REAL ANALYSIS IN METRIC SPACES

By

NASHAT FARIED MOHAMED

1992

CHAPTER 0

Least upper bound and Greatest lower bound

These two notions seem to be the most important among those which characterize modern analysis and a very efficient tool in many areas of mathematics.

Definition: A subset E of real numbers is called bounded from above if there exists a real number α such that:

$$x \leq \alpha \qquad \forall\, x \in E.$$

The subset E is called bounded from below if there exists a real number β such :

$$x \geq \beta \qquad \forall\, x \in E.$$

The subset E is called bounded if it is both bounded from above and from below .

Examples : 1) Any finite subset of real numbers such as the subset

$$\{1,\ 2.5,\ 7.9,\ 10,\ 12\}$$

is bounded from above and from below. Moreover it has a greatest element 12 and a least element which is 1.

2) The set of natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$ is not bounded from above but it is bounded from below.

3) The set $\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \ldots\}$ of all integers is neither bounded from above nor from below.

4) Any interval of the form :

$$[a, b] = \{x \in \mathbb{R} \ :\ a \leq x \leq b\}$$

is bounded from above and from below.

5) The infinite set $\left\{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots, \frac{n}{n+1}, \dots \right\}$ is bounded from above and from below.

Remarks :

1) Not each subset has a greatest element (see the last example number (5)).
2) While each finite subset of real numbers is bounded from above and below, we can not say any thing for infinite subsets. Sometimes they are bounded like example (4) and sometimes they are neither bounded from above nor from below like example (3).
3) If there exists an upper bound α for a subset E then each number greater than or equal to the number α is also an upper bound for the subset E.
4) Any increase of an upper bound for a subset E yields another upper bound for it.
5) Either a subset E has an infinite number of upper bounds or it has none.
6) Similarly, Either a subset E has an infinite number of lower bounds or it has no lower bounds at all.

Definition: Let E be a bounded from above subset of real numbers. The least upper bound α_0 of E which is denoted by $\alpha_0 = \sup E$ is a real number satisfying the following two conditions:

1) $x \le \alpha_0 \quad \forall\, x \in E$.
2) if $x \le \alpha \ \forall\, x \in E$, then $\alpha_0 \le \alpha$.

Remarks :

1) The existence of the number α_0 is an axiom of the real number system

2) These two conditions can be formulated in another equivalent form to give another equivalent definition for sup E:

Equivalent definition:

For a bounded from above subset E, $\alpha_0 = \sup E$ iff (if and only if)

1) α_0 is an upper bound for E.
2) $\forall\, \varepsilon > 0$ the number $\alpha_0 - \varepsilon$ is not an upper bound for E.

Another formulation of the definition:

For a bounded from above subset E, $\alpha_0 = \sup E$ iff (if and only if)

1) α_0 is an upper bound for E.
2) $\forall\, \varepsilon > 0$ $\exists$ an element $x_0 \in E$ such that :

$$x_0 > \alpha_0 - \varepsilon.$$

This last formulation of the definition is an easy way of handling with the definition.

Example 1: Prove that $\sup \left\{ \frac{n}{n+1} : n \in \mathbb{N} \right\} = 1$.

Solution:

We only verify the satisfaction of the two last conditions (conditions of the least upper bound). In fact: Clearly

$$\frac{n}{n+1} \le 1 \quad \forall\, n \in \mathbb{N} \qquad (1)$$

Thus the number 1 is an upper bound for the set of elements of the form $\frac{n}{n+1}$ $n = 1, 2, 3, \dots$

To verify the satisfaction of the second condition we take any arbitrary $\varepsilon > 0$ and try to find an element of the form $\frac{n_0}{n_0+1}$ for some fixed $n_0 \in \mathbb{N}$ such that :

$$\frac{n_0}{n_0+1} > 1 - \varepsilon$$

i.e.

$$1 - \frac{1}{n_0+1} > 1 - \varepsilon$$

i.e.

$$\frac{1}{n_0+1} < \varepsilon$$

i.e.

$$n_0 + 1 > \frac{1}{\varepsilon}$$

but we can easily find this natural number $n_0 \in \mathbb{N}$ if we take it such that :

$$n_0 > \frac{1}{\varepsilon} - 1$$

Hence the solution.

Remark :

The least upper bound of a bounded from above subset E of real numbers need not belong to the subset E. For example

$$\sup \frac{n}{n+1} = 1$$

while there is no natural number n that satisfy :

$$\frac{n}{n+1} = 1$$

Definition : For a bounded from above subset E of real numbers if

$$\sup E \in E$$

then we write

$$\sup E = \max E$$

In this case the maximum of the set E exists.

Remark :

We notice that the least upper bound of a bounded from above subset is one of its upper bounds and therefore we can say that it is the minimum of its upper bounds and write

$$\sup E = \min \left\{ \beta : x \leq \beta \ \ \forall x \in E \right\}$$

Definition: Let E be a bounded from below subset of real numbers. The greatest lower bound β_0 of E which is denoted by :

$$\beta_0 = \inf E$$

is a real number satisfying the following two conditions:

1) $x \geq \beta_0 \quad \forall x \in E$.

2) if $x \geq \beta \quad \forall x \in E$, then $\beta_0 \geq \beta$.

Remark:

These two conditions can be formulated in another equivalent form to give another equivalent definition for inf E :

Equivalent definition:

For a bounded from below subset E, $\beta_0 = \inf E$ iff

1) β_0 is a lower bound for E.

2) $\forall \varepsilon > 0$ the number $\beta_0 + \varepsilon$ is not a lower bound for E.

Another equivalent definition:

For a bounded from below subset E, $\beta_0 = \inf E$ iff

1) β_0 is a lower bound for E.

2) $\forall \varepsilon > 0 \ \exists$ an element $x_0 \in E$ such that:

$$x_0 < \beta_0 + \varepsilon.$$

This last form of the definition makes it easy to handle with.

EXAMPLE : Let E be a bounded subset of real numbers. If we define a subset $\neg E$ (only temporarly) by:

$$\neg E = \{ -x : x \in E \},$$

then prove that $\neg E$ is also bounded and for which we have:

1) $\sup \neg E = - \inf E$ 2) $\inf \neg E = - \sup E$.

Proof of 1)

Let $\inf E = \beta_0$ then from the definition we get:

i) $\beta_0 \leq x \quad \forall x \in E$ and

ii) $\forall \varepsilon > 0 \ \exists$ an element $x_0 \in E$ such that $x_0 < \beta_0 + \varepsilon$

Multiplying by -1 we get :

i) $-x \leq -\beta_0 \quad \forall \ -x \in \neg E$ and

ii) $\forall \varepsilon > 0 \ \exists$ an element $-x_0 \in \neg E$ such that $-x_0 > -\beta_0 - \varepsilon$.

The last two conditions are exactly the conditions of the supremum of the set $\neg E$. Thus we have:

$$\sup \neg E = -\beta_0 = -\inf E.$$

Proof of 2) is similar.

The following example shows how the definition of the supremum can be used to find and verify the result of a countable union of sets.

Example : Prove that $\bigcup_{n \in N}]0, \frac{n}{n+1}[=]0, 1[$

Proof:

Clearly $]0, \frac{n}{n+1}[\subseteq]0, 1[$. So it is enough to prove the other direction. One can use the definition of the supremum to get the required relation.

Since $\sup \frac{n}{n+1} = 1$ then $\forall 1 > \varepsilon > 0 \ \exists$ an element $\frac{n_0}{n_0+1}$ such that

$$0 < 1-\varepsilon < \frac{n_0}{n_0+1}$$

Hence for every element $x \in]0, 1[$ we take ε such that:

$$1 > \varepsilon = 1 - x > 0$$

and consequently $\exists$ an element $\frac{n_0}{n_0+1}$ such that :

$$0 < x < \frac{n_0}{n_0+1}.$$

Hence

$$x \in]0, \frac{n_0}{n_0+1}[$$

Hence the proof.

Example: Let $\{a_i\}$ and $\{b_i\}$, $i = 1, 2, 3, \ldots, n$ be real numbers.

Then :

1) $\sup (a_i + b_i) \leq \sup a_i + \sup b_i$,

2) $\inf a_i + \inf b_i \leq \inf (a_i + b_i)$.

In fact, let $\alpha = \sup a_i$, $\beta = \sup b_i$ and $\gamma = \sup (a_i + b_i)$.

It is then required to show that :

$$\gamma \leq \alpha + \beta .$$

From the definitions we get;

$$a_i \leq \alpha \quad \forall i \qquad \text{and} \qquad b_i \leq \beta \quad \forall i$$

Thus,

$$a_i + b_i \leq \alpha + \beta \quad \forall i = 1, 2, 3, \ldots, n$$

Hence $\alpha + \beta$ is an upper bound for the numbers $(a_i + b_i)$ and consequently is greater than or equal to the least upper bound γ.

The proof of (2) is similar

Example: Let E be a bounded from below subset of real numbers such that:

$$\inf E > 0$$

Denote by $\frac{1}{E}$ the subset:

$$\frac{1}{E} = \left\{ \frac{1}{x} : x \in E \right\}$$

Prove that :

$$\sup \frac{1}{E} = \frac{1}{\inf E}$$

Proof :

Let $\inf E = \beta$. Then from the definition of the infimum we get:

1) $0 < \beta \leq x \ \forall x \in E$

and 2) $\forall \eta > 0 \ \exists$ a number $x_0 \in E$ such that: $\beta \leq x_0 < \beta$.

Consequently we get;

1)' $\frac{1}{x} \leq \frac{1}{\beta} \quad \forall \frac{1}{x} \in \frac{1}{E}$

and 2)' $\forall \eta > 0 \quad \exists$ a number $\frac{1}{x_0} \in \frac{1}{E}$ such that :

$$\frac{1}{\beta + \eta} < \frac{1}{x_0} \leq \frac{1}{\beta}$$

Now for each $\frac{1}{\beta} \varepsilon > 0$ we can choose $\eta > 0$ to satisfy :

$$\frac{1}{\beta + \eta} = \frac{1}{\beta} \varepsilon$$

This choice can be done if we take

$$\eta = \frac{\beta}{1 - \beta \varepsilon} - \beta$$

Then we get:

1)' $\frac{1}{x} \leq \frac{1}{\beta} \quad \forall \frac{1}{x} \in \frac{1}{E}$

and 2)' $\forall \varepsilon > 0 \quad \exists$ a number $\frac{1}{x_0} \in \frac{1}{E}$ such that :

$$\beta - \varepsilon < \frac{1}{x_0} \leq \frac{1}{\beta}$$

These are the two conditions for the supremum of the subset $\frac{1}{E}$ and hence,

$$\sup \frac{1}{E} = \frac{1}{\beta} = \frac{1}{\inf E}$$

Proposition : Let E and F be bounded subsets of real numbers such that $E \subseteq F$. Then we get;

$$\inf F \leq \inf E \leq \sup E \leq \sup F$$

Proof :

Let $\inf F = \beta_F$ then from the definition we get :

$$\beta_F \leq x \qquad \forall x \in F$$

Consequently,

$$\beta_F \leq x \qquad \forall x \in E$$

Hence β_F is a lower bound for the set E. It follows then that

$$\beta_F \leq \inf E$$

Clearly,

$$\inf E \leq x \leq \sup E \qquad \forall x \in E .$$

Finally, let $\sup F = \alpha_F$ then

$$\alpha_F \geq x \qquad \forall x \in F$$

Therefore,

$$\alpha_F \geq x \qquad \forall x \in E$$

Hence, α_F is an upper bound for E and then $\alpha_F \geq \sup E$.

Proposition is completely proved.

Exercises:

1) Let E be a subset of a bounded from above set F of real numbers satisfying the condition :

$$\forall x \in F \ \exists \text{ an element } y \in E \text{ such that } y \geq x.$$

Prove that: $\sup F = \sup E$.

2) Let E be a subset of a bounded from below set F of real numbers satisfying the condition :

$$\forall x \in F \ \exists \text{ an element } y \in E \text{ such that } y \leq x.$$

Prove that : $\inf F = \inf E$

3) Let E be a bounded subset of real numbers. for any real number c let us define the set $E \circledast \{c\}$ as the translation of E by the constant c:

$$E \circledast \{c\} =: \left\{ x + c \ : \ x \in E \right\}$$

Prove that:

i) $\sup E \circledast \{c\} = c + \sup E$

ii) $\inf E \circledast \{c\} = c + \inf E$

4) Let E be a bounded subset of real numbers. For any real number $\mu \in \mathbb{R}$ let us define the set $\mu \odot E$ as the multiplication of the set E by the constant μ:

$$\mu \odot E =: \left\{ \mu x \ : x \in E \right\}$$

For positive numbers μ prove that:

i) $\sup \mu \odot E = \mu \sup E$

ii) $\inf \mu \odot E = \mu \inf E$

And for negative numbers μ prove that:

i) $\sup \mu \odot E = \mu \inf E$

ii) $\inf \mu \odot E = \mu \sup E$

Countable and uncountable sets

Definition: Two sets E and F are said to have the same cardinal numbers if there exists a one to one correspondance between them.

Remark :

Having the same cardinal number defines an equivalence relation between sets.

It is written $E \sim F$ *to indicate that the two sets E and F are equivalent in the sense that they have the same cardinal numbers.*

Definition: A subset E is said to be infinite if there exists a one to one correspondence between the set E and one of its proper subsets.

