings as morphisms forms a category denoted by $(\textbf{\textit{BProx}}(\textbf{\textit{L}}, \textbf{2}))\textbf{\textit{Prox}}(\textbf{\textit{L}}, \textbf{2})$. In [6], it was shown that the family of (basic) proximities as objects and fuzzy proximity mappings as morphisms forms a category denoted by $(\textbf{\textit{FBProx}}(\textbf{\textit{L}}, \textbf{2}))\textbf{\textit{FProx}}(\textbf{\textit{L}}, \textbf{2})$ and it was shown that $(\textbf{\textit{BProx}}(\textbf{\textit{L}}, \textbf{2}) \subset \textbf{\textit{FBProx}}(\textbf{\textit{L}}, \textbf{2}))$ and $\textbf{\textit{Prox}}(\textbf{\textit{L}}, \textbf{2}) \subset \textbf{\textit{FProx}}(\textbf{\textit{L}}, \textbf{2})$.

Definition 2.9. The fuzzy subset $P \in L^X$ is called a *fuzzy point* if there exists $x_0 \in X$, such that $P(x) = 0$; for all $x \neq x_0$ and $P(x_0) = \lambda \neq 0_L$.

Definition 2.10 [31]**.** Let Σ be L-basic proximity on X. The operator C_Σ on L^X which is defined by: $C_\Sigma(A) = \vee\{P \in L^X : P\Sigma A\}$, where P is a fuzzy point, is called the *closed operator*.

Theorem 2.2 [14]**.** *The collection* $\delta_\Sigma = \{V \in L^X : C_\Sigma(V^c) = V^c\}$ *of* L^X *forms* $(L, 2)$*-fuzzy topology on X, which is called the induced fuzzy topology by the proximity* Σ.

Definition 2.11 [14]**.** Let (X, Σ) be a proximity on L^X and $U, V \in L^X$. Then V is called a Σ-*neighborhood* of U if $U\overline{\Sigma}V^c$.

The family of all Σ-neighborhood of U is denoted by $N_\Sigma(U)$. Moreover, it is clear that $C_\Sigma(V) = \bigcap_{U \in N_\Sigma(V)} U$.

The *I*-fuzzy proximity is defined on I^X by [32] and it is generalized to the *L*-fuzzy proximity on L^X [19].

Definition 2.12 [19]**.** A mapping $\delta : L^X \times L^X \to L$ is said to be *L-fuzzy proximity* on *X*, which satisfied the following conditions:

(LFP1) $\delta(\mathbf{1}_X, \mathbf{0}_X) = \delta(\mathbf{0}_X, \mathbf{1}_X) = 0_L$,

(LFP2) $\delta(\lambda \vee \mu, \omega) = \delta(\lambda, \omega) \vee \delta(\mu, \omega)$,

(LFP3) if $\delta(\lambda, \mu) \neq 1_L$, then $\lambda \leq \mu'$,

(LFP4) for any $\lambda, \mu \in L^X$; then there exists $\rho \in L^X$ such that

$$\delta(\lambda, \mu) \geq \wedge_{\rho \in L^X} (\delta(\lambda, \rho) \vee \delta(\rho', \mu)),$$

(LFP5) $\delta(\lambda, \mu) = \delta(\mu, \lambda)$.

δ is called *L-fuzzy basic proximity* if it satisfied (LFP1), (LFP2), (LFP3) and (LFP5).

The pair (L^X, δ) is called an *L-fuzzy proximity space*.

Definition 2.13 [19]**.** A mapping $F : X \rightarrow Y$ is called *proximity map* relative to the *L*-fuzzy (basic) proximity δ_1 on L^X and *L*-fuzzy (basic) proximity δ_2 on L^Y, if for all $A, B \in L^X$, the inequality $\delta(A, B) \leq \delta_1(F^{\rightarrow}(A), F^{\rightarrow}(B))$ holds.

3. Extended and Restricted Proximities

Let L^X, L^Y be fuzzy spaces and $Y \subset X$. In this section, it will be discussed how to extend a given proximity on the fuzzy space L^Y to a proximity on the fuzzy space L^X. Conversely, it will be shown that every basic proximity on L^X induces a basic proximity on L^Y. In this article, we shall use the following notations.

Notations 3.1. (a) It is known that the open subset is the complement of a closed subset. Letting $A \in L^X$, we shall use the notation A^c for the complement of *A*.

(b) Let *X*, *Y* be given ordinary sets and $Y \subset X$ and *L* be a given lattice.

If $U \in L^X$, then $U_{\downarrow Y} \in L^Y$ denotes the restriction of U on $Y : U_{\downarrow Y}(x)$ $= U(x)$; if $x \in Y$. And if $A \in L^Y$, then $A_{\uparrow X} \in L^X$ denotes the extension of A on $X : A_{\uparrow X}(x) = A(x)$; $x \in Y$ and $A_{\uparrow X}(x) = 0_L$; $x \in X - Y$.

It is clear that for every $U, V \in L^X$ and for every $A, B \in L^Y$:

(i) $A_{\uparrow X} \wedge B_{\uparrow X} = (A \wedge B)_{\uparrow X}$, $A_{\uparrow X} \vee B_{\uparrow X} = (A \vee B)_{\uparrow X}$; $A, B \in L^Y$,

(ii) $U_{\downarrow X} \wedge V_{\downarrow Y} = (U \wedge V)_{\downarrow Y}$, $U_{\downarrow Y} \vee V_{\downarrow Y} = (U \vee V)_{\downarrow Y}$; $U, V \in L^X$,

(iii) If $A|_{\uparrow X} = B|_{\uparrow X}$, then $A = B$.

Theorem 3.1. *Each basic proximity Σ on L^Y; $Y \subset X$ can be extended to a basic proximity Σ^* on L^X as follows*: *for every $U, V \in L^X$: $U\Sigma^*V$ if $U \wedge V \neq \mathbf{0}_X$, or $U_{\downarrow Y}\Sigma V_{\downarrow Y}$.*

Proof.

(1) Let $U\Sigma^*V$.

- If $U \wedge V \neq \mathbf{0}_X$, then $V \wedge U \neq \mathbf{0}_X$ and $V\Sigma^*U$.

- If $U_{\downarrow Y}\Sigma V_{\downarrow Y}$, then $V_{\downarrow Y}\Sigma U_{\downarrow Y}$. It follows that $V\Sigma^*U$.

(2) Let $(U \vee V)\Sigma^*W$.

- If $(U \vee V) \wedge W \neq \mathbf{0}_X$, then $U \wedge W \neq \mathbf{0}_X$ or $\wedge W \neq \mathbf{0}_X$, which means that $U\Sigma^*W$ or $V\Sigma^*W$.

- If $(U \vee V)_{\downarrow Y}\Sigma W_{\downarrow Y} = (U_{\downarrow Y} \vee V_{\downarrow Y})\Sigma W_{\downarrow Y}$, then it follows that $U_{\downarrow Y}\Sigma W_{\downarrow Y}$ or $V_{\downarrow Y}\Sigma W_{\downarrow Y}$. Therefore, $U\Sigma^*W$ or $V\Sigma^*W$.

(3) Let $U\Sigma^*V$.

- If $U \wedge V \neq \mathbf{0}_X$, then $U \neq \mathbf{0}_X$ and $V \neq \mathbf{0}_X$.

- If $U_{\downarrow Y}\Sigma V_{\downarrow Y}$, then $U_{\downarrow Y} \neq \mathbf{0}_Y$ and $V_{\downarrow Y} \neq \mathbf{0}_Y$ and consequently, $U \neq \mathbf{0}_X$ and $V \neq \mathbf{0}_X$.

(4) If $U \wedge V \neq \mathbf{0}_X$, then it follows from the definition directly that $U\Sigma^* V$.

Remark 3.1. If $U, V \in L^X$, then $U\Sigma^* V$ iff $U_{\downarrow Y}\Sigma^* V_{\downarrow Y}$.

Theorem 3.2. *If Σ is proximity on L^Y; $Y \subset X$, then Σ^* is a proximity on L^X.*

Proof. (a) Let $U\overline{\Sigma^*}V$. Then $U \bigcap V = \mathbf{0}_X$ and $U_{\downarrow Y}\overline{\Sigma}V_{\downarrow Y}$. Then there exists $C \in L^Y$ for which $U_{\downarrow Y}\overline{\Sigma}C$ and $C^c\overline{\Sigma}V_{\downarrow Y}$, where the complement C^c of the fuzzy subset C is relative to L^Y. Therefore, $U\overline{\Sigma^*}W_1$ and $W_2\overline{\Sigma^*}V$; where $W_1, W_2 \in L^X$ and $(W_1)_{\downarrow Y} = C$ and $(W_2)_{\downarrow Y} = C^c$.

(b) Define $W_1(x) = C(x)$; for every $x \in Y$, $W_1(x) = 0_L$; for every $x \notin Y$ and define $W_2 = W_1^c$, where the complement W_1^c of the fuzzy subset W_1 is relative to the fuzzy space L^X.

(c) Therefore, $W_2(x) = C^c(x)$; $x \in Y$ and $W_2(x) = 1_L$; $x \notin Y$. Therefore, for $U\overline{\Sigma^*}V$, there exists $W_1 \in L^X$ such that $U\overline{\Sigma^*}W_1$ and $W_1^c\overline{\Sigma^*}V$.

