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# Pseudospherical Surfaces and Evolution Equations in Higher Dimensions

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*Abstract : In this paper, the study of evolution equations with two independent variables which are related to pseudospherical surfaces in $R^3$, is extended to evolution equations with more than two independent variables. Equations of the type*

$$u_{xt} = \psi(u, u_x, \dots\dots, \frac{\partial^k u}{\partial x^k}, u_y, \dots\dots, \frac{\partial^{k'} u}{\partial y^{k'}})$$

*are studied and characterized. Some features and results on properties of these equations are given via this study.*

**Keywords-Evolutionequations,Pseudospherical surfaces,Riemannian manifold andSolitons.**

## I. Introduction

The studyof non-linear evolution equations has been closely related to the study of soliton phenomena. In particular, many non-linear evolution equations of one spatial variable plus the time variable,which admit.soliton solutions, have been extensively studied in the last two decades or so [v,ii]. Many interesting Features of solitons, accordingly to evolution equations which admit, these soliton solutions, have been disclosed, [xi-vi,x,ii].On the contrary, for the higher dimensional case, the studies of solitons arc less developed and remain one of the interesting, and challenging, present and future research subjects, [vi,i]. This is also, the case for non-linear evolution equations with two or more spatial variables plus the time variable, [iv,xiii,xiv].However, one of main geometrical techniques, motivated in part, by Sasaki [x],EI-Sabbagh[vi],Chern and Tenenblat[xii],is the notion of a differential equation which describes a pseudo spherical surface(P.S.S).With this concept,a systematic procedure has begun to obtain linear systems associated to the non-linear differential equations as well. These linear systems are essential in order to apply the inverse scattering method to obtain solutions of the non-linear differential equation, [viii,ix].

In this paper, we shall extend the notion of P.S.P to higher dimensions i.e. 3-dim plane of constant sectional curvature-1 imbedded in $R^5$. Conditions for equations of the type

$$u_{xt} = \psi(u, u_x, u_{xx}, \dots\dots, \frac{\partial^k u}{\partial x^k}, u_y, u_{yy}, \dots\dots, \frac{\partial^{k'} u}{\partial y^{k'}})$$

To describe a two-parameter 3-dim P.S.P, will .be given in section III. While, in section II, we give basic notations and definitions as well as necessary preliminaries.

## II. Basic notations and Preliminaries

Let M be an n-dimensional Riemannian manifold with constant curvature, isometric cally immersed in $\bar{M}^{2n-1}$with constant curvature $\bar{K}$,with$K<\bar{K}$. Let $e_1, e_2, \dots, e_{2n-1}$be a moving orthonormal frame on an open set of $\bar{M}$,so that at points ofM,$e_1, e_2, \dots, e_n$are tangents to M.Let $\omega_A$be the dual orthonormal coframe and consider $\omega_{AB}$ defined by

$$de_A = \sum_B \omega_{AB} e_B$$

The structure equations of $\bar{M}$are

$$d\omega_A = \sum_B \omega_B \wedge \omega_{BA} \quad , \quad \omega_{AB} + \omega_{BA} = 0 \tag{2.1}$$

$$d\omega_{AB} = \sum_C \omega_{AC} \wedge \omega_{CB} - \bar{K}\omega_A \wedge \omega_B \text{ with} 1 \le A, B, C \le 2n-1 \tag{2.2}$$

Restricting these forms to M we have $\omega_\alpha = 0$ ,so (2.1) gives with $n+1 \le \alpha, \beta, \gamma \le 2n-1$ and $1 \le I, J, L \le n$,

$$d\omega_\alpha = \sum_I \omega_I \wedge \omega_{I\alpha} = 0 \tag{2.3}$$

$$d\omega_I = \sum_J \omega_J \wedge \omega_{JI} \tag{2.4}$$

from (2.2) we obtain, Gauss equation

$$d\omega_{IJ} = \sum_L \omega_{IL} \wedge \omega_{LJ} + \sum_\alpha \omega_{I\alpha} \wedge \omega_{\alpha J} - \bar{K}\omega_I \wedge \omega_J \tag{2.5}$$

and Codazzi equation

$$d\omega_{I\alpha} = \sum_A \omega_{IA} \wedge \omega_{A\alpha} \tag{2.6}$$

Mhas constant sectional curvature K if and only if

$$\Omega_{IJ} = d\omega_{IJ} - \sum_L \omega_{IL} \wedge \omega_{LJ} = -K\,\omega_I \wedge \omega_J \tag{2.7}$$

$$\sum_\alpha \omega_{I\alpha} \wedge \omega_{\alpha J} = (\bar{K} - K)\,\omega_I \wedge \omega_J \tag{2.8}$$

Also, equation (2.2) implies that

$$d\omega_{\alpha\beta} = \sum_\gamma \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} + \Omega_{\alpha\beta} \quad \text{With} \Omega_{\alpha\beta} = \sum_I \omega_{\alpha I} \wedge \omega_{I\beta}$$

The forms $\Omega_{\alpha\beta}$ give the normal curvature of M and $I = \sum_I (\omega_I)^2$ is its first fundamental form.

For our purpose in this paper, we write these equations when $\bar{M}$ is taken to be $R^5$ and M is a 3-dimensional submanifold with constant sectional curvature $K = -1$(i.e. pseudo spherical 3-plane in $R^5$).

The equations take the forms

$$\left.\begin{aligned} d\omega_1 &= \omega_4 \wedge \omega_2 + \omega_5 \wedge \omega_3 \\ d\omega_2 &= -\omega_4 \wedge \omega_1 + \omega_6 \wedge \omega_3 \\ d\omega_3 &= -\omega_5 \wedge \omega_1 - \omega_6 \wedge \omega_2 \\ d\omega_4 &= \omega_1 \wedge \omega_2 \\ d\omega_5 &= \omega_1 \wedge \omega_3 \\ d\omega_6 &= \omega_2 \wedge \omega_3 \end{aligned}\right\} \tag{2.9}$$

where we have written