Example : The set $\mathbb{Z}$ of integers is an infinite set since one can define a one to one correspondance between the set $\mathbb{Z}$ and the set of natural numbers $\mathbb{N}$ which is one of its proper subsets.

In fact, the one to one correspondence f can be defined as follows :

$$f(n) = \begin{cases} 2n & \text{if n is positive} \\ -2n + 1 & \text{if n is negative} \end{cases} \qquad \forall n \in \mathbb{Z}$$

Clearly the inverse mapping f^{-1} for f is defined by:

$$f^{-1}(n) = \begin{cases} \frac{n}{2} & \text{if n is even} \\ -\frac{n-1}{2} & \text{if n is odd} \end{cases} \qquad \forall n \in \mathbb{N}$$

Hence the set $\mathbb{Z}$ *is infinite.*

Definition: An infinite set E is called countable if and only if there exists a one to one correspondence between this set E and the set of natural numbers $\mathbb{N}$.

Remark :

The last example shows that the set $\mathbb{Z}$ is a countable set because it can be put in one to one correspondence with the set $\mathbb{N}$ of natural numbers.

Example: There exists a one to one correspondence between the closed intervals $[0,1]$ and $[0,2]$ as follows :

$$f : [0,1] \rightleftarrows [0,2] : f^{-1}$$

$$f(x) = 2x \qquad \forall\, x \in [0,1]$$

and

$$f^{-1}(y) = \frac{y}{2} \qquad \forall\, y \in [0,2]$$

Hence the two sets $[0, 1]$ and $[0, 2]$ are equivalent. Moreover the closed interval $[0,2]$ (and hence $[0, 1]$) is infinite since it is equivalent to one of its proper subsets.

Proposition 1: The union of two countable sets is also countable.

Proof :

Let A_1, A_2 be two countable sets . We will show that their union

$$A_1 \cup A_2$$

is also countable. Clearly we can write each of the sets in the form of a sequence as :

$$A_1 = \{a_{11}, a_{12}, a_{13}, a_{14}, a_{15}, a_{16}, \dots\}$$

$$A_2 = \{a_{21}, a_{22}, a_{23}, a_{24}, a_{25}, a_{26}, \dots\}$$

Hence we can write their union in the form of a sequence as follows:

$$A_1 \cup A_2 = \{a_{11}, a_{21}, a_{12}, a_{22}, a_{13}, a_{23}, a_{14}, a_{24}, \dots\}$$

Hence the proposition

Proposition 2: Countable union of countable sets is also countable.

Proof:

Let $\{X_n\}$ be a countable collection of countable sets (i.e. each of the sets X_n is countable) then we show that the union

$$\bigcup_{n \in \mathbb{N}} X_n \quad \text{is a countable set.}$$

To prove this we only write this union in the form of the array.

$$X_1 = \{x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}, \dots\}$$

$$X_2 = \{x_{21}, x_{22}, x_{23}, x_{24}, x_{25}, x_{26}, \dots\}$$

$$X_3 = \{x_{31}, x_{32}, x_{33}, x_{34}, x_{35}, x_{36}, \dots\}$$

$$X_4 = \{x_{41}, x_{42}, x_{43}, x_{44}, x_{45}, x_{46}, \dots\}$$

... ...

... ...

... ...

The elements of the union can be written in the form of a sequence as follows:

$$\bigcup_{n\in\mathbb{N}} X_n = \{ x_{11};\ x_{21},\ x_{12};\ x_{31},\ x_{22},\ x_{13};\ x_{41},\ x_{32},\ x_{23},\ x_{14};\ ;\ x_{51},\ x_{42},\ x_{33},\ x_{24},\ x_{15};\ \ldots \}$$

We only ommit any repeated element. In this way we have written the elements of the union as collections (separated by semicolons) of elements x_{nm} having the same value of addition of the indices. The element x_{nm} belongs to the group number $(n + m - 1)$.

Proposition is proved

Proposition 3:

Cartesian product of two countable sets is also countable.

proof:

Let X and Y be two countable sets. We will prove that $X \times Y$ is also countable. Since the two sets are countable then each of them is written in a form of a sequence as follows:

$$X = \{ x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6,\ \ldots \}$$

and

$$Y = \{ y_1,\ y_2,\ y_3,\ y_4,\ y_5,\ y_6,\ \ldots \}$$

Then

$$X \times Y = \bigcup_{n\in\mathbb{N}} \{ (x_n,\ y_m) :\ m = 1,2,3,\ \ldots \}$$

is a countable union of countable sets and hence is also countable.

In fact the cartesian product of these two sets can be written in the form:

$$X \times Y = \{ (x_1, y_1), (x_1, y_2), (x_1, y_3), (x_1, y_4), (x_1, y_5), \ldots,$$
$$(x_2, y_1), (x_2, y_2), (x_2, y_3), (x_2, y_4), (x_2, y_5), \ldots,$$
$$(x_3, y_1), (x_3, y_2), (x_3, y_3), (x_3, y_4), (x_3, y_5), \ldots,$$
$$(x_4, y_1), (x_4, y_2), (x_4, y_3), (x_4, y_4), (x_4, y_5), \ldots,$$
$$(x_5, y_1), (x_5, y_2), (x_5, y_3), (x_5, y_4), (x_5, y_5), \ldots,$$
$$\ldots\ \ldots\ \ldots\ \ldots\ \ldots$$
$$\ldots\ \ldots\ \ldots\ \ldots\ \ldots \}$$

Proposition 4:

The set S of all sequences with coordinates either zoros or ones is uncountable.

proof:

Let us consider that the set S is countable. In this case we can write the elements of this set in the form of a sequence as follows:

$$S = \{ s_1,\ s_2,\ s_3,\ s_4,\ s_5,\ \ldots \}$$

where each element s_i is a sequence with terms zeros or ones only like the sequence

$$s_i = \{ 1, 0, 1, 0, 0, 0, 1, 1, 0, 1, 0, 1, \ldots \}$$

Let us write all the elements of the set S in thew form:

$$s_1 = \{ 1, 0, 1, 0, 0, 0, 1, 1, 0, 1, 0, 1, \ldots \}$$

$$s_2 = \{ 1, 1, 1, 0, 0, 1, \text{[illegible]}, 1, 0, 0, 1, 1, \ldots \}$$

EXAMPLE : Let E be a bounded subset of real numbers. If we define a subset $\neg E$ (only temporarily) by:

$$\neg E = \{ -x : x \in E \},$$

then prove that $\neg E$ is also bounded and for which we have:

1) $\sup \neg E = -\inf E$ 2) $\inf \neg E = -\sup E$.

Proof of 1)

Let $\inf E = \beta_0$ then from the definition we get:

i) $\beta_0 \le x \quad \forall x \in E$ and

16

$$s_3 = \{0, 1, 0, 0, 1, 0, 1, 1, 0, 1, 0, 0, \dots\}$$

$$s_4 = \{0, 0, 1, 1, 0, 1, 1, 0, 0, 1, 1, 1, \dots\}$$

$$s_5 = \{1, 0, 0, 0, 0, 0, 1, 1, 0, 0, 1, 0, \dots\}$$

$$s_6 = \{ \quad \dots \quad \dots \dots \dots \dots \quad \}$$

... ...

We construct a new sequence s_0 with coordinates zeros and ones as follows: s_0 differs from the sequence s_i in its ith coordinate like the case

$$s_0 = \{0, 0, 1, 0, 1, \dots\}$$

Then,

$$s_0 \ne s_i \qquad \forall\, i = 1,2,3,4 \dots.$$

Hence,

$$s_0 \notin S$$

while it is also a sequence of the same type as those of the set S. This gives a contradiction. Hence the set S is un countable.

Proposition is then proved

Proposition 1.6:

The set S^* of all finite sequences with coordinates either

Clearly, the set S^*_n is a finite set (in fact it has 2^n elements). Moreover, since we have:

$$S^* = \bigcup_{n \in N} S^*_n$$

then S^*_n which is a countable union of finite sets is also countable.

Proposition is then proved

Corollary:

The set $S \setminus S^*$ is uncountable. In fact, if the set $S \setminus S^*$ was countable then the set

$$S = (S \setminus S^*) \cup S^*$$

should have been countable which is false.

Proposition 1.7:

The closed interval $[0,1]$ is uncountable.

Proof:

We make a one to one correspondence between the set $[0,1]$ and the set $S \setminus S^*$. As a matter of fact each number $x \in [0,1]$ can be written in the form

$$x = \sum_{i=1}^{\infty} \frac{s_i}{2^i} = \frac{s_1}{2} + \frac{s_2}{4} + \frac{s_3}{8} + \frac{s_4}{16} + \dots$$

with s_i's ones or zeros with infinite number of ones.

We put the number x in a correspondence with the sequence

$$x \longleftrightarrow s = (s_1, s_2, s_3, \dots s_n, \dots) \in S \setminus S^*$$

Proposition is then proved

Some Logical Rules

Truth tables:

Truth table for the conjunction (and)

P	Q	P and Q
true	true	true
true	false	false
false	false	false
false	true	false

Truth table for the conjunction (or)

P	Q	P or Q
true	true	true
true	false	true
false	false	false
false	true	true

Truth table for implication

P	Q	$P \Longrightarrow Q$
true	true	true
true	false	false
false	false	true
false	true	true

From the last truth table for implication one can see that:

$\left\{ \text{FALSE} \Longrightarrow \text{ANY STATEMENT} \right\}$ IS ALWAYS TRUE

If we have three logical statements **P**, **Q**, and **R** such that :

$$P \Longrightarrow Q \Longrightarrow R$$

Then the weakest condition **R** is a necessary condition for the statement **Q** and hence a necessary condition for the statement **P**.

Also the statement **Q** is a necessary condition for the statement **P**.

On the other hand the strongest statement **P** is a sufficient condition for both statements **Q** and **R**. The statement **Q** is a sufficient one for **R** and is a necessary condition for the statement **P**.

FIRST CHAPTER
ABSTRACT METRIC SPACES

A metric space is a set on which there is defined a distance. In fact studying abstract metric spaces is studying different consequences of defining a distance on a nonempty set. What is a distance ?. We see the definition of a distance in the following:

Definition 1.1:

Let X be a non-empty set and d be a function from X × X into the set of real numbers ℝ;

$$d : X \times X \longrightarrow \mathbb{R}$$
$$(x, y) \longrightarrow d(x,y) \in \mathbb{R}$$

The pair (X, d) is called a metric space if the flowing conditions are satisfied :

d_1) $d(x, y) \geq 0$ with equality iff $x = y$.

d_2) $d(x, y) = d(x, y) \quad \forall \; x,y \in X$

d_3) (called the triangle inequality):

$d(x, y) \leq d(x, z) + d(z, y) \quad \forall \; x,y,z \in X$

The function d is called the distance function between the elements of the set X.

d_1) means that the distance cannot be negative. If the distance between two points is zero then thetwo poionts coincide.

d_2) says that the distance is independent of the starting point ant the end point.

d_3) Is the generalization of the famous fact about triangles. That the distace between two vertices (the length of an arc) cannot exceed the addition of the lenghes of the other two ones

Examples of metric spaces

Example 1: The set of real numbers ℝ with the distance

$$d(x, y) = |x - y|$$

form a metric space. In fact,

d_1) $d(x, y) = |x - y| \geq 0$ and $|x - y| = 0$ iff $x = y$.

d_2) $d(x, y) = |x - y| = |y - x| = d(y,x)$.

d_3) $d(x, y) = |x - y| = |x - z + z - y| \leq$

$\leq |x - z| + |z - y| =$

$= d(x, z) + d(z, y)$.

Example 2:

Let $\mathbb{R}^n$ denotes the set of all n-tuples of real numbers. For $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ we define the distance function as follows:

$$d(x, y) = \sqrt{\sum_{i=1}^{n} |y_i - x_i|^2}$$

Then $\mathbb{R}^n$ with this distance form a metric space.

Proof :

d_1) Since $|y_i - x_i|^2 \geq 0$ then $d(x, y) \geq 0$. If $d(x, y) = 0$ then:

$|y_i - x_i| = 0 \quad \forall i$. Then $x_i = y_i \quad \forall i$. Hence $x = y$.

d_2) $d(x, y) = \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2} = \sqrt{\sum_{i=1}^{n} |y_i - x_i|^2} = d(y, x).$

d_3) $d(x, y) = \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2} = \sqrt{\sum_{i=1}^{n} |x_i - z_i + z_i - y_i|^2} \leq$

$$\leq \sqrt{\sum_{i=1}^{n} |x_i - z_i|^2} + \sqrt{\sum_{i=1}^{n} |z_i - y_i|^2} =$$

$$= d(x, z) + d(z, y).$$

Example 3 : <u>The space</u> $C_{[0,1]}$

Let $C_{[0,1]}$ be the set of all continuous functions defined on the closed interval [0,1] . On $C_{[0,1]}$ we define a metric by putting

$$d(x, y) = \max |x(t) - y(t)|$$

$$\forall x(t), y(t) \in C_{[0,1]}$$

The space $C_{[0,1]}$ forms a metric space .