Theorem 3.3. *If C_Σ and C_{Σ^*} are the closure operators on the (basic) proximity Σ on L^Y and the extended basic proximity Σ^* on L^X, respectively, then:*

$$C_{\Sigma^*}(A_{\uparrow X}) = C_\Sigma(A)|_{\uparrow X}; \quad A \in L^Y.$$

Proof. From the definition of the closure operator:

$C_\Sigma(A) = \vee\{Q \in L^Y : Q\Sigma A\}$, where Q is a fuzzy point of L^Y and

$C_{\Sigma^*}(U) = \vee\{P \in L^Y : P\Sigma^*U\}$, where P is a fuzzy point of L^X. For every fuzzy point $Q \in L^Y$, the fuzzy subset $Q_{\uparrow X}$ is a fuzzy point of L^X. It follows that if $Q\Sigma A$, then $Q_{\uparrow X}\Sigma^*A_{\uparrow X}$ and consequently, $C_{\Sigma}(A)|_{\uparrow X} \subset C_{\Sigma^*}(A_{\uparrow X})$; $A \in L^Y$.

Let $P \in L^X$ be a fuzzy point and $P\Sigma^*A_{\uparrow X}$:

(a) If $P(x_0) \neq 0_L$; for some point $x_0 \in Y$, then $Q = P_{\downarrow Y}$ is a fuzzy point in L^Y. It follows that $Q\Sigma A$, since the negative of this relation contradicts $P\Sigma^*A_{\uparrow X}$.

(b) If $P(x_0) \neq 0_L$ and $x_0 \in X - Y$, then $P \wedge A_{\uparrow X} = \mathbf{0}_X$, $P_{\downarrow Y} = \mathbf{0}_X$ and $P_{\downarrow Y}\overline{\Sigma}A$, which implies that $P\overline{\Sigma^*}A_{\uparrow X}$.

It follows from (a), (b) that the fuzzy point $P \in L^X$ satisfies that $P\Sigma^*A_{\uparrow X}$ iff $P_{\downarrow Y}\Sigma A$, then $C_{\Sigma^*}(A_{\uparrow X}) = C_{\Sigma}(A)|_{\uparrow X}$.

Theorem 3.4. *If Σ is a (basic) proximity on L^Y and Σ^* is the extended (basic) proximity on L^X, then from Theorem 3.3, $A \in L^Y$ is Σ-closed fuzzy subset iff $A_{\uparrow X}$ is Σ^*-closed fuzzy subset.*

Corollary 3.1. *Let Σ be a (basic) proximity on L^Y, Σ^* be the extended (basic) proximity on L^X and $Y \subset X$. Then:*

(1) *The fuzzy topology $\delta_{\Sigma} = \{A^c \in L^Y : C_{\Sigma}(A) = A\}$, which is induced by the basic fuzzy proximity Σ on L^Y, defines a family of open fuzzy subsets $\{A_{\uparrow X}^c \in L^X : C_{\Sigma}(A) = A\} = \{A_{\uparrow X}^c \in L^X : C_{\Sigma^*}(A_{\uparrow X}) = A_{\uparrow X}\}$ of the fuzzy topology δ_{Σ^*}, which is induced by the basic fuzzy proximity Σ^* on L^X.*

(2) *The fuzzy subset* $D \in L^Y$ *is a* Σ*-neighborhood of the fuzzy subset* $B \in L^Y$ *iff* $D_{\uparrow X} \in L^X$ *is a* Σ^**-neighborhood of the fuzzy subset* $B_{\uparrow X} \in L^X$.

Theorem 3.5. *Each basic proximity* Σ *on* L^X *defines a restricted basic proximity* $\Sigma^\star$ *on* (*or* Σ/Y), *where* $Y \subset X$ *as follows*: *for every* $A, B \in L^Y$:

$A\Sigma^\star B$ *if* $A_{\uparrow X}\Sigma B_{\uparrow X}$.

Theorem 3.6. *If* Σ *is proximity on* L^X, *then* $\Sigma^\star$ *is a restricted proximity on* L^Y; *where* $Y \subset X$.

The following theorem gives interesting relation between the closure operators on the basic proximity Σ on L^X and the restricted basic proximity $\Sigma^\star$ on Y, where $Y \subset X$.

Theorem 3.7. *If* C_Σ *and* $C_{\Sigma^\star}$ *are the closure operators on the* (*basic*) *proximity* Σ *on* L^X *and the restricted* (*basic*) *proximity* $\Sigma^\star$ *on* L^Y, *then the following relations are valid*: $C_\Sigma(U)|_{\downarrow Y} = C_{\Sigma^\star}(U_{\downarrow Y}); U \in L^X$.

Theorem 3.8. *If* Σ *is a* (*basic*) *proximity on* L^X *and* $\Sigma^\star$ *is the restricted* (*basic*) *proximity on* L^Y, *then if* $U \in L^X$ *is a* Σ*-closed fuzzy subset, then* $U_{\downarrow Y}$ *is a* $\Sigma^\star$*-closed fuzzy subset.*

Corollary 3.2. *Let* Σ *be a* (*basic*) *proximity on* L^X *and* $\Sigma^\star$ *be the restricted* (*basic*) *proximity on* L^Y; $Y \subset X$. *Then*:

(1) *The fuzzy topology* $\delta_\Sigma = \{U^c \in L^X : C_\Sigma(U) = U\}$, *which is induced by the* (*basic*) *fuzzy proximity* Σ *on* L^X, *defines a family of open fuzzy subsets*

$$\{(U_{\downarrow Y})^c \in L^Y : C_\Sigma(U) = U\} = \{(U_{\downarrow Y})^c \in L^Y : C_{\Sigma^\star}(U_{\downarrow Y}) = U_{\downarrow Y}\}$$

of the fuzzy topology $\delta_{\Sigma^{\star}}$, *which is induced by the* (*basic*) *fuzzy proximity*

$\Sigma^{\star}$ *on* L^{Y}.

(2) *The fuzzy subset* $U \in L^{X}$ *is a* Σ-*neighborhood of the fuzzy subset*

$V \in L^{X}$ *iff* $U_{\downarrow Y} \in L^{Y}$ *is a* $\Sigma^{\star}$-*neighborhood of the fuzzy subset* $V_{\downarrow Y} \in L^{Y}$.

4. The Lattice of the Power Set $P(\Lambda)$

Let Λ be a given nonempty set and $P(\Lambda)$ be the power set of Λ. It is known that $(P(\Lambda), \cup, \cap, ')$ is a complete and completely distributive lattice with order-reversing involution, whose minimal element $0_{P(\Lambda)} = \varnothing$, the maximal element $1_{P(\Lambda)} = \Lambda$ and the complement of $\Lambda_{1} \subset \Lambda$ is $\Lambda_{1}' = \Lambda - \Lambda_{1}$.

The definition of the fuzzy function can be reformulated for the $P(\Lambda)^{X}$ as follows:

Definition 4.1. Let X, Y be given universal sets. The fuzzy function from $P(\Lambda)^{X}$ into $P(\Lambda)^{Y}$ is a fuzzy function $\boldsymbol{F} = (F, f_{x}): X \rightarrow Y$; where $F : X \rightarrow Y$ is an ordinary function and $f_{x} : \Lambda \rightarrow \Lambda$; $x \in X$ are a family of onto ordinary functions (comembership functions), which induces a family of functions $f_{x} : P(\Lambda) \rightarrow P(\Lambda)$; $x \in X$ (using the same notations f_{x} for the induced functions) satisfies the following conditions:

(i) $f_{x}(\varnothing) = \varnothing$ and $f_{x}(\Lambda) = \Lambda$; $x \in X$,

(ii) clear that f_{x} is a nondecreasing function on $P(\Lambda)$:

$$f_{x}(\lambda) \subset f_{x}(\mu), \text{ if } \lambda \subset \mu, \text{ where } \lambda, \mu \in P(\Lambda); x \in X.$$

The action of the fuzzy function $\boldsymbol{F} = (F, f_{x})$ on the $P(\Lambda)$-fuzzy subsets A of X and the inverse image of the $P(\Lambda)$-fuzzy subset B of Y are defined as follows:

$$F^{\rightarrow}(A)(y) = \begin{cases} \bigcup_{y=F(x)} f_x(A(x)), & \text{if } F^{-1}(y) \neq \varnothing \\ 0_{P(\Lambda)}, & \text{if } F^{-1}(y) = \varnothing; \end{cases} \quad y \in Y \text{ and } A \in P(\Lambda)^X,$$

$$F^{\leftarrow}(B)(x) = \bigcup f_x^{-1}(B(F(x))); \quad x \in X \text{ and } B \in P(\Lambda)^Y.$$

The family of ordinary functions $F : X \to Y$ is embedded in the family of fuzzy functions $\boldsymbol{F} = (F, f_x) : X \to Y$, since each ordinary function F corresponds to the uniform fuzzy function $\boldsymbol{F} = (F, f_x = id)$, where id is the identity function.

4.1. Lattices on Λ

There are different partial ordered relations, each of which can be defined on the nonempty set Λ, where Λ contains at least two elements, and convert Λ into a lattice L. Denote the minimal element of L by 0_L and the maximal element for all these lattices is 1_L. Denote by $\mathcal{L}(\Lambda)$ the family of all lattices on Λ. Each lattice $L \in \mathcal{L}(\Lambda)$ is isomorphic to a sublattice of $P(\Lambda)$, which is defined by the correspondence $a \in L \leftrightarrow \{b \in L; 0_L \leq b \leq a\} = [0_L, a]$.

In Section 2, it is defined $P^*(L)$ as the collection of subsets of L containing its 0_L. If L is a lattice on Λ, then $P^*(L) \subset P(\Lambda)$; for every $L \in \mathcal{L}(\Lambda)$.

It is clear that $0_{P^*(L)} = \{0_L\}$ and $1_{P^*(L)} = L$, while $0_{P(\Lambda)} = \varnothing$ and $1_{P(\Lambda)} = \Lambda$, for every lattice L defined on Λ.

