ON L-FUZZY TOPOGENOUS ORDERS

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Abstract.

In this paper we study L-fuzzy topogenous orders, where L represents a completely distributive lattice. We shall investigate the level decomposition of L-fuzzy topogenous on X and the corresponding L-fuzzy topogenous continuous maps. In addition, we shall establish the representation theorems of L-fuzzy topogenous on X.

Keywords.

L-fuzzy topogenous; L-topogenous; L-fuzzy topogenous continuous map; L-topogenous continuous map

1. Introduction

The concept of fuzzy topology was first define in 1968 by Chang [3] and later redefined in a somewhat different way by Lowen [16] and by Hutton [8]. According to Šostak[27], these definitions, a fuzzy topology is a crisp subfamily of family of fuzzy sets and fuzziness in the concept of openness of a fuzzy set has not been considered, which seems to be a drawback in the process of fuzzification of the concept of topological spaces. Therefore, Šostak introduced a new definition of fuzzy topology in 1985 [27], Later on he has developed the theory of fuzzy topological spaces in [28]. After that, several authors [5-8,15,16,23-30] have reintroduced the same definition and studied fuzzy topological spaces being unaware of Šostak's work. In [31] Zhang et al. investigated the level decomposition of L-fuzzy topology and the corresponding L-fuzzy continuous maps also, established the representation theorems of L-fuzzy topology. There have been all kinds of studies about the theory of topogenous in a fuzzy set theory (see [1,2,4,9-14,17-22], etc.). Now in the present paper, we study the level decomposition of an L-fuzzy topogenous and the corresponding L-fuzzy topogenous continuous maps. In addition, we also establish some representation theorems for L-fuzzy topogenous on X. The main results of this paper are several representation theorems for L-fuzzy topogenous on X, where L represents a completely distributive lattice. Based on the results of this paper, we have also developed representation theorems for the category L-FP which consists of L-fuzzy topogenous orders and L-fuzzy topogenous continuous maps.

2. Preliminaries

Throughout this paper, L represents a completely distributive lattice with the smallest element $\bot$ and the greatest element $\top$, where $\bot \neq \top$. We define $M(L)$ to be the set of all non-zero $\vee$-irreducible (or coprime) elements in L such that $a \in M(L)$ iff $a \leq b \vee c$ implies $a \leq b$ or $a \leq c$. Let $P(L)$ be the set of all non-unit prime elements in L such that $a \in P(L)$ iff $a \geq b \wedge c$ implies $a \geq b$ or $a \geq c$. Finally, let X be a non-empty usual set, and L^X be the set of all L-fuzzy sets on X. For each $a \in L$, let $\underline{a}$ denote a constant-valued L-fuzzy set with a as its value. Let $\underline{\bot}$ and $\underline{\top}$ be the smallest element and greatest element in L^X, respectively. for the empty set $\phi \subset L$, we define $\wedge\phi = \top$ and $\vee\phi = \bot$.

Defnition 2.1[29].

Suppose that $a \in L$ and $A \subset L$.

(1) A is called a maximal family of a if

(a) inf $A = a$,

(b) $\forall B \subset L$, inf $B \leq a$ implies that $\forall x \in A$ there exists $y \in B$ such that $y \leq x$.

(2) A is called a minimal family of a if

(a) sup $A = a$,

(b) $\forall B \subset L$, sup $B \geq a$ implies that $\forall x \in A$ there exists $y \in B$ such that $y \geq x$.

Remark 2.2.[8].

Hutton proved that if L is a completely distributive lattice and $a \in L$, then there exists $B \subset L$ such that

(i) $a = \bigvee B$, and

(ii) if $A \subset L$ and $a = \bigvee A$, then for each $b \in B$ there is a $c \in A$ such that $b \leq c$.

However, if $\forall a \in L$, and if there exists $B \subset L$ satisfying (i) and (ii), then in general L is not a completely distributive lattice. To this end,Wang [23] introduced the following modification of condition (ii), *(ii') if $A \subset L$ and $a \leq \bigvee A$, then for each $b \in B$ there is a $c \in A$ such that $b \leq c$.*

Wang proved that a complete lattice L is completely distributive if and only if for each element $a \in L$, there exists $B \subset L$ satisfying (i) and (ii). Such a set B is called a minimal set of a by Wang [31].The concept of maximal family is the dual concept of minimal family, and a complete lattice L is completely distributive if and only if for each element $a \in L$, there exists a maximal family $B \subset L$.

Let $\alpha(a)$ denote the union of all maximal families of a. Likewise, let $\beta(a)$ denote the union of all minimal sets of a. Finally, let $\alpha^*(a) = \alpha(a) \cap P(L)$ and $\beta^*(a) = \beta(a) \cap M(L)$. One can easily see that both $\alpha(a)$ and $\alpha^*(a)$ are maximal sets of a. Likewise, both $\beta(a)$ and $\beta^*(a)$ are minimal sets of a. Also, we have $\alpha(\top) = \phi$. and $\beta(\bot) = \phi$.

Definition 2.3. *(Katsaras [9])*

A binary relation η on L^X is an L-semi-topogenous order on X, if it satisfies the following axioms:

(T1) $(\underline{\top}, \underline{\top}), (\underline{\bot}, \underline{\bot}) \in \eta$,

(T2) If $(\lambda, \mu) \in \eta$, then $\lambda \leq \mu$,

(T3) If $\lambda \leq \lambda_1$, $\mu_1 \leq \mu$ and $(\lambda_1, \mu_1) \in \eta$, then $(\lambda, \mu) \in \eta$,

An L-fuzzy semi-topogenous order is called

(I) L-fuzzy topogenous if

(T4) $(\lambda \vee \mu, \gamma) \in \eta$ iff $(\lambda, \gamma) \in \eta$, $(\mu, \gamma) \in \eta$

(T5) $(\lambda, \mu \wedge \gamma) \in \eta$ iff $(\lambda, \mu) \in \eta$, $(\lambda, \gamma) \in \eta$.

(II) Perfect if

(T6) $(\lambda_i, \mu) \in \eta$ for any $\{\mu, \lambda_i \mid i \in \Delta\} \subset L^X$ implies $(\bigvee_{i\in\Delta} \lambda_i, \mu) \in \eta$,

(III) Co-perfect if

(T7) $(\lambda, \mu_i) \in \eta$ for any $\{\mu_i, \lambda \mid i \in \Delta\} \subset L^X$ implies $(\lambda, \bigwedge_{i\in\Delta} \mu_i) \in \eta$,

(IV) Biperfect if it is perfect and biperfect.

As in (Shi [23-25] and Wang [29]) we give the following Lemma:

Lemma 2.4.

For $a \in L$ and a map $\eta : L^X \times L^X \to L$, we define

$$\eta_{[a]} = \{(\lambda, \mu) \in L^X \times L^X \mid \eta(\lambda, \mu) \geq a\}$$

and

$$\eta^{[a]} = \{(\lambda, \mu) \in L^X \times L^X \mid a \notin \alpha(\eta(\lambda, \mu))\}$$

Let η be a map from $L^X \times L^X$ to L and $a, b \in L$. Then

(1) $a \in \beta(b) \Rightarrow \eta_{[b]} \subset \eta_{[a]}$; $a \in \alpha(b) \Rightarrow \eta^{[a]} \subset \eta^{[b]}$.