Proof :

d1) $|x(t) - y(t)| \geq 0$ then $d(x, y) = \max |x(t) - y(t)| \geq 0.$

$\max |x(t) - y(t)| = 0 \Longleftrightarrow |x(t) - y(t)| = 0 \ \forall t \Longleftrightarrow$

$\Longleftrightarrow = y(t) \ \forall t \in [0,1] \Longleftrightarrow x = y$

d2) $d(x, y) = \max_t |x(t) - y(t)| = \max_t |y(t) - x(t)| = d(y, x)$

d3) $|x(t) - y(t)| = |x(t) - z(t) + z(t) - y(t)| \leq |x(t) - z(t)| + |z(t) - y(t)| \leq \max_t |x(t) - z(t)| + \max_t |z(t) - y(t)| = d(x, z) + d(z, y)$

Since this relation is true for every t , then proceeding to the maximum we get :

$$d(x, y) = \max_t |x(t) - y(t)| \leq d(x, z) + d(z, y)$$

Topological Notions In Metric Spaces

Definition 1.2:

Let (X,d) be a metric space. Let x be any point in X. By an open neighbourhood $N_r(x)$ of the point x with radius $r>0$ we denote the set:

$$N_r(x) = \{ y : d(x,y) < r \}$$

(somtimes it is called an open sphere with center x and raduis r).

The set $\overline{N}_r(x) = \{ y : d(x,y) \leq r \}$ is called a closed sphere (or closed neighbourhood) with center a and raduis r.

Remark:

It seems from the definition that the point x_0 has infinite number of neighbourhoods as much as $r > 0$. However this statement is not always true. There are some cases in which the point do not have more than two neighbourhoods (see the discrete metric).

Definition 1.3:

A point x_0 is called an interior point for a subset E in a metric space (X,d) if and only if there exists a neighbourhood $N_r(x_0)$ such that

$$N_r(x_0) \subseteq E$$

Remark:

From the definition it is clear that any interior point of a subset must belong to it. So it is not a condition to require that a subset E contain its interior points. This is trivially *satisfied. All subsets contain all its interior points.*

Definition 1.4: (open sets)

A subset E is called an open subset in the metric space (X,d) if all its points are interior points.■

Equivalent forms of the definition:

1) A subset E is called an open subset in the metric space (X,d) if and only if:

$$x \in E \text{ implies that } x \text{ is an interior point to } E$$

2) A subset E is open if and only if :

$$\forall\, x \in E \;\; \exists \;\; r>0 : N_r(x) \subseteq E$$

3) An open subset is a subset which do not contain any point except its interior points.

Example:

The open interval

$$]a,b[\; = \{\, x \in \mathbb{R} : a < x < b \,\}$$

is an open subset of the straight line $\mathbb{R}$. In fact, for each point $x \in\,]a,b[$ we can choose a neighbourhood $N_r(x)$ such that

$$0 < r \le \min \{\, (b - x),(x - a) \,\}.$$

In this case we have

$$N_r(x) = \,]\, x - r,\, x + r\, [\; \subseteq \,]a,b[$$

Definition 1.5:

A point x is called a limit point for a subset E in a metric space (X,d) if and only if each neighbourhood $N_r(x)$ of the point x contains a point y of the subset E other than x itself.■

i.e.

A point x is called a limit point for a subset E in a metric space (X,d) if and only if

$$\forall\, N_r(x) : \quad N_r(x) \setminus \{x\} \cap E \neq \phi$$

Remarks:

1) A limit point of a subset need not belong to the subset. The number 1 for example is a limit point of the set]0,1[.

2) A point x is not a limit point of the set E iff

$$\exists \;\; N_r(x) \;\; : \;\; N_r(x) \setminus \{x\} \cap E = \phi$$

3) A point x which does not belong to the subset E is not a limit point of E iff

$$\exists \;\; N_r(x) \;\; : \;\; N_r(x) \cap E = \phi$$

Definition 1.6:

A subset E in a metric space (x,d) is called closed if and only if it contains all its limit points.

Remark:

1) The set]0,1] is not closed in $\mathbb{R}$ because 0 is one of its limit points while $0 \notin\,]0,1]$.

2)In any metric space any singelton {x} is closed (a singelton is a set containing exactly one point). In fact any singelton has no limit points. Its set of limit points is empty.

Definition 1.7:

the interior E^o of a set E is the set of all its interior points.■

Clearly a set E is open iff $E = E^o$.

Definition 1.8:

A set A in a metric space (X,d) is called bounded iff there exists a neighbourhood $N_r(x)$ for some point x such that $A \subseteq N_r(x)$.■

We can easily see that a set A is bounded iff

$$\exists\, r > 0 \text{ such that: } d(x, y) \le r \quad \forall\, x, y \in A.$$

This means that the set A is bounded iff

$$\sup_{x,y \in A} d(x,y) \text{ exists}$$

Definition 1.9:

A point x is called an isolated point of a set E iff $x \in E$ and x is not a limit point of the set E.■

Definition 1.10:

A set E is called perfect iff it is closed and has no isolated points.■

Examples:(in the real line $\mathbb{R}$)

1) The set

$$E = [0,1] \cup \{2\}$$

is a closed set which is not perfect because it contains the isolated point 2.

2) The closed interval $[a,b]$ is a perfect set.

Definition 1.11: (closure of a subset)

The union $\bar{M} = M \cup M'$ of the set M and the set M' of all

limit points of M is called the closure of M.■

In other words,

$$\bar{M} = \left\{ x : \forall\, r > 0,\ N_r(x) \cap M \ne \phi \right\}.$$

Proposition 1.1:

For any subsets L and M of a metric space we have:

i) $M \subseteq \bar{M}$

ii) If $M \subseteq L$ then $\bar{M} \subseteq \bar{L}$

iii) $\overline{M \cup L} = \bar{M} \cup \bar{L}$

iv) $\overline{\cap M_i} \subseteq \cap \bar{M}_i$

v) $\bar{\phi} = \phi$, $\bar{X} = X$

vi) $\overline{(\bar{M})} = \bar{M}$

In fact;

i) $x \in M \Longrightarrow \forall\ \varepsilon > 0,\ x \in N(x,\varepsilon) \cap M \Longrightarrow$

$\Longrightarrow \forall\, \varepsilon > 0,\quad N(x,\varepsilon) \cap M \ne \phi \Longrightarrow x \in \bar{M}.$

ii) $x \in \bar{M} \Longrightarrow \forall\ \varepsilon > 0,\ x \in N(x,\varepsilon) \cap M \ne \phi \Longrightarrow$

$\Longrightarrow \forall\ \varepsilon > 0,\ N(x,\varepsilon) \cap L \supseteq$

$\supseteq N(x,\varepsilon) \cap M \ne 0 \Longrightarrow$

$\Longrightarrow \forall\ \varepsilon > 0,\ N(x,\varepsilon) \cap L \ne \phi \Longrightarrow x \in \bar{L}.$

iii) since $M \subseteq M \cup L$ and $L \subseteq M \cup L$

Then $\bar{M} \subseteq \overline{M \cup L}$ and $\bar{L} \subseteq \overline{M \cup L}$

Consequently, $\bar{M} \cup \bar{L} \subseteq \overline{M \cup L}$

On the other hand, $x \notin \bar{M} \cup \bar{L}$ implies that $x \notin \bar{M}$ and $x \notin \bar{L}$

Hence there exist two neighbourhoods $N(x,\varepsilon_1)$ and $N(x,\varepsilon_2)$ such that

$$N(x,\varepsilon_1) \cap M = \phi \quad \text{and} \quad N(x,\varepsilon_2) \cap L = \phi$$

Taking $\varepsilon = \min(\varepsilon_1, \varepsilon_2) > 0$, we get

$$N(x,\varepsilon) \cap M = \phi \ , \ N(x,\varepsilon) \cap L = \phi$$

Hence,

$$N(x,\varepsilon) \cap (M \cup L) = \phi$$

Thus,

$$x \notin \overline{M \cup L}.$$

Hence,

$$\overline{M \cup L} \subseteq \bar{M} \cup \bar{L}.$$

iv) $x \in \overline{\cap_{i \in I} M_i} \Longleftrightarrow \forall \varepsilon > 0, N(x,\varepsilon) \cap (\cap_i M_i) \neq \phi$

$$\Longleftrightarrow \forall \varepsilon > 0, \ \cap_i (N(x,\varepsilon) \cap M_i) \neq \phi \Longleftrightarrow$$

$$\Longleftrightarrow \forall \varepsilon > 0, \ \forall_{i \in I} (N(x,\varepsilon) \cap M_i \neq \phi) \Longleftrightarrow$$

$$\Longleftrightarrow \forall i \in I \ (\forall \varepsilon > 0, N(x,\varepsilon) \cap M_i \neq \phi) \Longrightarrow$$

$$\Longrightarrow \forall i \in I \ \ x \in \bar{M}_i \Longrightarrow x \in \cap \bar{M}_i.$$

v) Is trivial

vi) It is easy from i) to see that $\bar{M} \subset (\overline{M})$

On the other hand we have

$x \in (\overline{\bar{M}}) \Longrightarrow \forall \varepsilon > 0 \ \exists y \ : \ y \in N(x,\frac{\varepsilon}{2}) \cap \bar{M} \Longrightarrow$

$$\Longrightarrow \forall \varepsilon > 0 \ \exists y \ : d(x,y) < \frac{\varepsilon}{2}, \ y \in \bar{M} \Longrightarrow$$

$$\Longrightarrow \forall \varepsilon > 0 \ \exists y,z : d(x,y) < \frac{\varepsilon}{2}, \ z \in N(y,\frac{\varepsilon}{2}) \cap M \Longrightarrow$$

$$\Longrightarrow \forall \varepsilon > 0 \ \exists y,z : d(x,y) < \frac{\varepsilon}{2}, \ d(y,z) < \frac{\varepsilon}{2}, \ z \in M \Longrightarrow$$

$$\Longrightarrow \forall \varepsilon > 0 \ \ N(x,\varepsilon) \cap M \neq 0 \Longrightarrow x \in \bar{M}.$$

Remarks:

1) To prove that a set F is closed it is enough to prove that all points which do not belong to the set are not limit point for

it. So all its limit points are contained in the set F.

***2) To prove that a set F in a metric space (X,d) is closed it** is enough to prove that each convergent sequence of its elements has its limit also in F.*

Proposition 1.2:

A set $O \subseteq X$ is open iff its complement O^c is closed.

Proof:

Let O be open set. We will prove that O^c is closed. We only prove that all points which do not belong to O^c are not limit points of O^c.

Let $x \notin O^c$. Then $x \in O$. Since the set O is an open set the the point x is an interior point for O. Hence there exists a neighbourhood $N_\varepsilon(x)$:

$$N_\varepsilon(x) \subseteq O$$

Hence,

$$N_\varepsilon(x) \cap O^c = \phi$$

A priori,

$$N_\varepsilon(x) - \{ x \} \cap o^c = \phi$$

Thus x in not a limit point for the set o^c.

Conversely Let O^c be closed and we prove that O is an open set. Each point x which does not belong to o^c is not a limit point for it.

If $x \in O$, then $x \notin O^c$. Therefore x in not a limit point for the set O^c. Thus there exists a neighbourhood $N_\varepsilon(x)$ such that

$$N_\varepsilon(x) - \{ x \} \cap O^c = \phi$$

Since $x \in O$ then adding it to the left hand side of the last relation we get:

$$N_\varepsilon(x) \cap O^c = \phi$$

Thus,

$$\exists\, N_\varepsilon(x) : x \in N_\varepsilon(x) \subseteq O$$

Therefore x is an interior point of the set O. All points of the set O are interior points and the set O is then an open set.

Proposition is proved

Another proof

Let O be open set. We will prove that O^c is closed. We have:

$$x \in \overline{O^c} \implies \forall\, \varepsilon > 0 \quad N(x, \varepsilon) \cap O^c \neq \phi \implies$$

$$\implies \forall\, \varepsilon > 0,\ N(x,\varepsilon) \text{ is not contained in } O \implies$$

$$\implies x \notin O \implies x \in O^c.$$

Hence,

$$\overline{O^c} \subset O^c,$$

Since from the definition we have $O^c \subset \overline{O^c}$. Then $\overline{O^c} = O^c$, therefore O^c is closed.

Conversely, let O^c be closed hence,

$$x \in O \implies x \notin O^c \implies \exists\, \varepsilon > 0 : N(x, \varepsilon) \cap O^c = \phi \implies$$

$$\implies \exists\, \varepsilon > 0 \text{ such that } N(x,\varepsilon) \subseteq O.$$

Hence,

$$x \in O \implies \exists\, \varepsilon > 0,\ N(x,\varepsilon) \subseteq O \qquad (*)$$

Thus the set O is an open set.

Conversely, since (*) is equivalent to

$$\forall\, \varepsilon > 0,\ N(x,\varepsilon) \text{ is not contained in } O \implies x \notin O$$

Remarks:

1) a) A set F is closed iff

$$F = \left\{ x : N(x,\varepsilon) \cap F \neq 0 \quad \forall\, \varepsilon > 0 \right\}.$$

2) **A subset F of a metric space X is closed iff $\overline{F} = F$.**

Proposition 1.3:

The class τ of all open sets of a metric space X, satisfies the topological axioms:

1) The collection τ is closed under taking arbitrary unions i.e.

$$O_i \in \tau \ \forall\, i \in I \implies \bigcup_{i \in I} O_i \in \tau$$

2) The collection τ is closed under taking finite intersections, i.e.

$$O_1 \in \tau \text{ and } O_2 \in \tau \implies O_1 \cap O_2 \in \tau$$

3) The collection τ contains the whole space and the empty set i.e.

$$\phi \in \tau \text{ and } X \in \tau$$

Proof:

Proof of 1): Let $\{ O_i \}_{i \in I}$ be a collection of open sets. We prove that their union $\bigcup_{i \in I} O_i$ is an open set. To do this, let,

$$x \in \bigcup_{i \in I} O_i$$

be an arbitrary point. There exists a set O_{i_0} such that:

$$x \in O_{i_0} \subseteq \bigcup_{i \in I} O_i$$

Since the set O_{i_0} is an open set then the point x is one of its interior points. Hence there exists a neighbourhood $N(x,\varepsilon)$ such that

$$x \in N(x,\varepsilon) \subseteq O_{i_0} \subseteq \bigcup_{i \in I} O_i$$

Hence the point x is an interior point of the *union* $\bigcup_{i \in I} O_i$.