Example 4.1.1. Consider $\Lambda = \{a, b, c, d\}$. There are some lattices which are defined on Λ. Each lattice is defined by the corresponding ordered relation:

$$L_1 : a \leq b \leq c \leq d, \quad L_2 : a \leq b \leq d \leq c, \quad L_3 : b \leq a \leq c \leq d,$$

$$L_4 : a \leq b \leq d \text{ and } a \leq c \leq d.$$

Moreover, one can show that

$$P^*(L_1) = \{\{0_{L_1} = a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \Lambda\}$$

and

$$P^*(L_3) = \{\{0_{L_3} = b\}, \{b, a\}, \{b, c\}, \{b, d\}, \{b, a, c\}, \{b, a, d\}, \{b, c, d\}, \Lambda\}.$$

Theorem 4.1.1. $\bigcup_{L \in \mathcal{L}(\Lambda)} P^*(L) = P(\Lambda) - \varnothing.$

Proof. Since L is a lattice on Λ, each subset in $P^*(L)$ is a subset of Λ, containing 0_L. Therefore, $P^*(L) \subset P(\Lambda) - \varnothing$. Consequently, $\bigcup_{L \in \mathcal{L}(\Lambda)} P^*(L) \subset P(\Lambda) - \varnothing$.

On the other hand, let $A \in P(\Lambda) - \varnothing$. Choose $a \in A$ and $b \in \Lambda$ and $a \neq b$. Define the following order on $\Lambda : a \prec x \prec b;\ x \in \Lambda - \{a, b\}$. It is clear that the order $\preccurlyeq$ defines on Λ a lattice L and $A \in P^*(L)$.

4.2. Some relations between $P(\Lambda)^X$, L^X and 2^X

In [6], it is proved that the two operators i_X, j_X form adjunction functors. Moreover, it shown that i_X is the unique right adjoint of j_X and j_X is the unique left adjoint of i_X. Since $P^*(L) \subset P(\Lambda)$, the mappings i_X from L^X into $P^*(L)^X$ can be considered as an operator from L^X into $P(\Lambda)^X$; $L \in \mathcal{L}(\Lambda)$, using the same notation

$$i_X : L^X \to P^*(L)^X \subset P(\Lambda)^X,$$

$$i_X(A)(x) = [0_L, A(x)] \in P(\Lambda)^X;\ \forall x \in X,\ A \in L^X.$$

Moreover, the function $j_X : P^*(L)^X \to L^X$ can be extended to $k_X : P(\Lambda)^X \to L^X$, $k_X(V)(x) = \sup(V(x));\ \forall x \in X,\ V \in P(\Lambda)^X$, where the sup is taken over the lattice L.

It is clear that the operator $j_X : P^*(L)^X \to L^X$ is the restriction of the operator $k_X : P(\Lambda)^X \to L^X$ on $P^*(L)^X$.

On the other hand, the mapping $\chi_X : 2^X \to P(\Lambda)^X$ is defined as follows:

$$\chi_M(M)(x) = \begin{cases} \Lambda, & x \in M, \\ 0_{P(\Lambda)}, & x \notin M; \end{cases} \text{ for each } M \in 2^X \text{ and } x \in X.$$

And the mapping $w_X : P(\Lambda)^X \to 2^X$ is defined as follows:

$$w_X(V) = \text{supp}(V)(\text{support}(V)); \text{ for each } V \in P(\Lambda)^X.$$

Theorem 4.2.1. *The mappings i_X, k_X, χ_X and w_X are well defined and for all $\{A, B, A_k; k \in \Delta\} \subset L^X$, $\{M, N, M_k; k \in \Delta\} \subset 2^X$ and $\{V, U, U_k; k \in \Delta\} \subset P(\Lambda)^X$, the following properties are satisfied:*

(i) $i_X(A \vee B) = i_X(A) \cup i_X(B)$ *and* $i_X(\wedge_{k \in \Delta} A_k) = \cap_{k \in \Delta} i_X(A_k)$,

(ii) $A \le B$ *if and only if* $i_X(A) \subset i_X(B)$ *and* $V \subset U$ *implies* $k_X(V) \le k_X(U)$,

(iii) $k_X(\cup_{k \in \Delta} U_k) = \vee_{k \in \Delta} k_X(U_k)$ *and* $k_X(V \cap U) \le k_X(V) \wedge k_X(U)$,

(iv) $k_X(i_X(A)) = A$ *and* $U \subset i_X(k_X(U))$,

(v) $w_X(V \cap U) \subset w_X(V) \cap w_X(U)$ *and* $w_X(\cup_{k \in \Delta} U_k) = \vee_{k \in \Delta} w_X(U_k)$,

(vi) $M \subset N$ *if and only if* $\chi_X(M) \subset \chi_X(N)$ *and* $V \subset U$ *implies that* $w_X(V) \subset w_X(U)$,

(vii)

$\chi_X(\cup_{k \in \Delta} M_k) = \cup_{k \in \Delta} \chi_X(M_k)$ *and* $\chi_X(\cap_{k \in \Delta} M_k) = \cap_{k \in \Delta} \chi_X(M_k)$,

(viii) $w_X(\chi_X(M)) = M$ *and* $U \subset \chi_X(w_X(U))$.

Now, let $E = \{i_X(A) : A \in L^X\} \subset P^*(L)^X \subset P(\Lambda)^X$ and consider the restriction mapping $k_X / E : E \to L^X$. It is clear that the restriction mapping k_X / E satisfies that

$$(k_X / E)(i_X(A)) = k_X(i_X(A)) = A,\ i_X((k_X / E)(i_X(A))) = i_X(A);$$

$$\text{for all } A \in L^X.$$

Therefore, L^X is in one to one correspondence with the subfamily

$$E = \{i_X(A) : A \in L^X\} \subset P(L)^X, \text{ which means that}$$

$$L^X \text{ is embedded in } P(\Lambda)^X.$$

In addition, χ_X is an embedded operator, since 2^X is in one to one correspondence with the family of characteristic functions $\{\chi_X(M) : M \in 2^X\} \subset P(\Lambda)^X$.

Since $P(\Lambda)^X$ and L^X are two preordered sets with respect to $\subset$ and $\leq$, respectively, and i_X, k_X, χ_X and w_X are order preserving mappings, i_X, k_X, χ_X and w_X are categorical functors.

Theorem 4.2.2. (1) *The two mappings i_X, k_X form adjunction functors. Moreover, i_X is the unique right adjoint of k_X and k_X is the unique left adjoint of i_X.*

(2) *The two mappings χ_X, w_X form adjunction functors. Moreover, χ_X is the unique right adjoint of w_X and w_X is the unique left adjoint of χ_X.*

Proof. (1) Notice that L^X, $P(\Lambda)^X \in |\textbf{POSET}|$, L^X is closed under arbitrary supremum operation and $i_X : L^X \to P(\Lambda)^X$ preserves order. Then, using the adjoint functor theorems [6], i_X is a functor from L^X into $P(\Lambda)^X$

and $k_X : P(\Lambda)^X \to L^X$;   where $k_X(V)(x) = \mathrm{supp}(V(x))$; $\forall x \in X$, $V \in P(\Lambda)^X$ is the unique functor such that i_X and k_X form an adjunction $(i_X \dashv k_X)$.

(2) Similarly, 2^X, $P(\Lambda)^X \in |\boldsymbol{POSET}|$, 2^X is closed under arbitrary supremum operation and $\chi_X : 2^X \to P(\Lambda)^X$ preserves order. Then, using the adjoint functor theorems [6], χ_X is a functor from 2^X into $P(\Lambda)^X$ and $w_X : P(\Lambda)^X \to 2^X$; where $w_X(V) = \mathrm{supp}(V)$; for each $V \in P(\Lambda)^X$ is the unique functor such that i_X and k_X form an adjunction $(\chi_X \dashv w_X)$.

Now, define the functors $\boldsymbol{R}$, $\boldsymbol{S}$ and $\boldsymbol{T}$ from the category $\boldsymbol{SET}$ to the category $\boldsymbol{POSET}$ as follows:

$$\boldsymbol{T}(X) = 2^X, \; \boldsymbol{T}(F : X \to Y) = F^{\to} : 2^X \to 2^Y,$$

$$\boldsymbol{R}(X) = L^X, \; \boldsymbol{R}(F : X \to Y) = F_L^{\to} : L^X \to L^Y,$$

$$\boldsymbol{S}(X) = P(\Lambda)^X, \; \boldsymbol{T}(F : X \to Y) = (F, id_\Lambda)^{\to} : P(\Lambda)^X \to P(\Lambda)^Y$$

for all X, $Y \in |\boldsymbol{SET}|$ and for all $F \in \boldsymbol{SET}(X, Y)$.

Theorem 4.2.3. *Let $\boldsymbol{R}$, $\boldsymbol{S}$ and $\boldsymbol{T}$ be defined as above. Then*:

(1) *The mappings $\bar{i} : \boldsymbol{R} \to \boldsymbol{S}$ and $\bar{k} : \boldsymbol{S} \to \boldsymbol{R}$ with components* $\{i_X : X \in |\boldsymbol{SET}|\}$ *and* $\{k_X : X \in |\boldsymbol{SET}|\}$ *form natural transformations.*

(2) *The mappings $\bar{\chi} : \boldsymbol{T} \to \boldsymbol{S}$ and $\bar{w} : \boldsymbol{S} \to \boldsymbol{T}$ with components* $\{\chi_X : X \in |\boldsymbol{SET}|\}$ *and* $\{w_X : X \in |\boldsymbol{SET}|\}$ *form natural transformations.*

5. Proximities on $P(\Lambda)^X$, $P^*(L)^X$ and L^X; $L \in \mathcal{L}(\Lambda)$

In this section, we study some relations between the basic proximities on L^X, some induced proximities on $P^*(L)^X$; $L \in \mathcal{L}(\Lambda)$ and $P(\Lambda)^X$.