(2) $a \leq b \Leftrightarrow \beta(a) \subset \beta(b) \Leftrightarrow \beta^*(a) \subset \beta^*(b) \Leftrightarrow \alpha(b) \subset \alpha(a) \Leftrightarrow \alpha^*(b) \subset \alpha^*(a)$.

(3) $\alpha(\bigwedge_{i\in\triangle} a_i) = \bigcup_{i\in\triangle} \alpha(a_i)$ *and* $\beta(\bigvee_{i\in\triangle} a_i) = \bigcup_{i\in\triangle} \beta(a_i)$ *for any sub-family* $\{a_i\}_{i\in\triangle} \subset L$

3. Level decomposition of an L-fuzzy topogenous

Definition 3.1[13].

A map $\eta : L^X \times L^X \to L$ *is called an L-fuzzy semi-topogenous order on X if it satisfies the following conditions:*

(FT1) $\eta(\underline{\top}, \underline{\top}) = \eta(\underline{\bot}, \underline{\bot}) = \top$,

(FT2) If $\eta(\lambda, \mu) \neq \bot$, *then* $\lambda \leq \mu$,

(FT3) If $\lambda \leq \lambda_1$, $\mu_1 \leq \mu$ *then* $\eta(\lambda_1, \mu_1) \leq \eta(\lambda, \mu)$.

An L-fuzzy semi-topogenous order is called

(I) L-fuzzy topogenous if

(FT4) $\eta(\lambda \vee \mu, \gamma) \geq \eta(\lambda, \gamma) \wedge \eta(\mu, \gamma)$,

(FT5) $\eta(\lambda, \mu \wedge \gamma) \geq \eta(\lambda, \mu) \wedge \eta(\lambda, \gamma)$.

(II) Perfect if

(FT6) $\eta(\bigvee_{i\in\Delta} \lambda_i, \mu) \geq \bigwedge_{i\in\Delta} \eta(\lambda_i, \mu)$, *for any* $\{\mu, \lambda_i \mid i \in \Delta\} \subset L^X$.

(III) Co-perfect if

(FT7) $\eta(\lambda, \bigwedge_{i\in\Delta} \mu_i) \geq \bigwedge_{i\in\Delta} \eta(\lambda_i, \mu)$, *for any* $\{\lambda, \mu_i \mid i \in \Delta\} \subset L^X$.

(IV) Biperfect if it is perfect and biperfect.

The pair (X, η) *is said to be an L-fuzzy topogenous space. Just as an L-topogenous on X is an ordinary subset of* $L^X \times L^X$, *an L-fuzzy topogenous on X is a fuzzy subset of* $L^X \times L^X$.

Remark 3.2.

(1) If $\eta : 2^X \times 2^X \to I$ *where* $I = [0, 1]$ *such that the above conditions hold respectively, we call it a fuzzifying topogenous on X in a sense [17].*

(2) We easily show that every L-fuzzy topogenous order is a Katsaras's fuzzy topogenous order [9].

Theorem 3.3.

Let η *be a map* $\eta : L^X \times L^X \to L$. *Then the following conditions are equivalent:*

(1) η *is an L-fuzzy topogenous on X.*

(2) $\forall a \in M(L)$, $\eta_{[a]}$ *is an L-topogenous on X.*

(3) $\forall a \in L$, $\eta^{[a]}$ *is an L-topogenous on X.*

(4) $\forall a \in P(L)$, $\eta^{[a]}$ *is an L-topogenous on X.*

proof. (1) $\Rightarrow$ (2)*: this part is obvious.*

(2) $\Rightarrow$ (1)*: (FT1) For each $a \in M(L)$, we have $(\underline{\top}, \underline{\top}) \in \eta_{[a]}$, and $\eta(\underline{\top}, \underline{\top}) \geq a$. Accordingly,*

$$\eta(\underline{\top}, \underline{\top}) \geq \bigvee \{a \mid a \in M(L)\} = \top.$$

Thus, $\eta(\underline{\top}, \underline{\top}) = \top$. Similarly $\eta(\underline{\bot}, \underline{\bot}) = \top$.

(FT2) Directly from (T2).

(FT3) Let $\lambda, \lambda_1, \mu_1, \mu \in L^X$ with $\lambda \leq \lambda_1$, $\mu_1 \leq \mu$. Clearly, when $\eta(\lambda_1, \mu_1) = \bot$, we have $\eta(\lambda_1, \mu_1) \leq \eta(\lambda, \mu)$.Otherwise if $\eta(\lambda_1, \mu_1) > \bot$, then for each $\eta(\lambda_1, \mu_1) \geq a$, we have $\eta(\lambda_1, \mu_1) \in \eta_{[a]}$. Consequently, we have $\eta(\lambda, \mu) \in \eta_{[a]}$ or $\eta(\lambda, \mu) \geq a$. This further implies that

$$\eta(\lambda, \mu) \geq \bigvee \{a \in M(L) \mid \eta(\lambda_1, \mu_1) \geq a\} = \eta(\lambda_1, \mu_1).$$

(FT4) Let $\lambda, \mu, \gamma \in L^X$. Clearly, when $\eta(\lambda, \gamma) \wedge \eta(\mu, \gamma) = \bot$, we have $\eta(\lambda \vee \mu, \gamma) \geq \eta(\lambda, \gamma) \wedge \eta(\mu, \gamma)$ Otherwise if $\eta(\lambda, \gamma) \wedge \eta(\mu, \gamma) > \bot$,then for each $\eta(\lambda, \gamma) \wedge \eta(\mu, \gamma) \geq a$, we have $\eta(\lambda, \gamma) \geq a$ and $\eta(\mu, \gamma) \geq a$ or $\eta(\lambda, \gamma) \in \eta_{[a]}$ and $\eta(\mu, \gamma) \in \eta_{[a]}$. Consequently, we have $(\lambda \vee \mu, \gamma) \in \eta_{[a]}$ and so $(\lambda \vee \mu, \gamma) \geq a$ This further implies that

$$\eta(\lambda \vee \mu, \gamma) \geq \bigvee \{a \in M(L) \mid \eta(\lambda, \gamma) \wedge \eta(\mu, \gamma) \geq a\} = \eta(\lambda, \gamma) \wedge \eta(\mu, \gamma).$$

(FT5) Similar to (FT4).