Proof of 2):

Let O_1, O_2 be two open sets. We show that their intersection $O_1 \cap O_2$ is an open set. Let

$$x \in O_1 \cap O_2.$$

Then,

$$x \in O_1 \quad \text{and} \quad x \in O_2$$

Since the sets O_1 and O_2 are open sets then the point x is an interior point to both of them. Therefore there exists two neighbourhoods $N(x,\varepsilon_1)$ and $N(x,\varepsilon_2)$ such that:

$$x \in N(x,\varepsilon_1) \subseteq O_1 \quad \text{and} \quad x \in N(x,\varepsilon_2) \subseteq O_2.$$

Taking $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$ we get:

$$N(x,\varepsilon) = N(x,\varepsilon_1) \cap N(x,\varepsilon_2)$$

we get,

$$x \in N(x,\varepsilon) \subseteq O_1 \cap O_2.$$

This shows that the point x in an interior point to the set

$$O_1 \cap O_2.$$

Therefore the set $O_1 \cap O_2$ is an open set.

Proof of 3):

To show that the whole space X is an open set we must show that any point $x \in X$ is an is an interior point to X i.e.That the space X contains at least one neighbourhood for the point x. In fact the space X contains all neighbourhoods of the point x.

To see that the empty set ϕ is an open set we notice that a set fails to be open if there exists a point in this set which is not an interior point. This does not happen for the empty set ϕ. So ϕ is open.

Proposition is completely proved

This shows that each metric space (x,d) is a topological space.

Remarks:

1) In fact from any false statement any implication gives a true statement:

$$\left\{ \text{FALSE} \Longrightarrow \text{ANY STATEMENT} \right\} \text{ IS ALWAYS TRUE}$$

The empty set ϕ is open iff

$$x \in \phi \Longrightarrow x \text{ is an interior point to } \phi$$

This implication is always true because the statement ($x \in \phi$) is always false.

2) The last argumentation can also be mentioned to prove that the empty set ϕ is contained in any set E. In fact to show that $\phi \subseteq E$ we only show that the following statement is true

$$x \in \phi \Longrightarrow x \in E$$

3) A similar argumentation is uset to see that a set having no limit points is a closed set. In fact in order to prove that a set F mis closed we mut have the following implication

$$x \text{ is a limit point of } F \Longrightarrow x \in F$$

This statement is always true if ther set F has no limit points.

Exercises:

1) Prove that the closure $\overline{E}$ of a set E can be defined by:

$$\overline{E} = \cap \left\{ F : F \text{ is closed and } F \supseteq E \right\}$$

2) Prove that the interior E° of a set E can be defined by:

$$E^{\circ} = \cup \left\{ O : O \text{ is open and } O \subseteq E \right\}$$

Exercises 1.1 :

1) Let (X,ρ) , (X,d) be two metric spaces , and let $c > 0$, show that (X, ρ_i) , $i = 1,2,\ldots,4$ are metric spaces where

i) $\rho_1 = c\,\rho$ ii) $\rho_2 = \rho + d$

iii) $\rho_3 = \max(\rho,d)$ iv) $\rho_4 = \min(c,\rho)$

2) Let X be any set and let $c > 0$ prove that (X , d_c) is a metric space where $d_c : X \times X \longrightarrow R$ is defined by

$$d_c(x , y) = \begin{cases} 0 & \text{if } x = y \\ c & \text{if } x \neq y \end{cases}$$

(This space is called the discrete metric space, in the case when $X = R$ and $c = 1$ this space is denoted by R_d)

3) Let $X =]0,\infty[$ and let $\rho : X \times X \longrightarrow R$ be defind by

$$\rho(x, y) = \left| \frac{1}{x} - \frac{1}{y} \right|.$$

Show that (X,ρ) is a metric space.

4) Prove that if (X ,ρ) is a metric space, then for any $M \subset X$, (M ,ρ) is a metric space.

5) Let R^n be the set of all n-tuples of real numbers and let

$$\sigma : R^n \times R^n \longrightarrow R \text{ , and } \tau : R^n \times R^n \longrightarrow R$$

be defind for all vectors

$$x = (x_1, \ldots ,x_n) \text{ , and } y = (y_1 , \ldots , y_n) \text{ by:}$$

$$\sigma(x , y) = \sum_{i=1}^{n} |x_i - y_i| \quad \text{and} \quad \tau(x, y) = \max_{1 \le i \le n} |x_i - y_i| .$$

Prove that (R^n ,σ) , (R^n ,τ) are two metric spaces

6) Draw the open sphere $N_1(\theta)$ in the metric spaces: (R^2, ρ), (R^2, σ) , (R^2, τ) and (R^2, d_c), $c > 0$, where σ, τ are defined as in exercise 5 ,ρ is the usual metric

$$\rho(x ,y) = \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2}$$

Also describe the open spheres $N_r(x_o)$, $r > 0$ for any point x_o in the discrete metric space (X, d_c)

7) Let (X , ρ) be a metric space . Prove that

i) $\forall\, x , y , z \in X \quad |\rho(x ,y) - \rho(y ,z)| < \rho(x ,z)$

ii) if $\{x_n\}$, $\{y_n\}$ are two convergent sequences of elements of X then,

$$\lim_{n \to \infty} \rho(x_n, y_n) = \rho(\lim_{n \to \infty} x_n, \lim_{n \to \infty} y_n)$$

(i.e. the sequence of real numbers $\{\rho(x_n, y_n)\}_{n=1}^{\infty}$ converges to $\rho(x, y)$.

In particular for fixed y,

$$\lim_{n \to \infty} \rho(x_n, y) = \rho(x, y).$$

Hence the set $\{ \rho(x_n, y), n \in N \}$ is bounded.

SECONED CHAPTER
SEQUENCES IN METRIC SPACES

Definition 2.1:

By a sequence $\{ x_n \}$ of elements of a set X it is meant a function :

$$x : \mathbb{N} \longrightarrow X \quad \text{such that :}$$
$$x(n) = x_n \quad \forall \; n \in \mathbb{N}$$

When the set X is the set of real numbers $\mathbb{R}$ we say that $\{ x_n \}$ is a sequence of real numbers. When X is a metric space we say that $\{ x_n \}$ is a sequence of elements of the metric space.

Definition : 2.2 (convergent sequences)

A sequence $\{ x_n \}$ of elements of a metric space (X,d) is said to converge to the element $x_0 \in X$ if and only if for each positive number $\varepsilon > 0$ there exists a natural number $n_0 \in \mathbb{N}$ such that :

$$d(x_n, x_0) < \varepsilon \quad \text{for all} \quad n \ge n_0$$

i.e.

$$n \ge n_0 \quad \text{implies that} \quad d(x_n, x_0)$$

In this case we write :

$$\lim_{n \to \infty} x_n = x_0$$

Remark:

A point x is the limit of the sequence $\{x_n\}$ if and only if for every $\varepsilon > 0$ there exists a natural number n_0 *such that*

$$x_n \in N_\varepsilon(x) \quad \forall\, n \geq n_0.$$

This means that any neighbourhood of the limit x contains all the sequence except a finite number of its terms.

Proposition :

Each convergent sequence in any metric space (X,d) is bounded while not each bounded sequence is convergent.

Proof :

Let $\{ x_n \}$ be a sequence of elements of a metric space (X,d) which converges to an element x_0. Then from the definition of the convergence we get ;

$\forall\ \varepsilon > 0\ \exists\ n_0 \in \mathbb{N}$:

$$d(x_n,x_0) < \varepsilon \quad \text{for all} \quad n \geq n_0$$

Then for the specific value $\varepsilon = 1$ we get $n_0 \in \mathbb{N}$ such that :

$$d(x_n, x_0) < 1 \quad \text{for all} \quad n \geq n_0$$

This shows that the part of the sequence :

$$\left\{ x_{n_0}, x_{n_0}+1, x_{n_0}+2, x_{n_0}+3, x_{n_0}+4 \cdots\cdots \right\}$$

is bounded. So that it remains to prove boundedness for the terms with indices less than n_0 . In fact,

$$d(x_n,x_0) \leq \max \left\{ d(x_i,x_0) : 1 \leq i < n_0 \right\} \quad \text{for all} \quad n < n_0$$

Therefore,

$$d(x_n,x_0) \leq \max \left\{ \max \left\{ d(x_i,x_0) : 1 \leq i < n_0 \right\}, 1 \right\} \quad \text{for all} \quad n \in \mathbb{N}.$$

The proof of the first part is complete

For the second part we only mention as an example the sequence :

$\left\{ (-1)^n \right\}$ which is bounded and not convergent.

Another example of a bounded sequence which is not convergent is given by the sequence:

$$\left\{ \left[(-1)^n, (-1)^n \right] \right\} \text{ in the plane } \mathbb{R}^2$$

Remark :

The last proposition can be stated in other words as follows:

Boundedness of sequences is a necessary condition for their convergence but not a sufficient one.

Proposition :(Uniqueness of the limit):

A sequence $\{ x_n \}$ of elements of a metric space (X,d) can not converge to more than one point.

Proof :

, Let $\{ x_n \}$ be a sequence of elements of a metric space (X,d) which converges to two different limits x and y.Then

$\forall \varepsilon > 0 \quad \exists \; n_1 \in \mathbb{N}$ such that :

$$d(x_n, x) < \varepsilon \qquad \forall \; n \geq n_1$$

and at the same time we have :

$\forall \varepsilon > 0 \quad \exists \; n_2 \in \mathbb{N}$ such that :

$$d(x_n, y) < \varepsilon \qquad \forall \; n \geq n_2$$

Taking $0 < \varepsilon < \frac{d(x,y)}{2}$ and taking $n_0 = \max(n_1 ; n_2)$

we get :

$\forall \varepsilon > 0 \quad \exists \; n_0 \in \mathbb{N}$ such that :

$$d(x_n, x) < \varepsilon \qquad \forall \quad n \geq n_0$$

and at the same time we have :

$\forall \varepsilon > 0 \quad \exists \; n_0 \in \mathbb{N}$ such that :

$$d(x_n, y) < \varepsilon \qquad \forall \quad n \geq n_0$$

Consequently we get :

$$d(x,y) \leq d(x,x_n) + d(x_n,y) < \varepsilon + \varepsilon < d(x,y)$$

This gives a contradiction if $d(x,y) > 0$. Therefore, $d(x,y) = 0$ and we have $x = y$.

Proposition is prooved

Subsequences

Definition 2.3:

Let $x : \mathbb{N} \longrightarrow X$ be a sequence of elements of a set X denoted by $\{ x_n \}$. Let

$$n : \mathbb{N} \longrightarrow \mathbb{N}$$

be an increasing injective function (i.e. strictly

increasing function) which translates k into $n(k) = n_k \in \mathbb{N}$ for each $k \in \mathbb{N}$. By a subsequence $\{ x_{n_k} \}$ of the sequence $\{ x_n \}$ we mean the composition function :

$$x \circ n : \mathbb{N} \xrightarrow{\;n\;} \mathbb{N} \xrightarrow{\;x\;} X$$

Clearly $x \circ n(k) = x_{n_k}$.

Remark :

A subsequence $\{ x_{n_k} \}$ has its terms numbered by k as follows :

$$\left\{ x_{n_1}, x_{n_2}, x_{n_3}, x_{n_4}, \ldots\ldots\ldots \right\}$$

Proposition 4.2 :

A sequence $\{ x_n \}$ is convergent to a limit x_0 if and only if all its subsequences converge to the same limit.

Proof :

Let $\{ x_n \}$ be a convergent sequence which converges to a point x_0 and let $\{ x_{n_k} \}$ be one of its subsequences. We will prove that the subsequence $\{ x_{n_k} \}$ converges to the same limit x_0 . In fact from the definition we get :

$\forall \; \varepsilon > 0 \; \exists \; n_0 \in \mathbb{N}$ such that ;

$$n \geq n_0 \quad \text{implies that} \quad d(x_n, x_0) < \varepsilon$$

Choose a natural number $k_0 \in \mathbb{N}$ such that : $n_{k_0} \geq n_0$ then we get ;

$$k \geq k_0 \;\; \text{implies} \;\; n_k \geq n_{k_0} \geq n_0 \;\; \text{which implies that ;}$$

$$d(x_{n_k}, x_0) < \varepsilon$$

Hence,

$$\forall\ \varepsilon > 0\ \exists\ k_0 \in \mathbb{N} :$$

$$k \geq k_0 \implies d(x_{n_k}, x_0) < \varepsilon$$

Therefore the subsequence $\{ x_{n_k} \}$ converges to the same point x_0.

The proof is then complete if we notice that the sequence itself is a subsequence of itself and so if we require that all the subsequences of certain sequence be convergent then we are,in fact, requiring that the sequence itself is convergent.

Proposition is completely proved

The last proposition is very usefull in proving that certain sequence has no limit i.e. is not convergent.