Notations 5.1. In this section, we shall use the notations:

The empty L-fuzzy subset of X is denoted by $\mathbf{0}_{(L, X)}$, where $\mathbf{0}_{(L, X)}(x)$ $= 0_L$; $\forall x \in X$. The greatest element of L-fuzzy subset of X is denoted by $\mathbf{1}_{(L, X)}$, where $\mathbf{1}_{(L, X)}(x) = 1_L$, $\forall x \in X$.

Theorem 5.1. *Each L-basic proximity* Σ *on a set* L^X *induces* $P^*(L)$- *basic proximity* Σ^* *on X which is defined as follows*: *for every* $U, V \in$ $P^*(X)^X$, $U\Sigma^*V$ *if*: $U \bigcap V \neq \mathbf{0}_{(P^*(L), X)}$ *or* $j_X(U)\Sigma j_X(V)$, *where* $j_X(V)(x)$ $= \text{supp}(V(x))$; $x \in X$.

Proof. (1) Let $U\Sigma^*V$. Then:

- If $U \bigcap V \neq \mathbf{0}_{(P^*(L), X)}$, then it follows that $V \bigcap U \neq \mathbf{0}_{(P^*(L), X)}$ and $V\Sigma^*U$.

- If $j_X(U)\Sigma j_X(V)$, then $j_X(V)\Sigma j_X(U)$. Therefore, it follows that $(V\Sigma^*U)$.

(2) Let $(U \bigcup V)\Sigma^*W$. Then:

- If $(U \bigcup V) \bigcap W \neq \mathbf{0}_{(P^*(L), X)}$, then it follows that $U \bigcap W \neq \mathbf{0}_{(P^*(L), X)}$ or $V \bigcap W \neq \mathbf{0}_{(P^*(L), X)}$. Therefore, $U\Sigma^*W$ or $V\Sigma^*W$.

- If $j_X(U \bigcup V)\Sigma j_X(W)$, then $j_X(U)\Sigma j_X(W)$ or $j_X(V)\Sigma j_X(W)$. Consequently, $U\Sigma^*W$ or $V\Sigma^*W$.

(3) Let $U\Sigma^*V$. Then:

- If $U \bigcap V \neq \mathbf{0}_{(P^*(L), X)}$, then $U \neq \mathbf{0}_{(P^*(L), X)}$ and $V \neq \mathbf{0}_{(P^*(L), X)}$.

- If $j_X(U)\Sigma j_X(V)$, then $j_X(U) \neq \mathbf{0}_{(L, X)}$ and $j_X(V) \neq \mathbf{0}_{(L, X)}$. Consequently, $U \neq \mathbf{0}_{(P^*(L), X)}$ and $V \neq \mathbf{0}_{(P^*(L), X)}$.

(4) If $U \cap V \neq \mathbf{0}_{(P^*(L), X)}$, then $j_X(U) \cap j_X(V) \neq \mathbf{0}_{(L, X)}$ and $U\Sigma^* V$.

Corollary 5.1. *If* Σ^* *is* $P^*(L)$-*basic proximity on* $P^*(L)^X$, *which is induced by the L-basic proximity* Σ *on* L^X, *then* $i_X(A)\Sigma^* i_X(B)$ *for every* $A\Sigma B$; A, $B \in L^X$. *Since* $j_X(i_X(A)) = A$ *and* $j_X(i_X(B)) = B$.

Theorem 5.2. *Let* Σ *be a proximity on* L^X; *where L is a complete lattice and* Σ^* *be the induced proximity by* Σ *on* $P^*(L)^X$. *Then* $C_{\Sigma^*}(i_X(A)) = i_X(C_\Sigma(A))$; $A \in L^X$, *where* C_Σ *is the closure operator on the proximity* Σ *on* L^X *and* C_{Σ^*} *is the induced closure operator* Σ^* *on* $P^*(L)^X$.

Proof. (a) From the definition of the closure operator, we have:

$$C_{\Sigma^*}(i_X(A)) = \vee\{P \in P^*(L)^X : P\Sigma^* i_X(A)\},$$

where P is a fuzzy point in $P^*(L)^X$.

Let Q be a fuzzy point in L^X : $Q(x_0) = r \neq 0_L$ and $Q(x) = 0_L$; $x \neq x_0$, for some $x_0 \in X$. If $Q\Sigma A$, then we have $i_X(Q)\Sigma^* i_X(A)$, where $i_X(Q)$ is a fuzzy point in $P^*(L)^X$. Hence, $i_X(Q) \subset C_{\Sigma^*}(i_X(A))$. Therefore, each fuzzy point $Q \in C_\Sigma(A)$ defines a fuzzy point $i_X(Q) \in C_{\Sigma^*}(i_X(A))$. Consequently, $\bigcup_{Q\Sigma A} i_X(Q) \subset C_{\Sigma^*}(i_X(A))$ and $i_X(\vee_{Q\Sigma A} Q) \subset C_{\Sigma^*}(i_X(A))$, i.e., $i_X(C_\Sigma(A)) \subset C_{\Sigma^*}(i_X(A))$.

(b) Let $P \in P^*(L)^X$ be a fuzzy point and $P \in C_{\Sigma^*} i_X(A)$. Then there exists $x_0 \in X$ such that $P(x_0) \neq 0_{P^*(L)}$, $P(x) = 0_{P^*(L)}$; $x \neq x_0$ and $P\Sigma^* i_X(A)$. Let $\alpha = \vee P(x_0)$, where the sup is taken over $P(x_0) \subset L$. Consider, the fuzzy point $P_1 \in P^*(L)^X$, where $P_1(x) = 0_{P^*(L)}$; $x \neq x_0$ and $P_1(x_0) = [0_L, \alpha]$. It is clear that $P_1 \supset P$, then $P_1 \Sigma^* i_X(A)$ and

$i_X(Q_1)\Sigma^* i_X(A)$, where Q_1 is a fuzzy point in L^X for which $Q_1(x_0) = \alpha$. Hence, $Q_1 \Sigma A$ and it follows that $Q_1 \in C_\Sigma(A)$. Consequently, $P \subset i_X(Q_1) \subset i_X(C_\Sigma(A))$. Then $C_{\Sigma^*}(i_X(A)) \subset i_X(C_\Sigma(A))$.

(c) From (a), (b) it follows that $C_{\Sigma^*}(i_X(A)) = i_X(C_\Sigma(A));\ A \in L^X$.

Theorem 5.3. *Each Σ-closed fuzzy subset A corresponds to a Σ^*-closed fuzzy subset $i_X(A)$.*

Corollary 5.2. *Let Σ be a basic proximity on L^X and Σ^* be the induced basic proximity on $P^*(L)^X$. Then*:

(1) *The fuzzy topology $\delta_\Sigma = \{A^c \in L^X : C_\Sigma(A) = A\}$, which is defined by the basic fuzzy proximity Σ on L^X, defines a family of open fuzzy subsets $\{i_X(A)^c \in P^*(L)^X : C_{\Sigma^*}(i_X(A)) = i_X(A)\}$ of the fuzzy topology δ_{Σ^*}, which is induced by the basic fuzzy proximity Σ^* on $P^*(L)^X$.*

(2) *The fuzzy subset $D^c \in L^X$ is a Σ-neighborhood of the fuzzy subset $B \in L^X$ iff $i_X(D)^c \in P^*(L)^X$ is a Σ^*-neighborhood of the fuzzy subset $i_X(B) \in P^*(L)^X$.*

Theorem 5.1 shows that each basic proximity Σ on L^X induces basic proximity Σ^* on $P^*(L)$. If Λ is a given set, L is a lattice defined on Λ and i_X is the defined function from L^X to $P(\Lambda)^X$, then we can get the following theorems, which can be considered as extensions of Theorem 5.1 and Theorem 5.2.

Theorem 5.4. *Each L-basic proximity Σ on L^X; $L \in \mathcal{L}(\Lambda)$ induces $P(\Lambda)$-basic proximity Σ^Δ on $P(\Lambda)^X$ which is defined by*: *for every $U, V \in P(\Lambda)^X$, $U\Sigma^* V$ if*:

$U \cap V \neq \mathbf{0}_{(P^*(L), X)}$ *or $j_X(U)\Sigma j_X(V)$, where $j_X(V)(x) = \text{supp}\, v_x;\ x \in X$.*

Theorem 5.5. *Let* Σ *be a proximity on* L^X*; where* $L \in \mathcal{L}(\Lambda)$ *is a complete lattice and* Σ^Δ *be the induced proximity by* Σ *on* $P(\Lambda)^X$*. Then the closure operator* C_Σ *of the proximity* Σ *on* L^X *and the closure operator* C_{Σ^Δ} *of the proximity* Σ^Δ *on* $P(\Lambda)^X$ *satisfy the relation*: $C_{\Sigma^\Delta}(i_X(A)) = i_X(C_\Sigma(A))$; $A \in L^X$.

Theorem 5.4 and Theorem 5.5 give the following theorem.

Theorem 5.6. *For every nonempty set* Λ *and for every basic proximity* Σ *on* L^X*;* $L \in \mathcal{L}(\Lambda)$*, there exists a basic proximity* Σ^Δ *on* $P(\Lambda)^X$ *such that the family of* Σ*-closed fuzzy subsets* $\{A\}$ *in* L^X *corresponds to a subfamily* $\{i_X(A)\}$ *of* Σ^Δ*-closed fuzzy subsets in* $P(\Lambda)^X$*, which is defined by the correspondence* i_X.