(1) $\Rightarrow$ (3)*:*

(T1) $\forall a \in L$, since $\eta(\underline{\top}, \underline{\top}) = \eta(\underline{\bot}, \underline{\bot}) = \top$, and $\alpha(\top) = \phi$, we have $a \notin \alpha(\top) = \alpha(\eta(\underline{\top}, \underline{\top})) = \alpha(\eta(\underline{\bot}, \underline{\bot}))$. Thus $(\underline{\top}, \underline{\top}), (\underline{\bot}, \underline{\bot}) \in \eta^{[a]}$

(T2) Directly from (FT2)

(T3) Let $\lambda, \lambda_1, \mu_1, \mu \in L^X$ with $\lambda \leq \lambda_1$, $\mu_1 \leq \mu$ and $(\lambda_1, \mu_1) \in \eta^{[a]}$. Then

$$a \notin \alpha(\eta(\lambda_1, \mu_1)) \supset \alpha(\eta(\lambda, \mu))$$

Hence $(\lambda, \mu) \in \eta^{[a]}$.

(T4) Let $(\lambda, \gamma), (\mu, \gamma) \in \eta^{[a]}$. Then $a \notin \alpha(\eta(\lambda, \gamma))$ and $a \notin \alpha(\eta(\mu, \gamma))$. Hance

$$a \notin \alpha(\eta(\lambda, \gamma)) \cup \alpha(\eta(\mu, \gamma)) = \alpha(\eta(\lambda, \gamma) \wedge \eta(\mu, \gamma)) \supset \alpha(\eta(\lambda \vee \mu, \gamma).$$

Furthermore, since $a \notin \alpha(\eta(\lambda \vee \mu, \gamma))$*, we have* $(\lambda \vee \mu, \gamma) \in \eta^{[a]}$.

(T5) Similar to (T4).

(3) $\Rightarrow$ (4)*: this part is obvious.*

(4) $\Rightarrow$ (1)*:*

(FT1) $\forall a \in P(L)$*, it is clear that* $(\underline{\top}, \underline{\top}) \in \eta^{[a]}$*. Thus* $a \notin \alpha(\eta(\underline{\top}, \underline{\top}))$*. Then* $\alpha^*(\eta(\underline{\top}, \underline{\top})) = \phi$ *and*

$$\eta(\underline{\top}, \underline{\top}) = \bigwedge \alpha^*(\eta(\underline{\top}, \underline{\top})) = \top.$$

Similarly $\eta(\underline{\perp}, \underline{\perp}) = \top$.

(FT2) Directly from (T2).

(FT3) Let $\lambda, \lambda_1, \mu_1, \mu \in L^X$ *with* $\lambda \leq \lambda_1$*,* $\mu_1 \leq \mu$*. Clearly, when* $\eta(\lambda_1, \mu_1) = \perp$*, we have* $\eta(\lambda_1, \mu_1) \leq \eta(\lambda, \mu)$*.Otherwise if* $\eta(\lambda_1, \mu_1) > \perp$*, then for each* $a \in P(L)$ *and* $a \notin \alpha(\eta(\lambda_1, \mu_1))$*,we have* $(\lambda_1, \mu_1) \in \eta^{[a]}$*. Consequently, we have* $(\lambda, \mu) \in \eta^{[a]}$ *hence* $a \notin \alpha(\eta(\lambda, \mu))$*.Accordingly, we have* $\alpha^*(\eta(\lambda_1, \mu_1) \supset \alpha^*(\eta(\lambda, \mu)$ *or* $\eta(\lambda_1, \mu_1) \leq \eta(\lambda, \mu)$

(FT4) Let $\lambda, \mu, \gamma \in L^X$*. Clearly, when* $\eta(\lambda, \gamma) \wedge \eta(\mu, \gamma) = \perp$*, we have* $\eta(\lambda \vee \mu, \gamma) \geq \eta(\lambda, \gamma) \wedge \eta(\mu, \gamma)$ *Otherwise if* $\eta(\lambda, \gamma) \wedge \eta(\mu, \gamma) > \perp$*,then for each* $a \in P(L)$ *and* $a \notin \alpha(\eta(\lambda, \gamma) \wedge \eta(\mu, \gamma)) = \alpha(\eta(\lambda, \gamma) \cup \alpha(\eta(\mu, \gamma))$*, we have* $a \notin \alpha(\eta(\lambda, \gamma))$ *and* $a \notin \alpha(\eta(\mu, \gamma))$*, implise* $\eta(\lambda, \gamma) \in \eta^{[a]}$ *and* $\eta(\mu, \gamma) \in \eta^{[a]}$ *and so* $\eta(\lambda \vee \mu, \gamma) \in \eta^{[a]}$ *hence* $a \notin \alpha(\eta(\lambda \vee \mu, \gamma))$*. Accordingly, we have* $\alpha^*(\eta(\lambda, \gamma) \wedge \eta(\mu, \gamma)) \supset \alpha^*(\eta(\lambda \vee \mu, \gamma))$*, or* $\eta(\lambda \vee \mu, \gamma) \geq \eta(\lambda, \gamma) \wedge \eta(\mu, \gamma)$.

(FT5) Similar to (FT4).

Theorem 3.3.

Let η *be a map* $\eta : L^X \times L^X \to L$ *be an L-fuzzy topogenous on* X*. Then the following conditions are equivalent:*

(1) η *is an L-fuzzy biperfect topogenous on* X*.*

(2) $\forall a \in M(L)$*,* $\eta_{[a]}$ *is an L-biperfect topogenous on* X*.*

(3) $\forall a \in L$*,* $\eta^{[a]}$ *is an L-biperfect topogenous on* X*.*

(4) $\forall a \in P(L)$*,* $\eta^{[a]}$ *is an L-biperfect topogenous on* X*.*

proof. (1) $\Rightarrow$ (2)*: this part is obvious.*

(2) $\Rightarrow$ (1)*:*

(FT6) Let $\{\mu, \lambda_i \mid i \in \Delta\} \subset L^X$*. Then for each* $a \in M(L)$ *and* $a \leq \bigwedge_{i \in \Delta} \eta(\lambda_i, \mu)$*, we have* $\eta(\lambda_i, \mu) \geq a$ *and* $(\lambda_i, \mu) \in \eta_{[a]}$ *for each* $i \in \Delta$.

The proof follows because $(\bigvee_{i\in\Delta}\lambda_i,\mu)\in\eta_{[a]}$ *and*

$$\eta(\bigvee_{i\in\Delta}\lambda_i,\mu)\geq\bigvee\{a\in M(L)\mid a\leq\bigwedge_{i\in\Delta}\eta(\lambda_i,\mu)\}=\bigwedge_{i\in\Delta}\eta(\lambda_i,\mu).$$

(FT7) Let $\{\mu_i,\lambda\mid i\in\Delta\}\subset L^X$. *Then for each* $a\in M(L)$ *and* $a\leq\bigwedge_{i\in\Delta}\eta(\lambda,\mu_i)$, *we have* $\eta(\lambda,\mu_i)\geq a$ *and* $(\lambda,\mu_i)\in\eta_{[a]}$ *for each* $i\in\Delta$. *The proof follows because* $(\lambda,\bigwedge_{i\in\Delta}\mu_i)\in\eta_{[a]}$ *and*

$$\eta(\lambda,\bigwedge_{i\in\Delta}\mu_i)\geq\bigvee\{a\in M(L)\mid a\leq\bigwedge_{i\in\Delta}\eta(\lambda,\mu_i)\}=\bigwedge_{i\in\Delta}\eta(\lambda,\mu_i).$$