Example :

In Euclidean space R^2, for the sequence,

$$\{ ((-1)^n,1) \} = \left\{ (-1,1),\ (1,1),\ (-1,1),\ldots, \right\}$$

one can select the following two subsequences:

$$\left\{ (-1,1),\ (-1,1),\ (-1,1),\ldots \right\}$$

and

$$\left\{ (1,1),\ (1,1),\ (1,1),\ldots \right\}$$

which converge to the two different limits (-1,1) and (1,1) respectively. Therefore the original sequence is not convergent. If the original sequence was convergent then all its subsequences

would have converged to the same limit which is not the case.

Cauchy Sequences

Definition 2.4:

A sequence of elements of a metric space (X,d) is called a Cauchy sequence or a fundamental serquence or a (sequence which converges in itself) if it satisfies the following condition :

For every positive number $\varepsilon > 0$ we can get a natural number $n_0 \in \mathbb{N}$ such that :

$$d(x_n,x_m) < \varepsilon \quad \text{for all} \quad n,m \geq n_0$$

i.e.

$$d(x_n,x_m) \longrightarrow 0 \quad \text{as } n,m \longrightarrow \infty$$

Remark:

A sequence of elements of a metric space (X,d) fails to be a Cauchy sequence iff there exists a positive constant ε and two increasing sequences of indices n_k and m_k such that

$$d(x_{n_k}, x_{m_k}) \geq \varepsilon \quad \forall\ k \in N$$

Proposition :

Every convergent sequence is a Cauchy sequence.

Proof:

Let $\{ x_n \}$ be a convergent sequence and let its limit be x_0. We will show that this sequence $\{ x_n \}$ is a Cauchy sequence. From the definition we get :

$\forall\ \varepsilon > 0\ \ \exists\ n_0 \in \mathbb{N}$ such that :

$n \geq n_0$ implies that $d(x_n, x_0)$

Hence,

$m \geq n_0$ implies that $d(x_m, x_0)$

Therefore,

$\forall \varepsilon > 0 \ \exists n_0 \in \mathbb{N}$ such that :

$$d(x_n,x_m) \leq d(x_n,x_0) + d(x_0,x_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \forall \ n,m \geq n_0$$

Definition:

A metric space (X,d) is called complete iff every Cauchy sequence in this space has a limit in it.

Completeness of the real number system

The axiom of completeness of the real number system $\mathbb{R}$ *states that:*

Every Cauchy sequence of real numbers has a limit in the real number system $\mathbb{R}$.

Remarks :

1) The rational number system Q is not complete.

In fact,

The sequence $\left\{ \left(1 + \frac{1}{n} \right)^n \right\}$ is a Cauchy sequence in the ratinonal number system Q but not convergent to a limit in Q . In fact its limit is the number e which is not a rational number.

2) The axiom of copmleteness of the real number system

$\mathbb{R}$ is equivalent to the following axiom:

Each bounded from above subset of real numbers has a least upper bound.

3) In the rational number system Q we can find a subset which is bounded from above in Q but has no least upper bound in the system Q .

In fact, the subset :

$$E = \left\{ x \in Q \ : \ 0 \leq x \ \text{and} \ x^2 < 3 \right\}$$

is bounded in Q but it has no least upper bound in the system Q . its least upper boud is $\sqrt{3} \notin Q$.

Exercise :

Prove that each Cauchy sequence in any metric space (X,d) is bounded.

THIRD CHAPTER

COMPACTNESS IN METRIC SPACES

Definition 3.1:

By an open cover $\{ O_i \}$ of a set K in a metric space (X,d) it is meant a collection of open sets O_i in X whose union $\cup O_i$ contains K. i.e.

$$O_i \text{ is open } \forall\, i \text{ and } K \subseteq \cup\, O_i$$

Definition 3.2:

A set K in a metric space (X,d) is called compact in X iff every open cover $\{ O_i \}$ of K has a finite subcover $\{ O_{i_j} \}_{j=1}^{n}$.

i.e.

if $K \subseteq \cup\, O_i$ (O_i is open $\forall\, i$) then $\exists\ O_{i_1}, O_{i_2}, O_{i_3}, \ldots, O_{i_n}$:

$$K \subseteq O_{i_1} \cup O_{i_2} \cup O_{i_3} \cup \ldots \cup O_{i_n}$$

Remarks:

1)As a direct consequence of the definition we see that any finite set $\{x_1, x_2, x_3, \ldots, x_n\}$ in a metric space is compact. In fact, if $\{x_1, x_2, x_3, \ldots, x_n\} \subseteq \cup\, O_i$ then

$$\exists\ O_{i_1}, O_{i_2}, O_{i_3}, \ldots, O_{i_n} :$$

$$x_j \in O_{i_j} \quad j = 1,2,\ldots,n.$$

Hence,

$$\{x_1, x_2, x_3, \ldots, x_n\} \subseteq O_{i_1} \cup O_{i_2} \cup O_{i_3} \cup \ldots \cup O_{i_n}$$

2) The open interval $]0,1[$ is not compact in the real line $\mathbb{R}$. The open cover

$$\left\{]0, \frac{n}{n+1}[\ , n \in N \right\}$$

for this set, has no finite subcover. In fact,we have,

$$]0,1[\ \subseteq \cup\ ]0, \frac{n}{n+1}[$$

while no finite union of the sets $]0, \frac{n}{n+1}[$ can cover the set $]0,1[$.

3) The real line $\mathbb{R}$ is not compact. To show this we only consider the open cover $\left\{]-n,n[\right\}_{n \in N}$, for $\mathbb{R}$ which has no finite subcover for $\mathbb{R}$.

Theorem 3.1:

A closed subset F of a compact set K is compact.

Proof:

Let $\{ O_i \}$ be an open cover for the closed set F. i.e.

$$F \subseteq \cup\ O_i$$

Hence,

$$K \subseteq F \cup F^c \subseteq \left\{ \cup\ O_i \right\} \cup F^c$$

Then,

$$\left\{ F^c, \{ O_i \} \right\}$$

is an open cover for K. Since the set K is compact then the open cover $\left\{ F^c, \{ O_i \} \right\}$ for K has a finite subcover say,

$$F^c, O_{i_1}, O_{i_2}, O_{i_3}, \ldots, O_{i_n}$$

Hence,

$$F \subseteq K \subseteq F^c \cup O_{i_1} \cup O_{i_2} \cup O_{i_3} \cup \dots \cup O_{i_n}$$

$$F \subseteq F^c \cup O_{i_1} \cup O_{i_2} \cup O_{i_3} \cup \dots \cup O_{i_n}$$

Since F^c cannot contain any point of the set F then

$$F \subseteq O_{i_1} \cup O_{i_2} \cup O_{i_3} \cup \dots \cup O_{i_n}$$

Hence,

$$O_{i_1}, O_{i_2}, O_{i_3}, \dots, O_{i_n}$$

is a finite subcover for the set F selected from the open cover $\{ O_i \}$■

Theorem is completely proved

Theorem 3.2:

Every compact subset K in a metric space (X,d) is closed.

Proof:

Let K be a compact subset ina metric space (X,d). To prove that the set K is closed we only prove that each point $y \notin K$ is not a limit point for the set K. In this case each limit point of the set K (if exists) must belong to the set K itself.

Let x be an arbitrary point of the set K. Clearly $x \notin y$ ($x \in K$ and $y \notin K$). Thus $d(x,y) > 0$. Let us choose two disjoint neighbourhoods

$$N_{\frac{d(x,y)}{3}}(x) \text{ and } N_{\frac{d(x,y)}{3}}(y)$$

for x and y respectively.

Since,

$$K = \bigcup_{x \in K} \{x\} \subseteq \bigcup_{x \in K} N_{\frac{d(x,y)}{3}}(x)$$

Then the collection $\left\{ N_{\frac{d(x,y)}{3}}(x) \right\}_{x \in K}$ is an open cover for the compact set K. Hence it has a finite subcover

$$N_{\frac{d(x_1,y)}{3}}(x_1),\ N_{\frac{d(x_2,y)}{3}}(x_2),\ \dots,\ N_{\frac{d(x_n,y)}{3}}(x_n).$$

i.e.

$$K \subseteq N_{\frac{d(x_1,y)}{3}}(x_1) \cup N_{\frac{d(x_2,y)}{3}}(x_2) \cup \dots \cup N_{\frac{d(x_n,y)}{3}}(x_n)$$

Taking,

$$N(y) = N_{\frac{d(x_1,y)}{3}}(y) \cap N_{\frac{d(x_2,y)}{3}}(y) \cap \dots \cap N_{\frac{d(x_n,y)}{3}}(y)$$

We obtain a neighbourhood N(y) for the point y which does not contain any point ofthe set K. In fact,we have;

$$N(y) \subseteq N_{\frac{d(x_i,y)}{3}}(y) \qquad \forall\ i = 1,2 \dots, n$$

Thus

$$N(y) \cap N_{\frac{d(x_i,y)}{3}}(x_i) = \phi \quad \forall\ i = 1,2 \dots, n$$

Consequently,

$$N(y) \cap \left\{ \bigcup_{i=1}^{n} N_{\frac{d(x_i,y)}{3}}(x_i) \right\} = \phi \quad \forall\ i = 1,2 \dots, n$$

Therefore,

$$N(y) \cap K = \phi.$$

Hence each point $y \notin K$ is not a limit point for the set K. Thus the set K is closed.

Theorem is completely proved

Remark:

The proof of the last theorem needs that for each two points $x \neq y$ we can select two disjoint neighbourhoods $N(x)$, $N(y)$. This property is satisfied in metric spaces. It is called the separation Hausdorff property. All topological spaces satisfying this property are called Hausdorff spaces.

Central Families

Definition 3.3:

A family $\{E_i\}$ of subsets of a given set X is called a central family if anf only if each of its finite subfamilies $\{E_{i_j}\}_{j=1}^{n}$ has a nonemty intersection. i.e. $\bigcap_{j=1}^{n} E_{i_j} \neq \phi$.■

Theorem 3.3:

A metric space (X,d) is compact if and omly if every central family $\{E_i\}$ of closed subsets of X has a nonempty intersection.

Proof:

Let X be a compact metric space and let $\{E_i\}$ be a central family of closed subsets of X.

If $\bigcap_{i \in I} E_i = \phi$ then $\bigcup_{i \in I} E_i^c = X$. Consequently $\{E_i^c\}$ is an open cover for X. Since the space X is compact then this open cover has a finite subcover

$$E_{i_1}^c, E_{i_2}^c, E_{i_3}^c, \ldots, E_{i_n}^c$$

This means that:

$$E_{i_1}^c \cup E_{i_2}^c \cup E_{i_3}^c \cup \ldots \cup E_{i_n}^c = X$$

Consequently from De morgan's low we get:

$$E_{i_1} \cap E_{i_2} \cap E_{i_3} \cap \cdots \cap E_{i_n} = \bigcap_{j=1}^{n} E_{i_j} = \phi.$$

This contradicts the centrality of the family $\{E_i\}$. Hence every central family $\{E_i\}$ of closed subsets of X has a nonempty intersection.

On the other hand let X be a metric space possesing the property that every central family $\{E_i\}$ of closed subsets of X it has a nonempty intersection. We prove that the space X is compact.

Let $\{O_i\}$ be an open cover for the space X. i.e.

$$X \subseteq \bigcup O_i$$

Hence,

$$\bigcap_{i \in I} O_i^c = \phi$$

Therefore, the family $\{O_i^c\}$ of closed subsets of the space X, is not a central family. this means that there exists a finite subfamily $\{O_{i_j}^c\}_{j=1}^{n}$ which has an emty intersection. i.e.

$$\bigcap_{j=1}^{n} O_{i_j}^c = \phi.$$

Therefore,

$$\bigcup_{j=1}^{n} O_{i_j} = X.$$

theorem is completely proved

Corollary:

Intersection $F \cap K$ of a closed set F with a compact set K, is also compact.

Diameters of bounded sets

Definition 3.4:

The diameter of a bounded set E in a metric space (X,d). is defined by:

$$\delta(E) = \sup \{ d(x,y) : x,y \in E \}.\blacksquare$$

Definition 3.5:

Let E be a subset of a metric space (X,d). The distance between the set E and a point $x \in X$ is defined by:

$$d(x, E) = \inf \{ d(x,y) : y \in E \}.\blacksquare$$

Exercises

1)Let E be a subset of a metric space (X,d) and let $x \in X$.

Prove that:

a) x is a limit point of E iff $d(x, E) = 0$.

b) For fixed set E, the function $f(x) = d(x,E)\ \forall x \in X$, is a uniformly continuous function on ths space X.

2)Prove that finite union of compact sets is also compact,but infinite union of compact sets need not be compact.

3)Let $\{ x_n \}$ be a convergent sequence in a metric space (X,d). Prove that the set E consisting of the range of the sequence and the limit of the sequence,i.e.

$$E = R\{ x_n \} \cup \{ \lim x_n \}$$

is a compact set.