Corollary 5.3. *Let* Σ *be a basic proximity on* L^X *and* Σ^Δ *be the induced basic proximity on* $P(\Lambda)^X$*. Then*:

(1) *The fuzzy topology* $\delta_\Sigma = \{A^c \in L^X : C_\Sigma(A) = A\}$*, which is defined by the basic fuzzy proximity* Σ *on* L^X*, induces a subfamily of* Σ^Δ*-open fuzzy subsets*: $\{i_X(A)^c \in P^*(L)^X : C_{\Sigma^*}(i_X(A)) = i_X(A)\}$ *of the fuzzy topology* δ_{Σ^Δ} *on* $P(\Lambda)^X$.

(2) *The fuzzy subset* $A^c \in L^X$ *is a* Σ*-neighborhood of the fuzzy subset* $B \in L^X$ *if and only if* $i_X(A)^c \in P(\Lambda)^X$ *is a* Σ^Δ*-neighborhood of the fuzzy subset* $i_X(B) \in P(\Lambda)^X$.

6. (L, M)-proximity Spaces

Let *L*, *M* be complete lattices. In [18], it was defined the *L*-fuzzy proximity. In the rest of this article, we shall use the following definition of (L, M)-proximity, in which we replace the lattice *L* of the degree of

openness by a lattice M which can be chosen more suitably and simpler than the lattice L in some cases.

Definition 6.1. A mapping $\delta : L^X \times L^X \to M$ is said to be (L, M)-*proximity* on X if it satisfies the following conditions:

(1) $\delta(\mathbf{1}_X, \mathbf{0}_X) = \delta(\mathbf{0}_X, \mathbf{1}_X) = 0_M$,

(2) $\delta(\lambda \vee \mu, \omega) = \delta(\lambda, \omega) \vee \delta(\mu, \omega)$,

(3) if $\delta(\lambda, \mu) \neq 1$, then $\lambda \leq \mu^c$,

(4) for any $\lambda, \mu \in L^X$, then $\delta(\lambda, \mu) \geq \wedge_{\rho \in L^X} (\delta(\lambda, \rho) \vee \delta(\rho^c, \mu))$,

(5) $\delta(\lambda, \mu) = \delta(\mu, \lambda)$.

δ is called (L, M)-*basic proximity* if it satisfied the conditions (1), (2), (3) and (5). The pair (L^X, δ) is called an (L, M)-*proximity space*.

Definition 6.2. A mapping $F : X \to Y$ is called (L, M)-*proximity map* relative to the (L, M)-(basic) proximity δ_1 on L^X and (L, M)-(basic) proximity δ_2 on L^Y, if for all $A, B \in L^X$, the inequality $\delta_1(A, B) \leq \delta_2(F_L^{\to}(A), F_L^{\to}(B))$ holds.

Definition 6.3. Let δ be an (L, M)-basic proximity on L^X. The closure operator $C_\delta : L^X \to L^X$ is defined as follows: for every $A \in L^X$; $q \in C_\delta(A)$ if and only if $\delta(q, A) = 1_L$ for each L-fuzzy point q of $X : C_\delta(A) = \vee \{q \in L^X : \delta(q, A) = 1_L\}$.

This closure operator defines the induced fuzzy topology on X (Chang fuzzy topology) $\tau_\delta = \{A^c \in L^X : C_\delta(A) = A\}$.

7. Induced and Restricted (L, M)-fuzzy Proximities

Let L^X, L^Y be two families of fuzzy subsets and $Y \subset X$. In this article,

it is discussed how to extend a given (L, M)-proximity on L^Y to an (L, M)-proximity on L^X or how to restrict a given (L, M)-proximity on L^X to an (L, M)-proximity on L^Y.

In this section, we shall use the notations which are introduced in Section 3.

Theorem 7.1. *Each* (L, M)-*basic proximity* $\delta : L^Y \times L^Y \to M$ *on* L^Y; $Y \subset X$ *can be extended to an* (L, M)-*basic proximity* $\delta^* : L^X \times L^X \to M$ *on* L^X *as follows: for every* $U, V \in L^X$: $\delta^*(U, V) = 1_M$ *if* $U \wedge V \neq \mathbf{0}_X$, *or* $\delta^*(U, V) = \delta(U_{\downarrow Y}, V_{\downarrow Y})$.

Proof.

(1) $\delta^*(\mathbf{1}_X, \mathbf{0}_X) = \delta^*(\mathbf{0}_X, \mathbf{1}_X) = 0_M$.

- $\delta^*(\mathbf{1}_X, \mathbf{0}_X) = \delta(\mathbf{1}_Y, \mathbf{0}_Y) = 0_M$ and $\delta^*(\mathbf{0}_X, \mathbf{1}_X) = \delta(\mathbf{0}_Y, \mathbf{1}_Y) = 0_M$.

(2) $\delta^*(U \vee V, W) = \delta^*(U, W) \vee \delta^*(V, W)$.

- If $(U \vee V) \wedge W \neq \mathbf{0}_X$, then $U \wedge W \neq \mathbf{0}_X$ or $V \wedge W \neq \mathbf{0}_X$. Consequently, $\delta^*(U \vee V, W) = \delta^*(U, W) \vee \delta^*(U, W)$.

- If $(U \vee V) \wedge W = \mathbf{0}_X$, then

$$\delta^*(U \vee V, W) = \delta((U \vee V)_{\downarrow Y}, W_{\downarrow Y}) = \delta(U_{\downarrow Y} \vee V_{\downarrow Y}, W_{\downarrow Y})$$

$$= \delta(U_{\downarrow Y}, W_{\downarrow Y}) \vee \delta(U_{\downarrow Y}, W_{\downarrow Y}) = \delta^*(U, W) \vee \delta^*(U, W).$$

(3) If $\delta^*(U, V) \neq 1_M$, then $U \leq V^c$.

- If $\delta^*(U, V) = \delta(U_{\downarrow Y}, V_{\downarrow Y}) \neq 1_M$, then $U \wedge V = \mathbf{0}_X$ and consequently, $U \leq V^c$.

(4) $\delta^*(U, V) = \delta^*(V, U)$.

- If $U \wedge V \neq \mathbf{0}_X$, then $V \wedge U \neq \mathbf{0}_X$ and $\delta^*(U, V) = \delta^*(V, U)$.

- If $U \wedge V = \mathbf{0}_X$, then $\delta^*(U, V) = \delta(U_{\downarrow Y}, V_{\downarrow Y}) = \delta(V_{\downarrow Y}, U_{\downarrow Y}) = \delta^*(V, U)$.

It follows that if δ is an (L, M)-basic proximity on L^Y; $Y \subset X$, then δ^* is an (L, M)-fuzzy basic proximity on L^X.

Theorem 7.2. *If δ is an (L, M)-proximity on L^Y; $Y \subset X$, then δ^* is (L, M)-proximity on L^X.*

Proof. It is sufficient to prove that: for any $U, V \in L^X$, then there exists $W \in L^X$ such that $\delta^*(U, V) \geq \wedge_{W \in L^X} (\delta^*(U, W) \vee \delta^*(W^c, V))$.

(1) If $U \wedge V \neq \mathbf{0}_X$, then $\delta^*(U, V) = 1_M$. It follows that the above inequality is valid for every $W \in L^X$.

(2) If $\wedge V = \mathbf{0}_X$, then $\delta^*(U, V) = \delta(U_{\downarrow Y}, V_{\downarrow Y})$. Then there exists $w \in L^Y$ such that

$$\delta^*(U, V) = \delta(U_{\downarrow Y}, V_{\downarrow Y}) \geq \wedge_{w \in L^Y} (\delta(U_{\downarrow Y}, w) \vee \delta(w^c, V_{\downarrow Y})),$$

where the complement w^c is taken in the fuzzy space L^Y, denote by $\mathfrak{I} \subset L^Y$ the family of all w satisfying the above inequality.

(3) For each $w \in \mathfrak{I}$, we define the fuzzy subset $W \in L^X$ as follows:

$W(x) = w(x)$; $x \in Y$ and $W(x) = 0_L$; $x \notin Y$. The family of all defined W, for each $w \in L^Y$ will be denoted by $\mathfrak{I}$. It is clear that $W_{\downarrow Y} = w$ and therefore $\delta^*(U, W) = \delta(U_{\downarrow Y}, w)$. Moreover, W^c is the complement of W in L^X which satisfies that $W^c(x) = w^c(x)$; $x \in Y$, $W^c(x) = 1_L$ and $(W^c)_{\downarrow Y} = w^c$. It follows that $\delta^*(W^c, V) = \delta(w^c, V_{\downarrow Y})$. Therefore,

$$\delta^*(U, V) = \delta(U_{\downarrow Y}, V_{\downarrow Y}) \geq \wedge_{W \in \mathfrak{I}}(\delta^*(U, W) \vee \delta^*(W^c, V)).$$

Consequently, $\delta^*(U, V) \geq \wedge_{W \in L^X}(\delta^*(U, W) \vee \delta^*(W^c, V))$, since the family of all, which is satisfied this inequality, containing $\mathfrak{I}$.