(1) $\Rightarrow$ (3)*:*

(T6) Let $(\lambda_i,\mu)\in\eta^{[a]}$. *Then* $a\notin\alpha(\eta(\lambda_i,\mu))$ *for each* $i\in\Delta$ *and*

$$a\notin\bigcup_{i\in\Delta}\alpha(\eta(\lambda_i,\mu))=\alpha(\bigwedge_{i\in\Delta}\eta(\lambda_i,\mu))\supset\alpha(\eta(\bigvee_{i\in\Delta}\lambda_i,\mu)).$$

Consequently, we have $a\notin\alpha(\eta(\bigvee_{i\in\Delta}\lambda_i,\mu))$ *or* $(\bigvee_{i\in\Delta}\lambda_i,\mu)\in\eta^{[a]}$

(T7) Let $(\lambda,\mu_i)\in\eta^{[a]}$. *Then* $a\notin\alpha(\eta(\lambda,\mu_i))$ *for each* $i\in\Delta$ *and*

$$a\notin\bigcup_{i\in\Delta}\alpha(\eta(\lambda,\mu_i))=\alpha(\bigwedge_{i\in\Delta}\eta(\lambda,\mu_i))\supset\alpha(\eta(\lambda,\bigwedge_{i\in\Delta}\mu_i)).$$

Consequently, we have $a\notin\alpha(\eta(\lambda,\bigwedge_{i\in\Delta}\mu_i))$ *or* $(\lambda,\bigwedge_{i\in\Delta}\mu_i)\in\eta^{[a]}$

(3) $\Rightarrow$ (4)*: this part is obvious.*

(4) $\Rightarrow$ (1)*:*

(FT6) Let $\{\mu,\lambda_i\mid i\in\Delta\}\subset L^X$. *Obviously if* $\bigwedge_{i\in\Delta}\eta(\lambda_i,\mu)=\perp$, *then* $\eta(\bigvee_{i\in\Delta}\lambda_i,\mu)\geq\bigwedge_{i\in\Delta}\eta(\lambda_i,\mu)$. *Suppose now that* $\bigwedge_{i\in\Delta}\eta(\lambda_i,\mu)>\perp$. *Then* $\forall a\in P(L)$ *and* $a\notin\alpha(\bigwedge_{i\in\Delta}\eta(\lambda_i,\mu))=\bigcup_{i\in\Delta}\alpha(\eta(\lambda_i,\mu))$. *It follows that* $a\notin(\alpha(\eta(\lambda_i,\mu))$ *for each* $i\in\Delta$ *implies* $(\lambda_i,\mu)\in\eta^{[a]}$ *for each* $i\in\Delta$ *and so* $(\bigvee_{i\in\Delta}\lambda_i,\mu)\in\eta^{[a]}$. *Hence*

$$\alpha^*(\eta(\bigvee_{i\in\Delta}\lambda_i,\mu))\subset\alpha^*(\bigwedge_{i\in\Delta}\eta(\lambda_i,\mu)).$$

Therefore

$$\eta(\bigvee_{i\in\Delta}\lambda_i,\mu)\geq\bigwedge_{i\in\Delta}\eta(\lambda_i,\mu).$$

(FT7) Let $\{\mu_i,\lambda \mid i\in\Delta\}\subset L^X$. *Obviously if* $\bigwedge_{i\in\Delta}\eta(\lambda,\mu_i)=\perp$, *then* $\eta(\lambda,\bigwedge_{i\in\Delta}\mu_i)\geq\bigwedge_{i\in\Delta}\eta(\lambda_i,\mu)$. *Suppose now that* $\bigwedge_{i\in\Delta}\eta(\lambda,\mu_i)>\perp$. *Then* $\forall a\in P(L)$ *and* $a\notin\alpha(\bigwedge_{i\in\Delta}\eta(\lambda,\mu_i))=\bigcup_{i\in\Delta}\alpha(\eta(\lambda,\mu_i))$. *It follows that* $a\notin(\alpha(\eta(\lambda,\mu_i))$ *for each* $i\in\Delta$ *implies* $(\lambda,\mu_i)\in\eta^{[a]}$ *for each* $i\in\Delta$ *and so* $(\lambda,\bigwedge_{i\in\Delta}\mu_i)\in\eta^{[a]}$. *Hence*

$$\alpha^*(\eta(\lambda,\bigwedge_{i\in\Delta}\mu_i))\subset\alpha^*(\bigwedge_{i\in\Delta}\eta(\lambda,\mu_i)).$$

Therefore

$$\eta(\lambda,\bigwedge_{i\in\Delta}\mu_i)\geq\bigwedge_{i\in\Delta}\eta(\lambda,\mu_i).$$

We can now state the following decomposition theorem of L-fuzzy topogenous. The proof is straightforward and therefore omitted.

Theorem 3.4..

Let η *be an L-fuzzy topogenous on* X. *Then*

$$\eta=\bigvee_{a\in L}(\underline{a}\wedge\eta_{[a]})=\bigvee_{a\in M(L)}(\underline{a}\wedge\eta_{[a]})=\bigwedge_{a\in L}(\underline{a}\vee\eta^{[a]})=\bigwedge_{a\in P(L)}(\underline{a}\vee\eta^{[a]})$$

Corollary 3.5..

Let η_1 *and* η_2 *be L-fuzzy topogenous's on* X, *then the following conditions are equivalent:*

(1) $\eta_1=\eta_2$.
(2) $\forall a\in L,\eta_{1_{[a]}}=\eta_{2_{[a]}}$.
(3) $\forall a\in M(L),\eta_{1_{[a]}}=\eta_{2_{[a]}}$.
(4) $\forall a\in L,\eta_1^{[a]}=\eta_2^{[a]}$.
(5) $\forall a\in P(L),\eta_1^{[a]}=\eta_2^{[a]}$.

Theorem 3.6.

Let η *be an L-fuzzy topogenous on* X, *then*

(1) $a\in L$, $\eta_{[a]}=\bigcap_{b\in\beta(a)}\eta_{[b]}$.
(2) $\forall a\in M(L)$, $\eta_{[a]}=\bigcap_{b\in\beta^*(a)}\eta_{[b]}$.
(3) $a\in L$, $\eta^{[a]}=\bigcap_{a\in\alpha(b)}\eta^{[b]}$.
(4) $\forall a\in P(L)$, $\eta^{[a]}=\bigcap_{a\in\alpha^*(a),b\in P(L)}\eta^{[b]}$.

Proof.

(1)By Lemma 2.4, we have that $\forall a \in L, \eta_{[a]} \subset \bigcap_{b\in\beta(a)} \eta_{[b]}$. *To show that* $\eta_{[a]} \supset \bigcap_{b\in\beta(a)} \eta_{[b]}$, *we take* $(\lambda,\rho) \in L^X \times L^X$ *and* $(\lambda,\rho) \in \bigcap_{b\in\beta(a)} \eta_{[b]}$. *Notice that* $\forall b \in \beta(a), \eta(\lambda,\rho) \geq b$. *Hence* $\eta(\lambda,\rho) \geq \bigvee \{b \mid b \in \beta(a)\} = a$, *which implies that* $(\lambda,\rho) \in \eta_{[a]}$.