Nested intervals and sets

Proposition 3.1:

Let $\left\{ [a_n,b_n] \right\}$ be a sequence of nested closed intervals,i.e.

$$[a_{n+1},b_{n+1}] \subseteq [a_n,b_n] \quad \forall n \in N$$

Then there exists a point $x_0 \in [a_n,b_n]\ \forall n \in N$. In this case

$$\bigcap_{n \in N} [a_n,b_n] \neq \phi$$

Proof:

Fron the inclusin $[a_{n+1},b_{n+1}] \subseteq [a_n,b_n]\ \forall n \in N$ we get:

$$a_1 \leq a_2 \leq a_3 \leq \dots a_n \leq \quad \leq b_n \leq \dots b_3 \leq b_2 \leq b_1$$

This conclusion is due to the relation

$$a_n \leq a_{n+m} \leq b_{n+m} \leq b_m \qquad \forall n,m \in N$$

Taking $x_o = \sup \{a_n : n \in N \}$ we get:

$$a_n \leq x_0 \leq \inf \{ b_n : n \in N \} \leq b_n \quad \forall n \in N .$$

Hence,

$$x_0 \in [a_n,b_n] \quad \forall n \in N.$$

proposition is proved

The following proposition shows that the last fact holds for nested sequences of compact sets. So the last proposition which deals with closed bounded intervals (which are compaqct sets) is a special case of the following one:

Proposition 3.2:

1) Any central family $\{ K_i \}$ of *compact sets* has a nonempty intersection.

2) Any nested sequence $\{ K_n \}$ of compact sets has a *nonempty* intersection.

Proof:

Let $\{ K_i \}_{i \in I}$ be a central family of compact sets which has an empty intersection.i.e.

$$\bigcap_{i \in I} K_i = \phi$$

Let K_{i_0} be one of the sets of this family. Then

$$K_{i_0} \bigcap_{i \neq i_o} K_i = \phi$$

then

$$\left\{ \bigcap_{i \neq i_o} K_i \right\}^c = \bigcup_{i \neq i_o} K_i^c \supseteq X \supseteq K_{i_0}$$

Consequently $\{ K_i^c \}_{i \neq i_o}$ is an opencover for the compact set K_{i_0}.

Therefore, we can select for the set K_{i_0} a finite subcover

$$K_{i_1}^c \cup K_{i_2}^c \cup K_{i_3}^c \cup ... \cup K_{i_n}^c \supseteq K_{i_0}$$

Therefore,

$$K_{i_1} \cap K_{i_2} \cap K_{i_3} \cap \cdots \cap K_{i_n} \subseteq K_{i_0}^c$$

$$K_{i_0} \cap K_{i_1} \cap K_{i_2} \cap K_{i_3} \cap \cdots \cap K_{i_n} = \phi$$

This contradicts the centrality of the family. Thus,

$$\bigcap_{i \in I} K_i \neq \phi$$

For the proof of 2) we only remark that any nested sequence of nonempty sets is a central family. The proof follows from 1).

Propositionis proved

Nested k-cells:

Definition 3.6:

By a k-cell I in the Euclidean space R^k it is meant a set ofthe form:

$$I = \{ x = (x_1, x_2 ... x_k) : x_i \in [a_i, b_i] \quad i = 1,2...k \}.■$$

Remarks:

1) We can write a k-cell I inthe form of cartesian product as:

$$I = \Pi_{i=1}^k [a_i, b_i]$$

2) On the real line 1-cells are bounded closed intervals.

3) On the plane 2-cells are rectangles.

4) Let $I = \Pi_{i=1}^k [a_i, b_i]$, $I' = \Pi_{i=1}^k [a_i', b_i']$ be two cells.

Then, $I \subseteq I'$ iff

$$[a_i, b_i] \subseteq [a_i', b_i'] \quad \forall i = 1,2...k \}.$$

Propositios 3.4:

Any nested sequence of k-cells has a nonempty intersection.

Proof:

Let Let $I_n = \Pi_{i=1}^k [a_i^n, b_i^n]$, $n = 1,2,...$be a nested sequence of k-cells. i.e.

$$I_n = \Pi_{i=1}^{k} [a_i^n, b_i^n] \supseteq I_{n+1} = \Pi_{i=1}^{k} [a_i^{n+1}, b_i^{n+1}]$$

Hence,

$$[a_i^n, b_i^n] \supseteq [a_i^{n+1}, b_i^{n+1}] \quad \forall\, i = 1,2,\dots,k, \forall\, n \in N.$$

Thus

$$\left\{ [a_i^n, b_i^n] \right\}$$

is a nested sequence of closed intervals for each $i = 1,2,\dots,k$.

Let $x_i^0 \in \bigcap_{n \in N} [a_i^n, b_i^n]$ $i=1,2,\dots,k$. Then the point

$$x^o = (x_1^o, x_2^o, x_3^o, \dots, x_k^o) \in \bigcap_{n \in N} \Pi_{i=1}^{k} [a_i^n, b_i^n]$$

Hence,

$$\bigcap_{n \in N} \Pi_{i=1}^{k} [a_i^n, b_i^n] \neq \phi$$

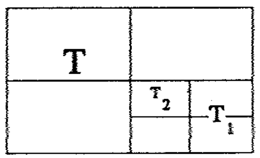
Proposition is then proved

Theorem 3.4:(Heine - Borel *Emile-french mathematician and politician ,Paris 1871-1956*)

Every k-cell is compact.

Proof:

Let $I = \Pi_{i=1}^{k} [a_i, b_i]$ be a k-cell which is not compact. Then there exists an open cover $\{ O_i \}$ for I which has no finite subcover for the cell I.



By dividing each interval $[a_i, b_i]$ into two intervals $[a_i, c_i]$ and $[c_i, b_i]$, where $c_i = \frac{a_i + b_i}{2}$ $i = 1,2,\dots,k$, we obtain 2^k

small intervals.

At least one of these small intervals cannot be covered by a finite number of the sets of the open cover $\{ O_i \}$. Otherwise if each small interval is covered by afinite number of sets of the open cover, then the interval I itsel is covered by a finite number of sets of the open cover.

Let us denote by I_1 one of these small k-cells which cannot be covered by a finite number of sets of the cover.

In the same manner we divide I_1 into 2^k small k-cells. One of these small k-cells which we denote by I_2 cannot be covered by a finite number of sets of the cover.

Continuing in this manner we obtain a sequence $\{ I_n \}$ of nested k-cells such that:

1) $I \supseteq I_1 \supseteq I_2 \supseteq \dots$

2) $d(x,y) < \frac{\delta}{2^n} \quad \forall\, x,y \in I_n \quad n = 1,2,\dots$

where δ is the diameter if the k-cell I defined by:

$$\delta = \sqrt{\sum_{i=1}^{k} |b_i - a_i|^2}$$

Using the principle of nested k-cells we get a point x_0:

$$\bigcap_{n \in N} I_n = \{ x_0 \}$$

Since,

$$x_0 \in I \subseteq \bigcup_{i \in I} O_i$$

then there exists an open set O_{i_0} containing the point x_0. Then x_0 is an interior point for the set O_{i_0}. Consequently there exists a neighbourhood $N_\varepsilon(x_0)$ such that:

$$N_\varepsilon(x_0) \subseteq O_{i_0}$$

Choosing n_0 such that $\frac{\delta}{2^{n_0}} < \varepsilon$ we get :

$$I_{n_0} \subseteq N_\varepsilon(x_0) \subseteq O_{i_0}$$

which shows that I_{n_0} is covered by a finite number of sets of the original cover (in fact only one set). This is a contradiction.
Hence the k-cell I is compact.

theoren is proved

Theorem 3.5:

Every infinite subset T of a compact set K have a limit point in K.

Proof

Let T be an infinite subset of the compact set K and let T has no limit points in K. In this case all points of the set K fail to be a limitpoints for K. Hence for each point $x \in K$ we can select a neighbourhood N(x) such that:

$$N(x) \setminus \{x\} \cap T = \phi$$

Since adding the point x to the left side of the intersection does not affect the intersection in the case $x \notin T$ we get:

$$\forall\, x \in K \setminus T \;\; \exists\; N(x) : \quad N(x) \cap T = \phi$$

and

$$\forall\, x \in T \;\; \exists\; N(x) : \{ N(x) \setminus T \} \cap T = \phi$$

Therefore, each point $x \in K$ has a neighbourhood N(x) which contains at most one point of the subset T.

Taking an open cover $\{ N(x) \}_{x \in K}$ for the compact set K consisting of these neighbourhoods,i.e.

$$K = \bigcup_{x \in K} \{ x \} \subseteq \bigcup_{x \in K} N(x)$$

we get a finite subcover

$$T \subseteq N(x_1) \cup N(x_2) \cup N(x_3) \ldots \cup N(x_n)$$

This cannot happen because the subset T is infinite and the number of neighbourhoods is finite while each neighbourhood contains at most one point of T.

Theorem is completely proved

Corollary:

If there is an infinite subset which has no limit points, then it cannot be contained in a compact set. For example the set $\mathbb{N}$ *of natural numbers has no limit points and at the same time it is contained in the real line* $\mathbb{R}$. *So the real line* $\mathbb{R}$ *is not compact.*

Definition 3.7:

A subset M of a metric space X is called sequentially compact if every infinite subset T of M has a limit point in M. ∎

Remarks:

1) If a set M has no infinite subsets (i.e. if it is finite) then it is sequentially compact.

2) Since a point x_0 *is a limit point of a set T (in a metric space) if and only if it is a limit of a sequence of elements of this set, then :*

A subset M of a metric space X is sequentially compact if every infinite subset T of M contains a convergent sequence whose limit belongs to M. 3) *In metric spaces the notions of compactness and sequential compactness are equivalent.*

Theorem 3.6:

Let K be a subset of the Euclidean space R^k. The folloing statements are equivalent:

1) K is closed and bounded,

2) K is compact,

3) Every infinite subset T of K has at least one limit point in K.

Proof:

1) implies 2):- Let K be a closed bounded set in R^k. Being bunded, the set K is contained in some neighbourhood. We can take a neighbourhood $N_c(\vartheta)$ of the zero vector containing the set K. Hence,

$$d(x,\vartheta) = \sqrt{\sum_{i=1}^{k} |x_i - 0|^2} < c \quad \forall \; x \in K$$

This neighbourhood $N_c(\vartheta)$ is contained in the k-cell I defined by:

$$I = \{ x = (x_1, x_2 \ldots x_k) : x_i \in [-c, c] \quad i = 1,2\ldots k \}.$$

Since every k-cell is compact then the set K is a closed subset of the compact set I. Therefore the set K is also compact.

2) implies 3):- : Theorem (3.4)

3) implies 1) $\equiv$ $\neg$ 1) implies $\neg$ 3):

Let 1) be false i.e. let K be a set whichis not bounded or

not closed. We show that in both cases there is an infinite subset T of the set K which has no limit points.

Case K is not bounbed:

In this case the set K cannot be contained in a neighbourhood of the zero vector ϑ. Hence for each natural number *n* we can find an element $x_n \in K$ such that $x_n \notin N_n(\vartheta)$. i.e.

$\forall \; n \in N \;\; \exists \;\; x_n \in K$ such that

$$d(x_n, \vartheta) \geq n$$

Clearly $T = \{ x_n \}$ is an infinite subset of K which have no limit points. In fact to have a limit point requires having a sequence with different terms from T which converges to the limit point. This is impossible because all infinite sequences of T are not bounded.

Case K is not closed:

In this case K has a limit point x_0 which does not belong to K ($x_0 \notin K$). In this case we can select a *sequence* $\{ x_n \}$, *of* different elements of the set K, which converges to the point x_0.

Choosing the range of the sequence $R\{x_0\}$ as an infinite set T. We see that this set T has no limit point other than x_0. While this limit point does not belong to the set K.

Thus in both cases where K is not bounded or not closed we get an infinite subset $T \subseteq K$ which has no limit point in the set K.

Theorem is completely proved

Cantor Set

The following example for a compact, perfect, uncountable, set which do not contain any interval is due to the German mathematician (of origin russian) George Cantor (1845-1918).

Consider the closed interval $E = [0,1]$. This interval which is closed and bounded in the real number system R is also compact. Let

$$E_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{3}{3}\right],$$

$$E_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{3}{9}\right] \cup \left[\frac{6}{9}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, \frac{9}{9}\right]$$

$$E_3 = \left[\frac{0}{27}, \frac{1}{27}\right] \cup \left[\frac{2}{27}, \frac{3}{27}\right] \cup \left[\frac{6}{27}, \frac{7}{27}\right] \cup \left[\frac{8}{27}, \frac{9}{27}\right] \cup$$

$$\cup \left[\frac{18}{27}, \frac{19}{27}\right] \cup \left[\frac{20}{27}, \frac{21}{27}\right] \cup \left[\frac{24}{27}, \frac{25}{27}\right] \cup \left[\frac{26}{27}, \frac{27}{27}\right],$$

...

$$E_n = \left[0, \frac{1}{3^n}\right] \cup \left[\frac{2}{3^n}, \frac{3}{3^n}\right] \cup \left[\frac{6}{3^n}, \frac{7}{3^n}\right] \cup \left[\frac{8}{3^n}, \frac{9}{3^n}\right] \cup$$

$$\ldots \cup \left[\frac{18}{3^n}, \frac{19}{3^n}\right] \cup \ldots$$

The Cantor set C is defined by:

$$C = \cap \{ E_n : n \in N \}$$

1) Cantor set is compact:

Each set E_n is the union of 2^n small closed bounded intervals and hence each E_n is a closed bounded set. Therefore each set E_n is compact. Thus family $\left\{ E_n : n \in N \right\}$ *forms a nested sequence of* compact sets. Thus its intersection, which is the Cantor set, is

not empty and is a compact set.

2) Cantor set cannot contain any interval:

In fact, Cantor set cannot contain any interval of the form

$$\left[\frac{2^i-1}{3^n}, \frac{2^i}{3^n}\right] \quad i=1,2,3,\ldots \quad ,n \in N$$

But each interval $[\alpha,\beta]$ must contain an interval of the last form.