Theorem 7.3. *If C_δ and C_{δ^*} are the closure operators on the (L, M)-(basic) proximity δ on L^Y and the extended (L, M)-(basic) proximity δ^* on L^X, respectively, then $C_{\delta^*}(A_{\uparrow X}) = C_\delta(A)|_{\uparrow X}; A \in L^Y$.*

Theorem 7.4. *Each C_δ-closed fuzzy subset $A \in L^Y$ corresponds to an C_{δ^*}-closed fuzzy subset $A_{\uparrow X} \in L^X$.*

Corollary 7.1. *Let δ be a (basic) proximity on L^Y, δ^* be the extended (basic) proximity on L^X and $Y \subset X$. Then:*

(1) The fuzzy topology $\tau_\delta = \{A^c \in L^Y : C_\delta(A) = A\}$, which is induced by the basic fuzzy proximity δ on L^Y, defines a family of open fuzzy subsets $\{A^c_{\uparrow X} \in L^X : C_\delta(A) = A\} = \{A^c_{\uparrow X} \in L^X : C_{\delta^*}(A_{\uparrow X}) = A_{\uparrow X}\}$ of the fuzzy topology τ_{δ^*}, which is induced by the basic fuzzy proximity δ^* on L^X.

(2) The fuzzy subset $D \in L^Y$ is a δ-neighborhood of the fuzzy subset $B \in L^Y$ iff $D_{\uparrow X} \in L^X$ is a δ^*-neighborhood of the fuzzy subset $B_{\uparrow X} \in L^X$.

Theorem 7.5. *Each (L, M)-basic proximity $\delta : L^X \times L^X \to M$ on L^X defines restricted (L, M)-basic proximity $\delta^\star : L^Y \times L^Y \to M$ on L^Y(or Σ/Y), where $Y \subset X$ as follows: $\delta^\star(A, B) = \delta(A_{\uparrow X}, B_{\uparrow X}); A, B \in L^Y$.*

Theorem 7.6. *If $\delta : L^X \times L^X \to M$ is an (L, M)-proximity on L^X, then it defines $\delta^\star : L^Y \times L^Y \to M$, which is an (L, M)-restricted proximity on L^Y.*

Theorem 7.7. *If* C_δ *and* $C_{\delta^\star}$ *are the closure operators of the (basic) proximity* δ *on* L^X *and the restricted (basic) proximity* $\delta^\star$ *on* L^Y, *then the following relations are valid:* $[C_\delta(A_{\uparrow X})]_{\downarrow Y} = C_{\delta^\star}(A);$ $A \in L^Y$ *and*

$C_\delta(U)|_{\downarrow Y} = C_{\delta^\star}(U_{\downarrow Y});$ $U \in L^X$.

Notice that the family of points P for which $P_{\downarrow Y} = \mathbf{0}_{(L,Y)}$ is deleted from the above supremum, since this family does not affect the supremum.

Theorem 7.8. *Each* C_δ-*closed fuzzy subset* $U \in L^X$ *corresponds to a* C_{δ^*}-*closed fuzzy subset* $U_{\downarrow X} \in L^Y$.

The following results are directly obtained.

Corollary 7.2. *Let* δ *be an* (L, M)-*(basic) proximity on* L^X *and* $\delta^\star$ *be the restricted* (L, M)-*(basic) proximity on* L^Y; $Y \subset X$. *Then*:

(1) *The fuzzy topology* $\tau_\delta = \{U^c \in L^X : C_\delta(U) = U\}$, *which is induced by the* (L, M)-*(basic) proximity* δ *on* L^X, *defines a family of open fuzzy subsets* $\{(U_{\downarrow Y})^c \in L^Y : C_\delta(U) = U\} = \{(U_{\downarrow Y})^c \in L^Y : C_{\delta^\star}(U_{\downarrow Y}) = U_{\downarrow Y}\}$ *of the fuzzy topology* $\tau_{\delta^\star}$, *which is induced by the (basic) proximity* $\delta^\star$ *on* L^Y.

(2) *The fuzzy subset* $U \in L^X$ *is a* δ-*neighborhood of the fuzzy subset* $V \in L^X$ *iff* $U_{\downarrow Y} \in L^Y$ *is a* $\delta^\star$-*neighborhood of the fuzzy subset* $V_{\downarrow Y} \in L^Y$.

8. (L, M)-fuzzy Proximities on $P(\Lambda)^X$, $P^*(L)^X$ and L^X; $L \in \mathcal{L}(\Lambda)$

In this section, we obtained some relations between the (L, M)-basic fuzzy proximities on L^X, $P(L)^X$; $L \in \mathcal{L}(\Lambda)$ and $P(\Lambda)^X$. It must be remarkable that, it is difficult to get any relationship between the (L, M)-proximities on these fuzzy spaces in case of the absence of any relationship

between the defined complement operations in these spaces. Due to this situation, we replace condition (3) in Definition 6.1 by condition (3*):

(3*) if $\lambda \wedge \mu \neq 0_{(L, X)}$, then $\delta(\lambda, \mu) = 1_M$, $\lambda, \mu \in L^X$.

This condition in $P^*(L)^X$ (or in $P(\Lambda)^X$) fuzzy spaces takes the form:

(3*) if $U \cap V \neq \mathbf{0}_{(P^*(L), X)}$, then $\delta(U, V) = 1_M$, $U, V \in P^*(L)^X$.

Theorem 8.1. *Each* (L, M)*-basic proximity* $\delta : L^X \times L^X \to M$ *on the fuzzy family* L^X *induces* $(P^*(L), M)$*-basic proximity* $\delta^* : P^*(L)^X \times P^*(L)^X \to M$ *on the fuzzy family* $P^*(L)^X$, *which is defined as follows*:

For every $U, V \in P^*(L)^X : \delta^*(U, V) = 1_M$, *if* $U \cap V \neq \mathbf{0}_{(P^*(L), X)}$ *and*

otherwise $\delta^*(U, V) = \delta(j_X(U), j_X(V))$; *where* $j_X(V)(x) = \text{supp}(V(x))$; $x \in X$.

Proof.

(1)

$$\delta^*(\mathbf{1}_{(P^*(L), X)}, \mathbf{0}_{(P^*(L), X)}) = \delta^*(\mathbf{0}_{(P^*(L), X)}, \mathbf{1}_{(P^*(L), X)}) = 0_M,$$

$$\delta^*(\mathbf{1}_{(P^*(L), X)}, \mathbf{0}_{(P^*(L), X)}) = \delta(j_X(\mathbf{1}_{(P^*(L), X)}), j_X(\mathbf{0}_{(P^*(L), X)}))$$

$$= \delta(\mathbf{1}_{(L, X)}, \mathbf{0}_{(L, X)}) = 0_M.$$

Similarly, one can show that $\delta^*(\mathbf{0}_{(P^*(L), X)}, \mathbf{1}_{(P^*(L), X)}) = 0_M$.

(2) $\delta^*(U \vee V, W) = \delta^*(U, W) \vee \delta^*(V, W)$:

• If $(U \vee V) \cap W \neq \mathbf{0}_{(P^*(L), X)}$, then $U \cap W \neq \mathbf{0}_{(P^*(L), X)}$ or $V \cap W \neq \mathbf{0}_{(P^*(L), X)}$. It follows that $\delta^*(U \vee V, W) = 1_M$ and $\delta^*(U, W) = 1_M$ or $\delta^*(V, W) = 1_M$. In this case, the equality is valid.

- Otherwise

$$\delta^*(U \cup V, W) = \delta(j_X(U \cup V), j_X(W)) = \delta(j_X(U) \cup j_X(V), j_X(W))$$

$$= \delta(j_X(U), j_X(W)) \vee \delta(j_X(V), j_X(W))$$

$$= \delta^*(U, W) \vee \delta^*(V, W).$$

(3) If $U \cap V \neq \mathbf{0}_{(P^*(L), X)}$, then $\delta^*(U, V) = 1_M$. This condition follows

directly from the definition δ^*.

(4) $\delta^*(U, V) = \delta^*(V, U)$:

- If $U \cap V \neq \mathbf{0}_{(P^*(L), X)}$, then $\delta^*(U, V) = 1_M = \delta^*(V, U)$.

- Otherwise

$$\delta^*(U, V) = \delta((j_X(U), j_X(V))) = \delta((j_X(V), j_X(U))) = \delta^*(V, U).$$

Corollary 8.1. *If* δ^* *is* $(P^*(L), M)$*-basic proximity on* $P^*(L)^X$, *which is induced by the* (L, M)*-basic proximity* δ *on* L^X, *then* $\delta^*(i_X(A), i_X(B))$ $= \delta(A, B)$, *where* $A, B \in L^X$.

Theorem 8.2. *Let* δ *be an* (L, M)*-basic proximity on* L^X; *where L is a complete lattice and* δ^* *be the induced* (L, M)*-basic proximity by* δ *on* $P^*(L)^X$. *Then the closure operator* C_δ *on the proximity* δ *on* L^X *and the closure operator* C_{δ^*} *on the induced proximity* δ^* *on* $P^*(L)^X$ *satisfy the relation*:

$$C_{\delta^*}(i_X(A)) = i_X(C_\delta(A)); \; A \in L^X.$$

Proof. (a) From the definition of the closure operator, we have:

$$C_{\delta^*}(i_X(A)) = \bigcup \{P \in P^*(L)^X : \delta^*(P, i_X(A)) = 1_M\},$$

where P is a fuzzy point in $P^*(L)^X$. Let $Q \in L^X$ be a fuzzy point, for which $Q(x_0) = r \neq 0_L$ and $P(x) = 0_L$; $x \neq x_0$, for some $x_0 \in X$. Since $C_\delta(A) = \vee\{Q \in L^X : \delta(Q, A) = 1_M\}$, each fuzzy point $Q \in C_\delta(A)$ defines a fuzzy point $i_X(Q) \in P^*(L)^X$, satisfying that $\delta^*(i_X(Q), i_X(A)) = \delta(Q, A)$ $= 1_M$, which means that $i_X(Q) \in C_{\delta^*}(i_X(A))$. Therefore, $\cup_{Q\delta A}\, i_X(Q)$ $\subset C_{\delta^*}(i_X(A))$ and $i_X(\vee_{Q\delta A}\, Q) \subset C_{\delta^*}(i_X(A))$, and this means $i_X(C_\delta(A)) \subset$ $C_{\delta^*}(i_X(A))$.