(2) The proof is similar to (1).

(3)By Lemma 2.4, we have that $\forall a \in L, \eta^{[a]} \subset \bigcap_{a\in\alpha(b)} \eta^{[b]}$. *To show that* $\eta^{[a]} \supset \bigcap_{a\in\alpha(b)} \eta^{[b]}$, *we take* $(\lambda,\rho) \in L^X \times L^X$ *and* $(\lambda,\rho) \in \bigcap_{a\in\alpha(b)} \eta^{[b]}$. *Notice that* $\forall b \in L$ *and* $a \in \alpha(b)$, *it follows that* $b \notin \alpha(\eta(\lambda,\rho))$. *We prove by contradiction as follows. Suppose that* $a \in \alpha(\eta(\lambda,\rho))$. *Notice that* $\eta(\lambda,\rho) = \bigwedge \{b \mid b \in \alpha(\eta(\lambda,\rho))\}$ *and* $\alpha(\eta(\lambda,\rho)) = \bigcup \{\alpha(b) \mid b \in \alpha(\eta(\lambda,\rho))\}$. *There must exist* $b \in \alpha(\eta(\lambda,\rho))$ *such that* $a \in \alpha(b)$. *But this is impossible.*

(4) The proof is similar to (3).

Remark 3.7.

(1) $b \in \beta(a)$ *implies* $b \ll a$, *where* $b \ll$ *is way-below relation [6], i.e.* $b \ll a$ *if and only if for every up-directed set* S *in* L, $\bigvee S \geq a$ *implies that there exists* $s \in S$ *such that* $s \geq b$;

(2) If $a \in M(L)$, *then* $b \in \beta^*(a)$ *if and only if* $b \ll a$.

(3) $\forall a \in M(L)$, $\eta_{[a]} = \bigcap_{b\in\beta^*} \eta_{[b]} \Leftrightarrow \eta_{[a]} = \bigcap_{b\ll a, b\in M(L)} \eta_{[b]}$.

Proof.

(1) Since $\beta(a)$ *is a minimal set of* a, *from Definition 2.1, we have that for every up-directed set* S *in* L, *if* $\bigvee S \geq a$, *then* $\forall b \in \beta(a)$ *there exists* $s \in S$ *such that* $s \geq b$. *It follows that* $b \ll a$.

(2) Let $a \in M(L)$ *and* $b \ll a$. *From Theorems 1.3.6 and 1.3.8 in [15] and Definition 2.1, we know that* $\beta^*(a)$ *is both an up-directed set and a lower set, and* $\bigvee \beta^*(a) = a$. *Hence, there exists* $b^{'} \in \beta^*(a)$ *such that* $a \geq b^{'} \geq b$. *In other words,* $b \in \beta^*(a)$. *Conversely, if* $b \in \beta^*(a)$, *then since* $\beta^*(a) \subset \beta(a)$ *and* $b \in \beta^*(a)$ *implies* $b \in \beta(a)$. *It follows that* $b \ll a$.

(3) It is obvious.

Theorem 3.8.

Let $\{\eta_{[a]} \mid a \in M(L)\}$ *be a family of* L*-topogenous's on* X. *Then the following conditions are equivalent:*

(1) There exists an L*-fuzzy topogenous* η *on* X *such that* $\eta_{[a]} = \eta_a$ *for each* $a \in M(L)$.

(2) $\forall a \in M(L)$, $\eta_a = \bigcap_{b\in\beta^*(a)} \eta_b$.

Proof. (1) $\Rightarrow$ (2)*: This holds because of Theorem 3.5.*

(2) $\Rightarrow$ (1)*: Let* $\eta = \bigvee_{a\in M(L)}(a \wedge \eta_a)$. *Obviously, we have* $\eta_a \subset \eta_{[a]}$. *For any* $(\lambda, \rho) \in \eta_{[a]}$, *we have* $\eta(\lambda, \rho) \geq a$ *and* $\bigvee \{b \in M(L) \mid (\lambda, \rho) \in \eta_b\} \geq a$. *Next, since* $\beta^*(a)$ *is a minimal family of* a, *for each* $b \in \beta^*(a)$, *there exists* $b' \in M(L)$ *such that* $b \geq b'$ *and* $(\lambda, \rho) \in \eta_{b'} \subset \eta_b$. *Therefore,* $\bigcap_{b\in\beta^*(a)} \eta_b = \eta_a$.

Similarly, we can state the following theorems.

Theorem 3.9. *Let* $\{\eta_a \mid a \in P(L)\}$ *be a family of* L*-topogenous's on* X*. Then the following conditions are equivalent:*

(1) There exists an L*-fuzzy topogenous* η *on* X *such that* $\eta^{[a]} = \eta_a$ *for each* $a \in P(L)$.

(2) $\forall a \in P(L)$, $\eta_a = \bigcap_{a\in\alpha^*(b)} \eta_b$.

Theorem 3.10. *Let* $\{\eta_a \mid a \in L\}$ *be a family of* L*-topogenous's on* X*. Then the following conditions are equivalent:*

(1) There exists an L*-fuzzy topogenous* η *on* X *such that* $\eta_{[a]} = \eta_a$ *for each* $a \in L$.

(2) $\forall a \in L$, $\eta_a = \bigcap_{b\in\beta(a)} \eta_b$.

Theorem 3.11. *Let* $\{\eta_a \mid a \in L\}$ *be a family of* L*-topogenous's on* X*. Then the following conditions are equivalent:*

(1) There exists an L*-fuzzy topogenous* η *on* X *such that* $\eta^{[a]} = \eta_a$ *for each* $a \in L$.

(2) $\forall a \in L$, $\eta_a = \bigcap_{a\in\alpha(b)} \eta_b$.

4. Representation theorems of L-fuzzy topogenous's

Let $LT[X]$ denote the family of all L-topogenous's on X. Let $LFT[X]$ denote the family of all L-fuzzy topogenous's on X. The order relation on $LFT[X]$ is defined as follow:

$$\forall \eta_1, \eta_2 \in LFT[X], \eta_1 \preceq \eta_2 \Leftrightarrow \forall (\lambda, \rho) \in L^X \times L^X, \eta_1(\lambda, \rho) \leq \eta_2(\lambda, \rho).$$

Theorem 4.1.

$(LFT[X], \preceq)$ *is a complete lattice. In fact, it is a complete sub-meet-semilattice of* $L^{L^X \times L^X}$, *i.e. closed under the* $\wedge$ *of* $L^{L^X \times L^X}$.