End of Example

In the following we see a general criterion for compactness in metric spaces. For this we are in need to the following notion.

Definition 3.8:

Let M be a subset of a metric space X. A subset $E \subseteq X$ is called an ε-net for the set $M \subseteq X$ if and only if for every $x \in M$ there exists a point $x_\varepsilon \in E$ such that $d(x, x_\varepsilon) < \varepsilon$.■

Remarks:

1)An ε-net E is for a set M also an ε'-net for every $\varepsilon < \varepsilon'$. for example.if E is $\frac{1}{4}$-net for a set M, then it is a $\frac{1}{3}$-net and a $\frac{1}{2}$-net for M.

2) If E is an ε-net for a subset M and F is a set containing E then F is also an ε-net for M.

3)An everywhere dense set E in a metric spase X is an ε-net for X for every $\varepsilon > 0$.

4)If E is an ε-net for a set M, then

$$\bigcup_{x\in E} N_{\varepsilon}(x) \supseteq M$$

5)A set E is everywhere dense in a metric space X *if and only if*

$$\bigcup_{x\in E} N_{\varepsilon}(x) = X \quad \forall\, \varepsilon > 0$$

Definition 3.9:

A subset E in a metric space is called relatively compact iff its clousure $\overline{E}$ is compact.■

Remark:

In metric spaces a subset E is relatively compact if and only if every sequence in E has a convergent subsequence. If we require that this subsequence converge to a point in E then the set E is compact

Criteria For Compactness

Theorem 3.7: (**Hausdorff** *Felix :German mathematician Breslau 1868-Bonn 1942*)

For a subset M of a metric space to be relatively compact, it is necessary that : for every $\varepsilon > 0$ there exists a finite ε-net for the set M. If the space X is complete, then this condition is also sufficient.

Proof: Necessity :

We assume that M is relatively compact. Let $\varepsilon > 0$, x_1 be an arbitrary point of M. If $d(x, x_1) < \varepsilon$ for all $x \in M$; then $\{x_1\}$ itself is an ε-net for M. If it is not so, then there exists a point $x_2 \in M$ such that $d(x_1, x_2) \geq \varepsilon$.

If either $d(x_1,x) < \varepsilon$ or $d(x_2,x) < \varepsilon$ for every point $x \in M$ then x_1, x_2 is an ε-net for M. If this does not hold, then there exists a point $x_3 \in M$:

$$d(x_1, x_3) \geq \varepsilon, \quad d(x_3, x_2) \geq \varepsilon.$$

We continue this process and obtain points $x_1, x_2, \ldots, x_n$ for which

$$d(x_i, x_j) \geq \varepsilon \quad \text{for } i \neq j$$

There are two possibilities. Either this process stops at the k-th stage i.e. there holds one of the inequalities

$$d(x_i, x) < \varepsilon \quad i = 1, 2, \ldots, k$$

for every $x \in M$ and in this case $\{x_1, x_2, \ldots, x_k\}$ form a finite ε-net of M; or the process continues indefinitely.

In the second possibility we get a sequence $\{x_n\}$ of points of M for which :

$$d(x_i, x_j) \geq \varepsilon \quad i \neq j\ ,\ i, \quad j \in N.$$

Neither this sequence nor any of its subsequences would converge. This contradicts that M is relatively compact.

Thus the process must stop and we get, a finite ε-net for M for every $\varepsilon > 0$.

Sufficiency :-

We assume that the space X is complete, and for every $\varepsilon > 0$ there exists a finite ε-net for M. We choose a sequence of positive numbers $\{ \varepsilon_n \}$, $\lim_{n\to\infty} \varepsilon_n = 0$. For every $\varepsilon_n > 0$ we construct a finite ε_n-net for M,

$$\{ x_1^n, x_2^n, x_3^n, \ldots, x_{k_n}^n \}.$$

We select an arbitrary infinite subset $T \subset M$ and get a sequence in T convergent to a point in X.

Let us construct a sphere $N_{\varepsilon_1}(x_i^1)$ of radius ε_1 around every point $x_1^1, x_2^1, x_3^1, \dots, x_{k_1}^1$ of the ε_1-net. We have

$$T \subseteq K \subseteq \bigcup_{i=1}^{k_1} N_{\varepsilon_1}(x_i^1)$$

Then every point of T is contained in one of the spheres. Since the number of spheres is finite, then there exists an infinite subset of T in one of the spheres. We denote this subset by T_1.

We construct spheres of radii ε_2 around all points

$$x_1^2, x_2^2, x_3^2, \dots, x_{k_2}^2$$

of the ε_2-net. By the same arguments as above, we obtain another infinite set $T_2 \subset T_1$ lying in one of the spheres constructed with radius ε_2. Continuing this process we obtain a sequence of infinite subsets T_i :

$$T_1 \supset T_2 \supset T_3 \supset T_4 \dots \supset T_n \dots$$

where T_n lies in a sphere of radius ε_n.

Hence the distance between arbitrary points of T_n is not greater than $2\varepsilon_n$ (The diameter of the sphere).

Now, we choose points:

$a_1 \in T_1$, $a_2 \in T_2$ ($a_2 \neq a_1$), $a_3 \in T_3$ (other than a_1, a_2) ... and so on. We, then, obtain a sequence

$$T = \{ a_1, a_2, a_3, \dots, a_n, \dots \}$$

This is a Cauchy sequence because $a_n, a_{n+p} \in T_n$ for every

natural number p, so that

$$d(a_{n+p}, a_n) < 2\varepsilon_n \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty$$

since X is complete then T converges to a point $a \in X$.

Which completes the proof.

Definition 3.10:

A subset M of a metric space is called precompact if and only if it has a finite ε-net for every $\varepsilon > 0$.■

Remarks:

1) Every compact set is precompact.

2) In complete metric spaces, relative compactness and precompactness are the same notions.

Exercises

1. Show that for every compact set A of a metric space (X, ρ) the space (A, ρ) is a complete metric space.
2. Let X, Y be two metric spaces and $A \subset X$ is compact set and $f : X \longrightarrow Y$ is continuous function then f(A) is compact.
3. Show that in each compact metric space X there exists a countable set E such that $\overline{E} = X$.

FOURTH CHAPTER

CONTINUOUS FUNCTIONS ON METRIC SPACES

Definition 4.1:

Let (X,d) and (y,ρ) be two metric spaces.A *function*

$$f : (X,d) \longrightarrow (Y,\rho)$$

is called continuous at a point $x_0 \in X$ if for each positive number ε there exists a positive number δ (depending in general on ε and on the point x_0) such that:

$$d(x,x_0) < \delta \Longrightarrow \rho(f(x),f(x_0)) < \varepsilon \quad *$$

Remark 1: In the case of real valued functions of real variables, (i.e. if $X = Y = R$ and $d(x,y)=\rho(x,y)=|x-y|$) the abvoe definition is reduced to the well known definition:

The function $f:R \longrightarrow R$ is continuous at the point $x_0 \in R$ iff

$$\forall \varepsilon > o \; \exists \delta > 0 : | x-x_0| \Longrightarrow | f(x) - f(x_0)| < \varepsilon$$

Remark 2:

The number δ in the above definition is not unique. In fact if we choose any other positive number δ' smaller than this δ will valid for the definition because :

$$d(x,x_0) < \delta' \Longrightarrow d(x,x_0) < \delta \Longrightarrow \rho((fx),f(x_0)) < \varepsilon \; .$$

Remark 3:

The last definition requires satisfaction of the implication * for all ε . So in order to prove that a function f(x) is not continuous at a point x_0 it is enough to show that the implication * is not satisfied for only one value for ε .

Remark 4:

According to remark 1) we can always diminue the number δ through any finite number of steps .

To explain this remark let us have two functions f_1 and f_2 which are continuous at the same point $x_0 \in X$ then for every $\varepsilon > 0$ $\exists \; \delta_1 > 0$ and $\delta_2 > 0$ such that:

$$d(x,x_0) < \delta_1 \Longrightarrow \rho(f_1(x),f_1(x_0)) < \varepsilon$$

and

$$d(x,x_0) < \delta_2 \Longrightarrow \rho(f_2(x),f_2(x_0)) < \varepsilon$$

Hence for the number

$$\delta = \min (\delta_1 ,\delta_2)$$

the two implications will be valid .

Remark 5:

Since the relation $d(x,x_0) < \delta$ is equivalent to :

$$x \in N_\delta (x_o)$$

and the relation $\rho (f(x),f(x_0)) < \varepsilon$ is equivalent to :

$$f(x) \in N_\varepsilon \{ f(x_0) \} ,$$

One can write the definition of continiity in terms of neighbourhouds as follows:

(Equivalent definition) :

A function f(x) from a metric space (X,d) into a *metric* space (Y,ρ) is called continuous at a point x_0 in X if the following condition is satisfied:

for every neighbourhoud $N_\varepsilon\{f(x_0)\}$ there exists a neibourhoud $N_\delta(x_0)$ such that :

$$x \in N_\delta(x_0) \Longrightarrow f(x) \in N_\varepsilon\{f(x_0)\}$$

(Equivalent definition) :

A function f(x) from a metric space (X,d) into a metric space (Y,ρ) is called continuous at a point x_0 in X if the following condition is satisfied:

for every neighbourhoud $N_\varepsilon\{f(x_0)\}$ there exists a neibourhoud $N_\delta(x_0)$ such that :

$$f\left\{N_\delta(x_0)\right\} \subseteq N_\varepsilon\{f(x_0)\}$$

or equivalently:

$$N_\delta(x_0) \subseteq f^{-1}\left\{N_\varepsilon\{f(x_0)\}\right\}$$

Definition 4.2:

A function f from a metric space (X,d) into a metric space (Y,ρ) is called continuous if it is continuous at each point x_0 in its domain X.

Terminology of limits

Definition 4.3:

Let (X,d), (Y,ρ) be two metric spaces, $a \in X$, $L \in Y$ and $f:X \longrightarrow Y$ be a function from X into Y. We say that L is the limit of f(x) as x tends to a and write

$$\lim_{x \to a} f(x) = L$$

iff the following definition is satisfied,

$$\forall \varepsilon > 0 \ \exists \delta > 0 : 0 < d(x,a) < \delta \Longrightarrow \rho(f(x), L) < \varepsilon.$$

Remark:

Let (X,d), (Y,ρ) be two metric spaces, a function $f:X \longrightarrow Y$ is continuous at a point $a \in X$ iff

$$\lim_{x \to a} f(x) = f(a)$$

Remark:

Continuity of a function at each point of its domain is, in fact, a global property . This will be clarified by the following theorem :

Theorem 4.1:

A function f from a metric space (X,d) into a metric space (Y,ρ) is continuous if and only if the inverse image $f^{-1}(O)$ is an open subset in X for any open subset O in Y .

Proof :

Let f be a continuous function at each point of its domain and let O be an open subset in Y.

we will prove that:

the subset f^{1} (O) is an open subset in X.

We will show that :

each point in f^{1} (O) is an interior point for it.

Let $x_0 \in f^{-1}(O)$. Then $f(x_0) \in O$. Since O is an open subset then $f(x_0)$ is an interior point for O. Thus there exists a neighbourhoud $N_\varepsilon\{f(x_0)\}$ such that:

$$N_\varepsilon\{f(x_0)\} \subseteq O$$

From the continuity of the function f there exists a neighbourhoud $N_\delta(x_0)$such that:

$$N_\delta(x_0) \subseteq f^{-1}(O)$$

This proves the first direction

Conversly let the inverse image $f^{-1}(O)$ be open in X for any open set $O \subseteq Y$ .We will prove that the function f(x) is continuous at any point $x_0 \in X$.

Since any neighbourhoud $N_\varepsilon\{f(x_0)\}$ is an open subset in the space Y then the set

$$f^{-1}\left\{N_\varepsilon\{f(x_0)\}\right\}$$

is an open subset in X which of course contains the point x_0. Therefore, the point x_0 is an interior point for the set,

$$f^{-1}\left\{N_\varepsilon\{f(x_0)\}\right\}$$

Consequently there exists a neibourhoud $N_\delta(x_0)$ such that:

$$x_0 \in N_\delta(x_0) \subseteq f^{-1}\left\{N_\varepsilon\{f(x_0)\}\right\}$$

Therefore,

$$d(x,x_0) < \delta \Longrightarrow \rho(f(x),f(x_0)) < \varepsilon$$

Theorem is completely proved

The following theorem gives a criteria for continuity of functions on metric spaces in terms of convergence of sequences.

Theorem 4.2:

A function f(x) from a metric space (X,d) into a metric space (Y,ρ) is continuous at a point $x_0 \in X$ if and only if it preserves convergence. i.e. if it translates every sequence $\{x_n\}$ which converges to the point $x_0 \in X$ to a convergent sequence $\{f(x_n)\}$ which converges to $f(x_0)$ in Y. In other words

$$\lim_n x_n = x_0 \Longrightarrow \lim_n f(x_n) = f(x_0)$$

Proof :

Suppose first that the function f is continuous at the point x_0 and suppose that $\{x_n\}$ is a sequence which converges to the point x_0. We will prove that the sequence $\{f(x_n)\}$ must converge to the point $f(x_0)$.