(b) Let $P \in P^*(L)^X$ be a fuzzy point and $P \in C_{\delta^*}(i_X(A))$. Then there exists $x_0 \in X$ such that $P(x_0) \neq 0_L$ and $P(x) = 0_L$; $x \neq x_0$ and $\delta^*(P, i_X(A)) = 1_M$. Let $\alpha = \vee P(x_0)$, where the sup is taken over $P(x_0) \subset L$. Denote by $Q_1 \in L^X$ the fuzzy point for which: $Q_1(x_0) = \alpha$ and $Q_1(x) = 0_L$; $x \neq x_0$. Consider the fuzzy point $P_1 \in P^*(L)^X$, where $P_1 = i_X(Q_1)$. One can show that $P_1 \supset P$. Notice that

$$1_M = \delta^*(P, i_X(A)) = \delta^*(P_1, i_X(A)) = \delta^*(i_X(Q_1), i_X(A)) = \delta(Q_1, A).$$

Therefore, the fuzzy point $P \in C_{\delta^*}(i_X(A))$ implies that the fuzzy point $Q_1 \in C_\delta(A)$. Consequently, $P \subset P_1 = i_X(Q_1) \subset i_X(C_\delta(A))$. And so $C_{\delta^*}(i_X(A))$ $\subset i_X(C_\delta(A))$.

From (a), (b), it follows that $C_{\delta^*}(i_X(A)) = i_X(C_\delta(A))$; $A \in L^X$.

Theorem 8.3. *The mapping i_X translates the family of δ-closed fuzzy subsets in L^X into a subfamily of δ^*-closed fuzzy subsets in $P^*(L)^X$.*

Corollary 8.2. *Let δ be an (L, M)-basic proximity on L^X and δ^* be the induced (L, M)-basic proximity on $P^*(L)^X$. Then:*

(1) *The fuzzy topology* $\tau_\delta = \{A^c \in L^X : C_\delta(A) = A\}$, *which is defined by the* (L, M)*-basic fuzzy proximity* δ *on* L^X, *defines a family of open fuzzy subsets* $\{i_X(A)^c \in P^*(L)^X : C_{\delta^*}(i_X(A)) = i_X(A)\}$ *of the fuzzy topology* τ_{δ^*}, *which is induced by the* (L, M)*-basic fuzzy proximity* δ^* *on* $P^*(L)^X$.

(2) *The fuzzy subset* $D^c \in L^X$ *is a* Σ*-neighborhood of the fuzzy subset* $B \in L^X$ *iff* $i_X(D)^c \in L^X$ *is a* δ^**-neighborhood of the fuzzy subset* $i_X(B) \in P^*(L)^X$.

Theorem 8.4. *Each* $(P^*(L), M)$*-basic proximity* $\delta : P^*(L)^X \times P^*(L)^X \to M$ *on the fuzzy family* $P^*(L)^X$ *induces* (L, M)*-basic proximity* $\delta^\Diamond : L^X \times L^X \to M$ *on the fuzzy family* L^X, *which is defined as follows*:

for every $A, B \in L^X$: $\delta^\Diamond(A, B) = \delta(i_X(A), i_X(B))$.

Theorem 8.1 shows that each (L, M)-basic fuzzy proximity $\delta : L^X \times L^X \to M$ induces $(P^*(L), M)$-basic proximity $\delta^\Delta : P^*(L)^X \times P^*(L)^X \to M$ on the fuzzy family $P^*(L)^X$. Let L be a lattice defined on a nonempty set Λ. Consider i_X as a function from L^X to $P(\Lambda)^X$. The following results can be considered as extensions of Theorem 8.1 and Theorem 8.2.

Theorem 8.5. *Each* (L, M)*-basic proximity* $\delta : L^X \times L^X \to M$ *on* L^X; $L \in \mathcal{L}(\Lambda)$ *induces* $(P(\Lambda), M)$*-basic proximity* $\delta^\Delta : P(\Lambda)^X \times P(\Lambda)^X \to M$ *on* $P(\Lambda)^X$, *which is defined by*: $\delta^\Delta(U, V) = 1_M$, *if* $U \cap V \neq \mathbf{0}_{(P(\Lambda), X)}$ *and otherwise* $\delta^\Delta(U, V) = \delta(j_X(U), j_X(V))$; *where* $j_X(V)(x) = \mathrm{supp}(V(x))$; $x \in X$, *where the supremum is taken in the lattice L.*

Theorem 8.6. *Let* $\delta : L^X \times L^X \to M$ *be an* (L, M)*-proximity on* L^X; $L \in \mathcal{L}(\Lambda)$ *is a complete lattice and* $\delta^\Delta : P(\Lambda)^X \times P(\Lambda)^X \to M$ *be the*

$(P(\Lambda),\ M)$-*induced proximity on* $P(\Lambda)^X$. *The closure operator on the proximity* δ *on* L^X *and the closure operator* δ^Δ *on* $P(\Lambda)^X$ *satisfy the relation*: $C_{\delta^\Delta}(i_X(A)) = i_X(C_\delta(A));\ A \in L^X$.

Theorem 8.7. *For every nonempty set* Λ *and for every* $(L,\ M)$-*basic proximity* δ *on* L^X; $L \in \mathcal{L}(\Lambda)$, *there exists a* $(P(\Lambda),\ M)$-*basic proximity* δ^Δ *on* $P(\Lambda)^X$ *such that the mapping* i_X *translates the family of* δ-*closed fuzzy subsets* $\{A\}$ *in* L^X *into the family of* δ^Δ-*closed fuzzy subsets in* $P(\Lambda)^X$.

Corollary 8.3. *Let* δ *be an* $(L,\ M)$-*basic proximity on* L^X *and* δ^Δ *be the induced* $(P(\Lambda),\ M)$-*basic proximity on* $P(\Lambda)^X$. *Then*:

(1) *The fuzzy topology* $\tau_\delta = \{A^c \in L^X : C_\delta(A) = A\}$, *which is defined by the* $(L,\ M)$-*basic proximity* δ *on* L^X, *corresponds to the subfamily of open fuzzy subsets*: $\{i_X(A)^c \in P(\Lambda)^X : C_{\delta^\Delta}(i_X(A)) = i_X(A)\}$ *of the fuzzy topology* τ_{δ^Δ} *on* $P(\Lambda)^X$, *which is induced by the basic proximity* Σ.

(2) *The fuzzy subset* $A^c \in L^X$ *is a* δ-*neighborhood of the fuzzy subset* $B \in L^X$ *iff* $i_X(A)^c \in P(\Lambda)^X$ *is a* δ^Δ-*neighborhood of the fuzzy subset* $i_X(B) \in P(\Lambda)^X$.

Theorem 8.8. *Each* $(P(\Lambda),\ M)$-*basic proximity* $\delta : P(\Lambda)^X \times P(\Lambda)^X \to M$ *on the fuzzy family* $P^*(L)^X$ *induces* $(L,\ M)$-*basic proximity* $\delta^\Diamond : L^X \times L^X \to M$ *on the fuzzy family* L^X, *which is defined as follows*: *for every* $A,\ B \in L^X$;

$$\delta^\Diamond(A,\ B) = \delta(i_X(A),\ i_X(B)).$$

9. Categories of the Proximity Spaces on
$$P(\Lambda)^X, \; P^*(L)^X \; \text{ and } \; L^X; \; L \in \mathcal{L}(\Lambda)$$

Definition 9.1. Let (X, δ_1) and (Y, δ_2) be two (L, M)-basic proximity spaces.

A function $F : X \to Y$ is called an (L, M)-*proximity mapping* if

$$\delta_1(A, B) \le \delta_2(F^{\to}(A), F^{\to}(B)), \quad \forall A, B \in L^X.$$

Definition 9.2. Let (X, δ_1) and (Y, δ_2) be two (L, M)-basic proximity spaces. A fuzzy function $\boldsymbol{F} = (F, f_X) : X \to Y$ is called an (L, M)-*fuzzy proximity mapping* if

$$\delta_1(A, B) \le \delta_2(\boldsymbol{F}^{\to}(A), \boldsymbol{F}^{\to}(B)), \quad \forall A, B \in L^X.$$

The family of all (L, M)-basic proximity spaces and (L, M)-proximity mappings from a category that will be denoted by $\boldsymbol{BProx}(\boldsymbol{L}, \boldsymbol{M})$, while the family of all (L, M)-basic proximity spaces and (L, M)-fuzzy proximity mappings from a category that will be denoted by $(\boldsymbol{FBProx}(\boldsymbol{L}, \boldsymbol{M}))$. Moreover, the subcategory of $\boldsymbol{FBProx}(\boldsymbol{L}, \boldsymbol{M})$ with (L, M)-basic proximity spaces and fuzzy functions with identity comembership functions on L will be denoted by $\boldsymbol{idFBProx}(\boldsymbol{L}, \boldsymbol{M})$.

Lemma 9.1. (1) $F \in \boldsymbol{BProx}(\boldsymbol{L}, \boldsymbol{M})((X, \delta_1), (Y, \delta_2))$ *implies*

$(F, id_L) \in \boldsymbol{BProx}(\boldsymbol{P}^*(\boldsymbol{L}), \boldsymbol{M})((X, \delta_1^*), (Y, \delta_2^*))$, *where* $\delta^*(U, V) = 1_M$,

if $U \bigcap V \ne \boldsymbol{0}_{(P^*(L), \, X)}$ *and otherwise* $\delta^*(U, V) = \delta(j_X(U), \, j_X(V))$.