Proof. *Let X be a set. Define two maps $\eta : L^X \times L^X \to L$ as follows:*

$$\eta_0(\lambda, \rho) = \begin{cases} \top, & \text{if } \lambda = \underline{\perp} \text{ or } \rho = \underline{\top}, \\ \perp, & \text{otherwise,} \end{cases}$$

$$\eta_1(\lambda, \rho) = \begin{cases} \top, & \text{if } \lambda \leq \rho, \\ \perp, & \text{otherwise.} \end{cases}$$

Clearly, we have $\eta_0, \eta_1 \in LFT[X]$, and they are the smallest element and the greatest element in $(LFT[X], \preceq^L)$, respectively. Next, let $\{\eta_i \mid i \in \Delta\} \subset LFT[X]$ and $\eta = \bigwedge_{i\in\Delta}^{\preceq^L} \eta_i$. Obvious $\eta \in LFT[X]$. Accordingly, $(LFT[X], \preceq)$ is a complete lattice.

To facilitate further illustration, let us define the following classes:

$$U^L[X] = \{F : L \to LT[X] \mid \forall a \in L, F(a) = \cap_{a\in\alpha(b)} F(b)\}$$

$$U_L[X] = \{F : L \to LT[X] \mid \forall a \in L, F(a) = \cap_{b\in\beta(a)} F(a)\}$$

$$U_{M(L)}[X] = \{F : M(L) \to LT[X] \mid \forall a \in M(L), F(a) = \cap_{b\in\beta^*(a)} F(b)\}$$

$$U_{P(L)}[X] = \{F : P(L) \to LT[X] \mid \forall a \in P(L), F(a) = \cap_{a\in\alpha^*(b)} F(b)\}$$

In addition, let us define the following order relations within the classes $U^L[X]$, $U_L[X]$,$U_{M(L)}[X]$ and $U_{P(L)}[X]$:

$$F_1, F_2 \in U^L[X], F_1 \preceq^L F_2 \Leftrightarrow \forall a \in L, F_1(a) \subset F_2(a)$$

$$F_1, F_2 \in U_L[X], F_1 \preceq_L F_2 \Leftrightarrow \forall a \in L, F_1(a) \subset F_2(a)$$

$$F_1, F_2 \in U_{M(L)}[X], F_1 \preceq_{M(L)} F_2 \Leftrightarrow \forall a \in M(L), F_1(a) \subset F_2(a)$$

$$F_1, F_2 \in U_{P(L)}[X], F_1 \preceq_{P(L)} F_2 \Leftrightarrow \forall a \in P(L), F_1(a) \subset F_2(a)$$

Theorem 4.2.
$(U^L[X], \preceq^L)$, $(U_L[X], \preceq_L)$, $(U_{M(L)}[X], \preceq_{M(L)})$ and $(U_{P(L)}[X], \preceq_{P(L)})$ are complete lattices. Obviously, $(U^L[X], \preceq^L)$ and $(U_L[X], \preceq_L)$ are complete sub-meet-semilattices of the lattice $(LT[X])^L$ (i.e., closed under the $\wedge$ of $(LT[X])^L$, when $\{F_i \mid i \in \Delta\} \subset U^L[X]$, $F = \bigwedge_{i\in\Delta}^{\preceq_L} F_i$ be defined as $\forall a \in L, F(a) = \bigcap_{i\in\Delta} F_i(a)$, $(U_{M(L)}[X], \preceq_{M(L)})$ is a complete sub-meet-semilattices of the lattice $(LT[X])^{M(L)}$, and $(U_{P(L)}[X]), \preceq_{p(L)})$ is a complete sub-meet-semilattices of the lattice $(LT[X])^{P(L)}$.

Proof. *$\forall a \in L$, let us define $F_\perp(a) = \{(\lambda, \rho) \mid \lambda = \underline{\perp}, \rho = \underline{\top}\}$ and $F_\top(a) = \{(\lambda, \rho) \mid \lambda \leq \rho\}$. Clearly, we have $F_\perp(a), F_\top(a) \in U^L[X]$, and they are the smallest element and the greatest element in $(U^L[X], \preceq^L)$, respectively. Next, let $\{F_i \mid i \in \Delta\} \subset U^L[X]$ and $F = \bigwedge_{i\in\Delta}^{\preceq^L} F_i$. Since*

$$F(a) = \bigcap_{i\in\Delta} F_i(a) = \bigcap_{i\in\Delta} \bigcap_{a\in\alpha(b)} F_i(b) = \bigcap_{a\in\alpha(b)} \bigcap_{i\in\Delta} F_i(b) = \bigcap_{a\in\alpha(b)} F(b),$$

it follows that $F \in U^L[X]$. Accordingly, $(U^L[X], \preceq^L)$ is a complete lattice. The same argument can be used to prove the rest of the theorem.

The following representation theorem of L-fuzzy topogenous follows naturally.

Theorem 4.3.
The map $f : LFT[X] \to U^L[X], \eta \mapsto F_\eta$ (for every $a \in L$ and $F_\eta(a) = \eta^{[a]}$ is an isomorphism in the category of complete meet-semilattices and $f^{\leftarrow} : U^L[X] \to LFT[X], F \mapsto \eta_F = \bigwedge_{a\in L}(\underline{a} \vee F(a))$.

Proof.
For each $\eta \in LFT[X]$, it is easy to verify that

$$F_\eta(a) = \eta^{[a]} = \bigcap_{a\in\alpha(b)} \eta^{[b]} = \bigcap_{a\in\alpha(b)} F_\eta(b)$$

Hence, $F_\eta \in U^L[X]$. Next, by Theorems 3.3, 3.4 and Corollary 3.5, it suffices to show that f is an injection. Since $(\lambda, \rho) \notin (\eta_F)^{[c]}$ iff

$$\alpha((\eta_F(\lambda, \rho)) = \bigcup_{a\in L} \alpha((\underline{a} \vee F(a))((\lambda, \rho)) = \bigcup \{\alpha(a) \mid a \in L, (\lambda, \rho) \notin F(a)\}$$

iff there exists $a \in L$ *such that* $c \in \alpha(a)$ *and* $(\lambda, \rho) \notin F(a)$ *iff* $(\lambda, \rho) \notin \bigcap_{c\in\alpha(a)} F(a) = F(c)$*, we have* $F_{\eta_F}(c) = \eta_F^{[c]} = F(c)$*. This shows that* $F_{\eta_F} = F$*. It follows that* f *is a surjection as well as a bijection, and*

$$f^{\leftarrow} : U^L[X] \to LFT[X], F \mapsto \eta_F = \bigwedge_{a\in L} (\underline{a} \vee F(a))$$

Next, let $\eta_1, \eta_2 \in LFT[X]$ *and* $\{\eta_i \mid i \in \Delta\} \subset LFT[X]$*. Then it is straightforward to show that* $f(\eta_1) \preceq^L f(\eta_2)$ *when* $\eta_1 \preceq \eta_2$*. Hence* $f(\bigwedge_{i\in\Delta} \eta_i) = \bigwedge^{\preceq^L}_{i\in\Delta} f(\eta_i)$ *and the proof is complete.*

The following Theorem follows directly from the above proof.

Theorem 4.4.
The map $f : LFT[X] \to U_{P(L)}[X], \eta \mapsto F_\eta$ *(for every* $a \in P(L)$ *and* $F_\eta(a) = \eta^{[a]}$ *is an isomorphism in the category of complete meet-semilattices and* $f^{\leftarrow} : U_{P(L)}[X] \to LFT[X], F \mapsto \eta_F = \bigwedge_{a\in P(L)}(\underline{a} \vee F(a))$*.*

Theorem 4.5.
The map $f : LFT[X] \to U_L[X], \eta \mapsto F_\eta$ *(for every* $a \in L$ *and* $F_\eta(a) = \eta_{[a]}$ *is an isomorphism in the category of complete meet-semilattices and* $f^{\leftarrow} : U^L[X] \to LFT[X], F \mapsto \eta_F = \bigwedge_{a\in L}(\underline{a} \vee F(a))$*.*

Proof.
For each $\eta \in LFT[X]$*, it is easy to verify that*

$$F_\eta(a) = \eta_{[a]} = \bigcap_{b\in\beta(a)} \eta_{[b]} = \bigcap_{b\in\beta(a)} F_\eta(b)$$

Hence, $F_\eta \in U_L[X]$*. Next, by Theorems 3.4 and Corollary 3.5, it suffices to show that* f *is an injection. It is proved easily that* $(\lambda, \rho) \in (\eta_F)_{[c]}$ *iff*

$$\eta_F((\lambda, \rho)) = \bigvee_{a\in L} (\underline{a} \wedge F(a))((\lambda, \rho)) = \bigvee \{a \mid (\lambda, \rho) \in F(a)\} \geq c$$

iff (because of Lemma 2.4)

$$\bigcup_{(\lambda,\rho)\in F(a)} \beta(a) = \beta(\bigvee \{a \mid (\lambda, \rho) \in F(a)\}) \supset \beta(c)$$

On the other hand, we can prove

$$(\lambda,\rho)\in F(c)=\bigcap_{a\in\beta(\alpha)}F(a)\Leftrightarrow \forall a\in\beta(\alpha),(\lambda,\rho)\in F(a)\Leftrightarrow \bigcup_{(\lambda,\rho)\in F(a)}\beta(a)\supset\beta(c)$$

Clearly, $\forall a\in\beta(\alpha),(\lambda,\rho)\in F(a)\Rightarrow\bigcup_{(\lambda,\rho)\in F(a)}\beta(a)\supset\bigcup_{a\in\beta(c)}=\beta(c)$. *Conversely, for each* $d\in\beta(c)\subset\bigcup_{(\lambda,\rho)\in F(a)}\beta(a)$, *then there exists* $a\in L$ *such that* $d\in\beta(a)$ *and* $(\lambda,\rho)\in F(a)=\bigcap_{b\in\beta(a)}F(b)$. *It show that* $(\lambda,\rho)\in F(d)$. *So, we conclude that* $(\lambda,\rho)\in(\eta_F)_{[c]}\Leftrightarrow(\lambda,\rho)\in F(c)$, *i.e.,* $F_{\eta_F}(c)=(\eta_F)_{[c]}=F(c)$. *This shows that* $F_{\eta_F}=F$. *It follows that* f *is a surjection as well as a bijection, and*

$$f^{\leftarrow}:U_L[X]\to LFT[X],F\mapsto\eta_F=\bigvee_{a\in L}(\underline{a}\wedge F(a))$$

Next, let $\eta_1,\eta_2\in LFT[X]$ *and* $\{\eta_i\mid i\in\Delta\}\subset LFT[X]$. *Then it is straightforward to show that* $f(\eta_1)\preceq^L f(\eta_2)$ *when* $\eta_1\preceq\eta_2$. *Hence* $f(\bigwedge_{i\in\Delta}\eta_i)=\bigwedge^{\preceq^L}_{i\in\Delta}f(\eta_i)$ *and the proof is complete.*

The following Theorem follows directly from the above proof.

Theorem 4.6.
The map $f:LFT[X]\to U_{M(L)}[X],\eta\mapsto F_\eta$ *(for every* $a\in M(L)$ *and* $F_\eta(a)=\eta_{[a]}$ *is an isomorphism in the category of complete meet-semilattices and* $f^{\leftarrow}:U_{M(L)}[X]\to LFT[X],F\mapsto\eta_F=\bigvee_{a\in M(L)}(\underline{a}\wedge F(a))$.

5. L-fuzzy continuous topogenous maps

Definition 5.1.
Let (X,η_1) *and* (Y,η_2) *be two L-fuzzy topogenous orders. Let* $f:X\to Y$ *be a map.* $f:(X,\eta_1)\to(Y,\eta_2)$ *is called L-fuzzy topogenous continuous map if for every* $(\lambda,\rho)\in L^Y\times L^Y$ *we have*

$$\eta_1(f^{\leftarrow}(\lambda),f^{\leftarrow}(\rho))\geq\eta_2(\lambda,\rho),$$

where $f^{\leftarrow}(\lambda)=\lambda\circ f$.

From Definition 5.1, obviously, $f:(X,\eta_1)\to(Y,\eta_2)$ is an L-fuzzy topogenous continuous if and only if $\forall a\in M(L)$, $f:(X,{\eta_1}_{[a]})\to(Y,{\eta_2}_{[a]})$ is an L-topogenous continuous map.

Excepting this, we have the followings equivalent conditions:

Theorem 5.2.

Let (X,η_1) and (Y,η_2) be L-fuzzy topogenous orders and $f : X \to Y$ be a map. Then the following conditions are equivalent:

(1) $f : (X,\eta_1) \to (Y,\eta_2)$ is an L-fuzzy topogenous continuous map.

(2) $\forall a \in M(L)$, $f : (X,\eta_{1_{[a]}}) \to (Y,\eta_{2_{[a]}})$ is an L-topogenous continuous map.

(3) $\forall a \in L$, $f : (X,\eta_1^{[a]}) \to (Y,\eta_2^{[a]})$ is an L-topogenous continuous map.

(4) $\forall a \in P(L)$, $f : (X,\eta_1^{[a]}) \to (Y,\eta_2^{[a]})$ is an L-topogenous continuous map.

Proof. (1) $\Rightarrow$ (2)*: This part is obvious.*

(2) $\Rightarrow$ (1)*: $\forall(\lambda,\rho) \in L^Y \times L^Y$, $a \in M(L)$ such that $a \leq \eta_2(\lambda,\rho)$, we have $(\lambda,\rho) \in \eta_{2_{[a]}}$ and $(f^{\leftarrow}(\lambda), f^{\leftarrow}(\rho)) \in \eta_{1_{[a]}}$ by the continuity of $f : (X,\eta_{1_{[a]}}) \to (Y,\eta_{2_{[a]}})$. Accordingly, $\eta_1(f^{\leftarrow}(\lambda), f^{\leftarrow}(\rho)) \geq a$ for each $\forall a \in M(L) \cap M(\eta_2(\lambda,\rho))$, where $M(\eta_2(\lambda,\rho)) = \{a \in M(L) \mid a \leq \eta_2(\lambda,\rho)\}$. It follows that $\eta_1(f^{\leftarrow}(\lambda), f^{\leftarrow}(\rho)) \geq \bigvee M(\eta_2(\lambda,\rho)) = \eta_2(\lambda,\rho)$.*