(2) $\boldsymbol{G} = (G, id_L) \in \boldsymbol{FBProx}(\boldsymbol{P}^*(\boldsymbol{L}), \boldsymbol{M})((X, \delta_1), (Y, \delta_2))$ *implies*

$G \in \boldsymbol{BProx}(\boldsymbol{L}, \boldsymbol{M})((X, \delta_1^\Diamond), (Y, \delta_2^\Diamond))$, *where* $\delta^\Diamond(A, B) = \delta(i_X(A), i_X(B))$.

(3) $\boldsymbol{G} = (G, id_L) \in \boldsymbol{FBProx}(P(\Lambda), \boldsymbol{M})((X, \delta_1), (Y, \delta_2))$ *implies*

$G \in \boldsymbol{BProx}(\boldsymbol{L}, \boldsymbol{M})((X, \delta_1^\Diamond), (Y, \delta_2^\Diamond))$, *where* $\delta^\Diamond(A, B) = \delta(i_X(A), i_X(B))$.

Theorem 9.1. *The mapping* i_X *(respectively* j_X*) generates a functor* $\mathbf{I_P}$ *(respectively* $\mathbf{J_P}$*) as follows*:

(1)

$$\mathbf{J_P} : \mathbf{BProx}(L, M) \to \mathbf{FBProx}(P^*(L), M),$$

where $\mathbf{J_P}(X, \delta) = (X, \mathbf{J_P}(\delta))$, *where* $\mathbf{J_P}(\delta) = \delta^*$, $\mathbf{J_P}(F) = (F, id_L)$.

(2)

$$\mathbf{I_P} : \mathbf{idFBProx}(P^*(L), M) \to \mathbf{BProx}(L, M),$$

where $\mathbf{I_P}(X, \delta) = (X, \mathbf{I_P}(\delta))$, *where* $\mathbf{I_P}(\delta) = \delta^\Diamond$, $\mathbf{I_P}(F, id_L) = F$.

Lemma 9.2. *Let* $(X, \delta) \in \mathbf{FBProx}(P^*(L), M)$ *and* $(X, \rho) \in \mathbf{BProx}(L, M)$. *Then we have the following*:

(1) $\rho = \mathbf{I_P}\mathbf{J_P}(\rho)$.

(2) $\delta \le \mathbf{J_P}\mathbf{I_P}(\delta)$.

Theorem 9.2. *The functor* $\mathbf{I_P}$ *is left adjoint to the functor* $\mathbf{J_P}$ $(\mathbf{I_P} \dashv \mathbf{J_P})$, *where* $\mathbf{I_P}$ *and* $\mathbf{J_P} : \mathbf{BProx}(L, M) \underset{\mathbf{I_P}}{\overset{\mathbf{J_P}}{\rightleftarrows}} \mathbf{idFBProx}(P^*(L), M)$ *are defined in Theorem* 9.1.

Proof. Let $(X, \delta) \in \mathbf{idFBProx}(P^*(L), M)$, $(Y, \rho) \in \mathbf{BProx}(L, M)$.

Lemma 9.2 implies that the identity fuzzy function

$$(id_X, id_L) : (X, \delta) \to (X, \mathbf{J_P}\mathbf{I_P}(\delta))$$

is a $(P^*(L), M)$-fuzzy proximity mapping. Therefore, it is sufficient to show that for every $(F, id_L) \in \mathbf{idFBProx}(P^*(L), M)$, $((X, \delta), (Y, \mathbf{J_P}(\rho)))$ there exists a unique $G \in \mathbf{BProx}(L, M)((X, \mathbf{I_P}(\delta)), (Y, \rho))$ making the following diagram commutes:

$$(Y, \boldsymbol{J_P}(\rho)) \overset{\boldsymbol{J_P}(G)}{\leftarrow} (X, \boldsymbol{J_P I_P}(\delta))$$

$$(F, id_L) \searrow \nearrow (id_X, id_L)$$

$$(X, \delta)$$

Let $G = F$, since (F, id_L) is a $(P^*(L), M)$-proximity mapping. Then

$$\boldsymbol{I_P}(\delta)(A, B) = \delta(i_X(A), i_X(B))$$

$$\leq \boldsymbol{J_P}(\rho)((F, id_L)^{\rightarrow}(i_X(A)), (F, id_L)^{\rightarrow}(i_X(B)))$$

$$= \boldsymbol{J_P}(\rho)(i_Y(F^{\rightarrow}(A)), i_Y(F^{\rightarrow}(B)))$$

$$= \rho(j_Y(i_Y(F^{\rightarrow}(A))), j_Y(i_Y(F^{\rightarrow}(B)))) = \rho(F^{\rightarrow}(A), F^{\rightarrow}(B)).$$

Thus, $G = F$ is an (L, M)-proximity mapping and is the unique function that makes the above diagram commutes. Therefore, $\boldsymbol{I_P} \dashv \boldsymbol{J_P}$ and the functor $\boldsymbol{I_P}$ is left adjoint to $\boldsymbol{J_P}$.

One can show that easily the category $\boldsymbol{BProx}(\boldsymbol{L}, \boldsymbol{M})$ is isomorphic to the category $\boldsymbol{idFSBProx}(\boldsymbol{L}, \boldsymbol{M})$ with the functors

$$\boldsymbol{BProx}(\boldsymbol{L}, \boldsymbol{M}) \overset{\boldsymbol{R_P}}{\underset{\boldsymbol{S_P}}{\rightleftarrows}} \boldsymbol{idFBProx}(\boldsymbol{L}, \boldsymbol{M}),$$

where

$$\boldsymbol{R_P}(X, \delta) = (X, \delta), \quad \boldsymbol{R_P}(F) = (F, id_L),$$

$$\boldsymbol{S_P}(X, \delta) = (X, \delta) \quad \text{and} \quad \boldsymbol{S_P}(F, id_L) = F.$$

Now, to prove that (L, M)-basic proximity spaces are embedded in $(P^*(L), M)$-basic proximity spaces, we define the subfamily E of $P^*(L)$ as the range of the mapping $i : L \to P^*(L)$, $E = \{[0, \alpha] : \alpha \in L\}$. The mapping i embeds E in $P^*(L)$. This restriction redefines the induced (E, M)-basic proximity δ^* on E^X of (L, M)-basic proximity δ on L^X as follows:

For every $A, B \in L^X$; $\delta(A, B) = \delta^*(i_X(A), i_X(B))$.

Moreover, each (E, M)-basic proximity $\delta : E^X \times E^X \to M$ on the fuzzy family E^X induces (L, M)-basic proximity $\delta^\diamond : L^X \times L^X \to M$ on the fuzzy family L^X, which is defined as: for every $A, B \in L^X$; $\delta^\diamond(A, B) = \delta(i_X(A), i_X(B))$.

Theorem 9.3. *The category* ***BProx**(L, M)* *is isomorphic to the category* ***idFBProx**(E, M)* *with the two functors* ***I_P**, **J_P***:

$$\textbf{\textit{BProx}}(L, M) \overset{\textbf{\textit{J}}_P}{\underset{\textbf{\textit{I}}_P}{\rightleftarrows}} \textbf{\textit{idFBProx}}(E, M).$$

Theorem 9.4. *The mapping* i_X *(respectively* k_X*) generates a functor* ***I_P*** *(respectively* ***K_P***) *as follows*:

(1)

$$\textbf{\textit{K}}_P : \textbf{\textit{BProx}}(L, M) \to \textbf{\textit{FBProx}}(P(\Lambda), M),$$

where $\textbf{\textit{K}}_P(X, \delta) = (X, \textbf{\textit{K}}_P(\delta))$, *where* $\textbf{\textit{K}}_P(\delta) = \delta^\Delta$, $\textbf{\textit{K}}_P(F) = (F, id_L)$.

(2)

$$\textbf{\textit{I}}_P : \textbf{\textit{idFBProx}}(P^*(L), M) \to \textbf{\textit{BProx}}(L, M),$$

where $\textbf{\textit{I}}_P(X, \delta) = (X, \textbf{\textit{I}}_P(\delta))$, *where* $\textbf{\textit{I}}_P(\delta) = \delta^\diamond$, $\textbf{\textit{I}}_P(F, id_L) = F$.

Lemma 9.3. *Let*

$$(X, \delta) \in \textbf{\textit{FBProx}}(P(\Lambda), M) \quad and \quad (X, \rho) \in \textbf{\textit{BProx}}(L, M).$$

Then we have the following:

(1) $\rho = \textbf{\textit{I}}_P \textbf{\textit{K}}_P(\rho)$.

(2) $\delta \subset \textbf{\textit{KI}}_P(\delta)$.

Theorem 9.5. *The functor* I_P *is left adjoint to the functor* K_P $(I_P \dashv K_P)$, *where* I_P, K_P *are defined in Theorem* 9.4 *and*

$$BProx(L, M) \overset{K_P}{\underset{I_P}{\rightleftarrows}} idFBProx(P(\Lambda), M).$$

From Theorem 9.3, $BProx(L, M)$ is isomorphic to the subcategory $idFBProx(E, M)$ of $BProx(P(\Lambda), M)$, moreover, the functor K_P is an injective. Therefore, the category $BProx(L, M)$ is embedded in the category $BProx(P(\Lambda), M)$, for every $L \in \mathcal{L}(\Lambda)$.

10. Conclusion

From the study of proximity spaces in some family of fuzzy subsets, we can advocate that every basic proximity in the category $BProx(L, M)$, for every $L \in \mathcal{L}(\Lambda)$, is isomorphic to at least one basic proximity in the category $BProx(P(\Lambda), M)$.

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