(1) $\Rightarrow$ (3)*: $\forall(\lambda,\rho) \in L^Y \times L^Y$, since $\eta_1(f^{\leftarrow}(\lambda), f^{\leftarrow}(\rho)) \geq \eta_2(\lambda,\rho)$, it follows from Lemma 2.4 that $a \notin \alpha(\eta_1(f^{\leftarrow}(\lambda), f^{\leftarrow}(\rho)))$ when $\forall a \in L$, if $a \notin \alpha(\eta_2(\lambda,\rho))$. In other words, if $(\lambda,\rho) \in \eta_2^{[a]}$, then $(f^{\leftarrow}(\lambda), f^{\leftarrow}(\rho)) \in \eta_1^{[a]}$. Thus $f : (X,\eta_1^{[a]}) \to (Y,\eta_2^{[a]})$ is a fuzzy topogenous continuous map.*

(3) $\Rightarrow$ (4)*: This is obvious.* (4) $\Rightarrow$ (1)*: For $\forall a \in P(L)$ and $(\lambda,\rho) \in L^Y \times L^Y$, if $a \notin \alpha(\eta_2(\lambda,\rho))$, then $(\lambda,\rho) \in \eta_2^{[a]}$. Thus $(f^{\leftarrow}(\lambda), f^{\leftarrow}(\rho)) \in \eta_1^{[a]}$ by the continuity of $f : (X,\eta_1^{[a]}) \to (Y,\eta_2^{[a]})$. In other words, $a \notin \alpha(\eta_1(f^{\leftarrow}(\lambda), f^{\leftarrow}(\rho)))$ and $\alpha^*(\eta_1(f^{\leftarrow}(\lambda), f^{\leftarrow}(\rho))) \subset \alpha^*(\eta_2(\lambda,\rho))$. It follows from Lemma 2.4 that*

$$\eta_1(f^{\leftarrow}(\lambda), f^{\leftarrow}(\rho)) \geq \eta_2(\lambda,\rho)$$

Hence the proof is completed.

Definition 5.3.

Let (X,η_1) and (Y,η_2) be two L-fuzzy topogenous orders. Let $f : X \to Y$ be a map. $f : (X,\eta_1) \to (Y,\eta_2)$ is called L-fuzzy topogenous open map if for every $(\lambda,\rho) \in L^Y \times L^Y$ we have

$$\eta_2(f^{\to}(\lambda), f^{\to}(\rho)) \geq \eta_1(\lambda,\rho),$$

Theorem 5.4.

Let (X, η_1) and (Y, η_2) be L-fuzzy topogenous orders and $f : X \to Y$ be a map. Then the following conditions are equivalent:

(1) $f : (X, \eta_1) \to (Y, \eta_2)$ is an L-fuzzy topogenous open map.

(2) $\forall a \in M(L)$, $f : (X, {\eta_1}_{[a]}) \to (Y, {\eta_2}_{[a]})$ is an L-topogenous open map.

(3) $\forall a \in L$, $f : (X, \eta_1^{[a]}) \to (Y, \eta_2^{[a]})$ is an L-topogenous open map.

(4) $\forall a \in P(L)$, $f : (X, \eta_1^{[a]}) \to (Y, \eta_2^{[a]})$ is an L-topogenous open map.

Proof. (1) $\Rightarrow$ (2)*: This part is obvious.*

(2) $\Rightarrow$ (1)*: For a given $(\lambda, \rho) \in L^Y \times L^Y$,if $\eta_1(\lambda, \rho) = \bot$,then clearly, $\eta_2(f^{\to}(\lambda), f^{\to}(\rho)) \geq \eta_1(\lambda, \rho)$,. If $\eta_1(\lambda, \rho) > \bot$ then since $\eta_1(\lambda, \rho) = \bigvee M(\eta_1(\lambda, \rho))$, we have $\eta_2(f^{\to}(\lambda), f^{\to}(\rho)) \geq a$ for each $a \in M(\eta_1(\lambda, \rho)$. Hence*

$$\eta_2(f^{\to}(\lambda), f^{\to}(\rho)) \geq \bigvee\{a \mid a \in M(\eta_1(\lambda, \rho))\} = \eta_1(\lambda, \rho).$$

(1) $\Rightarrow$ (3)*: $\forall(\lambda, \rho) \in L^Y \times L^Y$. From part (1) of the theorem and Lemma 2.4, we have $\alpha(\eta_2(f^{\to}(\lambda), f^{\to}(\rho))) \subset \alpha(\eta_1(\lambda, \rho))$. It follows that $a \notin \alpha(\eta_2(f^{\to}(\lambda), f^{\to}(\rho)))$ if for each $a \notin \alpha(\eta_1(\lambda, \rho))$. In other words,$(f^{\to}(\lambda), f^{\to}(\rho)) \in \eta_2^{[a]}$ if for each $(\lambda, \rho) \in \eta_1^{[a]}$ for each $a \in L$. Hence statement (3) holds.*

(3) $\Rightarrow$ (4)*: This is obvious.*

(4) $\Rightarrow$ (1)*: For $\forall a \in P(L)$ and $(\lambda, \rho) \in L^Y \times L^Y$, from part (4) of the theorem if $a \notin \alpha(\eta_1(\lambda, \rho))$, then $a \notin \alpha(\eta_2(f^{\to}(\lambda), f^{\to}(\rho)))$. Thus $\alpha^*(\eta_2(f^{\to}(\lambda), f^{\to}(\rho))) \subset \alpha^*(\eta_1(\lambda, \rho))$. We have from Lemma 2.4 that $\eta_2(f^{\to}(\lambda), f^{\to}(\rho)) \geq \eta_1(\lambda, \rho)$,. Hence the proof is completed.*

Definition 5.5.

Let (X, η_1) and (Y, η_2) be two L-fuzzy topogenous orders. Let $f : X \to Y$ be a map. $f : (X, \eta_1) \to (Y, \eta_2)$ is called an L-fuzzy topogenous homeomorphism if f is bijective and f and $f^{\leftarrow}$ are L-fuzzy continuous maps.

Theorem 5.6.

Let (X, η_1) and (Y, η_2) be L-fuzzy topogenous orders and $f : X \to Y$ be a bijective map. Then the following conditions are equivalent:

(1) $f : (X, \eta_1) \to (Y, \eta_2)$ is an L-fuzzy topogenous homeomorphism .

(2) $\forall a \in M(L)$, $f : (X, {\eta_1}_{[a]}) \to (Y, {\eta_2}_{[a]})$ is an L-topogenous homeomorphism .

(3) $\forall a \in L$, $f : (X, \eta_1^{[a]}) \to (Y, \eta_2^{[a]})$ is an L-topogenous homeomorphism .

(4) $\forall a \in P(L)$, $f : (X, \eta_1^{[a]}) \to (Y, \eta_2^{[a]})$ *is an L-topogenous homeomorphism .*

Proof. *It follows from Definitions 5.1, 5.3, 5.5 and Theorems 5.2 and 5.4.*

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