In fact from the continuity of the function f(x) we get

$\forall\ \varepsilon>0\ \exists\ \delta>0$ such that :

$$d(x_n, x_0) < \delta \Longrightarrow \rho(f(x_n), f(x_0)) < \varepsilon \quad (1)$$

Since the sequence $\{ x_n \}$ is convergent to the point $x_0 \in X$ then

$\forall \ \delta > 0 \ \exists$ a natural number $n_0 \in \mathbb{N}$ such that :

$$n \geq n_0 \Longrightarrow d(x_n, x_0) < \delta \qquad (2)$$

Combining (1) , (2) we get :

$\forall \ \varepsilon > 0 \ (\exists \ \delta > 0) \ \exists \ n_0 \in \mathbb{N}$ such that :

$$n \geq n_0 \Longrightarrow d(x_n, x_0) < \delta \Longrightarrow \rho(f(x_n), f(x_0)) < \varepsilon$$

Hence the first direction is proved

Now suppose the the function f(x) is not continuous at the point x_0 . Then :

$\exists \ \varepsilon_0 > 0$ such that $\forall \ \delta > 0 \ \exists$ an element $x_\delta \in X$

such that:

$$d(x_\delta , x_0) < \delta \quad \text{while} \quad \rho(f(x_\delta) , f(x_0)) \geq \varepsilon_0$$

Hence

$\exists \ \varepsilon_0 > 0$ such that $\forall \ n \in \mathbb{N} \ \exists \ x_n \in X$ such that :

$$d(x_n, x_0) < \frac{1}{n} \quad \text{while} \quad \rho (f(x_n) , f(x_0)) \geq \varepsilon_0$$

Thus the sequence $\{ x_n \}$ is convergent to the point x_0 while the sequence $\{ f(x_n)$ is not convergent to the point $f(x_0)$.

Theorem is completely proved

Remarks:

1) The above mentioned theorem is very usefull especially to show that a certain function is not continuous at certain point.

2) As a consequence of the last theorem one can easily prove that a function is not continuous in the following two ways :

1) The first way is to select a sequence $\{x_n\}$ which converges to the point x_0 while at the same time $f(x_n)$ is not convergent.

2) The second way is to choose two sequences $\{ x_n \}$ and $\{x_n'\}$ such that:

$$\lim_n x_n = x_0 = \lim_n x_n'$$

while

$$\lim_n f(x_n) \neq \lim_n f(x_n')$$

both limits exist but they are not equal.

Example: The function $f(x) = \text{sign } x$ defined by :

$$f(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ -1 & \text{for } x < 0 \end{cases}$$

is not continuous at $x = 0$. In fact, the sequence $\left\{ \frac{(-1)^n}{n} \right\}$ converges to zero while the sequence $\left\{ f\left[\frac{(-1)^n}{n}\right] \right\} = = \left\{(-1)^n\right\}$ is not convergent to any limit.

Example: The function

$$f(x) = \begin{cases} \sin(\frac{1}{x}) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

is not continuous at $x = 0$ because we can select the two sequences

$$\{x_n\} = \left\{ \frac{1}{2n\pi} \right\} \quad \text{and} \quad \{x'_n\} = \left\{ \frac{1}{2n\pi + \pi/2} \right\}$$

which converge to zero while,

$$f(x_n) = 0 \longrightarrow 0 \quad \text{and} \quad f(x'_n) = 1 \longrightarrow 1$$

Continuity and compactness

Definition 4.3:

If a function (a mapping) maps a metric space X or any space into the set of real numbers R then it is called a functional on X.■

Remark:

A functional f(x) is continuous on a set M iff

$$x_n \longrightarrow x \text{ in } M \Longrightarrow f(x_n) \longrightarrow f(x) \text{ in } R$$

Theorem 4.3:

Let X be a metric space and M is a sequentially compact set and f(x) is a continuous functional defined on M then

1. The functional f(x) is bounded on M. i.e its range is a bounded subset of real numbers.
2. The functional f(x) attains its least upper bound and greatest

lower bound on M. This means that there exists two points $x_0, y_0 \in M$ such that:

$$f(x_0) = \sup \{ f(x), x \in M \} = \max \{ f(x), x \in M \}$$

and

$$f(y_0) = \inf \{ f(x), x \in M \} = \min \{ f(x), x \in M \}$$

Proof :-

We shall show that the functional f(x) is bounded from above (boundness below can be shown similarly).

Suppose that f(x) is not bounded from above, then there exists a sequence $\{x_n\}$ of points of M such that

$$f(x_n) \geq n \quad \forall n \in N.$$

Since the set M is sequentially compact, then the sequence $\{x_n\}$ contains a subsequence $\{ x_{n_k} \}$ which converges to a point $x_o \in M$.

Since $f(x_{n_k}) > n_k \quad \forall k \in N$, then the sequence $\{ f(x_{n_k}) \}$ is not bounded.

On the other hand, since f(x) is continuous. Then,

$$f(x_{n_k}) \longrightarrow f(x_o) \text{ as } k \longrightarrow \infty$$

This leads to a contradiction. And hence f(x) is bounded from above.

First part is then proved

Let $\alpha = \sup \{ f(x): x \in M \}$. This means that:

1) $f(x) \leq \alpha$ for all $x \in M$

and

2) for every $\varepsilon > 0$ there exists a point $x_0 \in M$ such that

$$\alpha - \varepsilon < f(x_0) \leq \alpha.$$

Let $\varepsilon = \frac{1}{n}$. Hence we can choose a sequence $\{x_n\}$ in M :

$$\alpha - \frac{1}{n} < f(x_n) \le \alpha \qquad (1)$$

Since M is compact, the sequence $\{x_n\}$ contains a subsequence $\{ x_{n_k} \}$ convergent to a point $x_o \in M$.

Then, because of (1) we have

$$\alpha - \frac{1}{n_k} < f(x_{n_k}) \le \alpha$$

Taking the limit as $k \longrightarrow \infty$ we get :

$$\lim_{k \to \infty} f(x_{n_k}) = \alpha$$

On the other hand we have $\lim_{k \to \infty} f(x_{n_k}) = f(x_o)$ because f(x) is continuous on M . Hence :

$$f(x_o) = \alpha = \sup \{ f(x),\ x \in M \}$$

We can similarly prove that : If $\beta = \inf \{ f(x),\ x \in M \}$, then there exists a point $y_o \in M$ for which $f(y_o) = \beta$.

Theorem is completely proved.

UNIFORM CONTINUITY AND COMPACTNESS

Theorem 4.4:

Any continuous function f from a metric space (X,d) into a metric space (Y,ρ) is uniformly continuous on any compact set K.

Proof:

Let f be a continuous function on a compact subset K of a metric space X.

To prove that f is uniformly continuous on K, let $\varepsilon > 0$ be arbitrary and we prove that there exists $\delta > 0$ which is independent of the point x and depends only on ε, such that:

$$d(x,y) < \delta(\varepsilon) \Longrightarrow \rho(f(x),f(y)) < \varepsilon$$

Since f is continuous at any point $x \in K$ then,

$\forall \varepsilon > 0 \ \exists \delta = \delta(\varepsilon,x) > 0$ such that:

$$d(x,y) < \delta(\varepsilon,x) \Longrightarrow \rho(f(x),f(y)) < \frac{\varepsilon}{2}$$

Since we have

$$K \subseteq \cup_{x\in K} \{ x \} \subseteq \cup_{x\in K} N_{\delta(\varepsilon,x)/2}$$

then the family

$$\{ N_{\delta(\varepsilon,x)/2} \}_{x\in K}$$

is an open cover for the set K.

Since the set K is compact then we can select from that cover a finite subcover

$$N_{\delta(\varepsilon,x_1)/2} \cup N_{\delta(\varepsilon,x_2)/2} \cup \ldots \cup N_{\delta(\varepsilon,x_n)/2} \supseteq K \qquad (*)$$

Taking

$$\delta(\varepsilon) = \min \{ \delta(\varepsilon,x_i)/2 : 1 \le i \le n \}$$

we get:

$$d(x_i,y) < 2\,\delta(\varepsilon) \Longrightarrow \rho(f(x_i),f(y)) < \frac{\varepsilon}{2}$$

Now let x,y be two points such that $d(x,y) < \delta(\varepsilon)$. From (*) we get a neighbourhood $N_{\delta(\varepsilon,x_i)/2}$ such that:

$$x \in N_{\delta(\varepsilon,x_i)/2} \subseteq N_{\delta(\varepsilon,x_i)}$$

Since we have

$$d(x_i,y) \le d(x_i,x) + d(x,y) <$$
$$< \delta(\varepsilon,x_i)/2 + \delta(\varepsilon) < \delta(\varepsilon,x_i)$$

then,

$$x, y \in N_{\delta(\varepsilon,x_i)}$$

Thus,

$$d\{f(x), f(y)\} \le d\{f(x), f(x_i)\} + d\{f(x_i), f(y)\} <$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Theorem is proved

Sequences of functions Point-wise And Uniform Convergences

Definition 4.4:

A sequence of functions $\{f_n(x)\}$ having the same domain of definition D and taking values in a metric space (Y,ρ) is said to converge pointwisely to a function $f(x)$ with the same domain of definition D and takes values in Y if and only if for each (fixed) point $x \in D$ we get :

$$\forall \varepsilon > 0 \quad \exists \quad n_0 = n_0(x,\varepsilon) \in \mathbb{N} :$$

$$n \ge n_0 \Longrightarrow \rho\{f_n(x), f(x)\} < \varepsilon$$

We write in this case

$$\lim_{n \longrightarrow \infty} f_n(x) = f(x)$$

If for each $\varepsilon > 0$ we can find a number $n_0 = n_0(\varepsilon) \in \mathbb{N}$ which is independent of the point $x \in D$ i.e. if we can get n_0 which

valid for all points $x \in D$, then we say that the convergence of the sequence of functions is a uniform convergence.

Remarks:

1) The uniform convergence of a sequence of functions $\{f_n(x)\}$ to a function $f(x)$ on a domain D implies its pointwise convergence.

We say that the uniform convergence of a sequence of functions $\{f_n(x)\}$ is stronger than its pointwise convergence.

2) If the domain of definition of the sequence of functions is finite then the two notions of convergence (pointwise and uniform convergence) are the same.

Proposition 4.1:

On a finite domain, pointwise convergence implies uniform convergence.

Proof :

In fact, let the domain of definition of the sequence of functions $\{f_n(x)\}$ and $f(x)$ be a set with only m - elements

$$D = \{x_1, x_2, x_3, \ldots\ldots, x_m\}.$$

Let the convergence be a pointwise convergence on this set D.

Then,

$$\forall \varepsilon > 0 \quad \exists \quad n_1 \in \mathbb{N} :$$

$$n \ge n_1 \Longrightarrow \rho\{f_n(x_1), f(x_1)\} < \varepsilon$$

and

$$\forall \ \varepsilon > 0 \quad \exists \quad n_2 \in \mathbb{N} :$$

$$n \geq n_2 \Longrightarrow \rho\{f_n(x_2), f(x_2)\} < \varepsilon$$

Finally,

$$\forall \ \varepsilon > 0 \quad \exists \quad n_m \in \mathbb{N} :$$

$$n \geq n_m \Longrightarrow \rho\{f_n(x_m), f(x_m)\} < \varepsilon$$

Then taking,

$$n_0 = \max \{n_i \ : \ 1 \leq i \leq m\}$$

we get :

$$\forall \ \varepsilon > 0 \quad \exists \quad n_0 \in \mathbb{N} :$$

$$n \geq n_0 \Longrightarrow \rho\{f_n(x), f(x)\} < \varepsilon \quad \forall \ x \in D$$

Hence the convergence is a uniform convergence.

Proposition is proved

Criteria for uniform convergence

Proposition 4.2:

A sequence of functions $\{f_n(x)\}$ converges to a function $f(x)$ uniformly on a domain D if and only if :

$$c_n = \sup_{x \in D} \rho\{ f_n(x), f(x)\} \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty$$

i.e.

$$\lim_{n \to \infty} \sup_{x \in D} \rho\{ f_n(x), f(x)\} = 0$$

Proof :

Let $\{f_n(x)\}$ be a sequence of continuous functions which converge pointwisely to a function $f(x)$ on a set D.

This means that:

$$\forall \ \varepsilon > 0 \quad \exists \quad n_0 = n_0(\varepsilon) \in \mathbb{N} :$$

$$n \geq n_0 \Longrightarrow \rho\{f_n(x), f(x)\} < \varepsilon \quad \forall \ x \in D.$$

Here ε is an upper bound for the set :

$$\left\{ \rho\{f_n(x), f(x)\} \ : x \in D \right\}$$

Hence,

$$\forall \ \varepsilon > 0 \quad \exists \quad n_0 = n_0(\varepsilon) \in \mathbb{N} :$$

$$n \geq n_0 \Longrightarrow \sup_{x \in D} \rho\{f_n(x), f(x)\} < \varepsilon$$

i.e.

$$c_n = \sup_{x \in D} \rho\{ f_n(x), f(x)\} \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty$$

Conversely, let

$$c_n = \sup_{x \in D} \rho\{ f_n(x), f(x)\} \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty$$

Then,

$$\forall \ \varepsilon > 0 \ \exists \ n_0 = n_0(\varepsilon) \in \mathbb{N} :$$

$$n \geq n_0 \Longrightarrow \rho\{f_n(x), f(x)\} \leq \sup_{x \in D} \rho\{f_n(x), f(x)\} < \varepsilon \qquad \forall \ x \in D.$$

Proposition is completely proved

Convergence of sequences in the space $C_{[0,1]}$ of continuous functions

Proposition :

Convergence in the metric space $C_{[0,1]}$ of all continuous fucnctions defined on the interval [0,1] is a uniform convergence of functions.

Proof :

Let $\{x_n(t)\}$ be a convergent *sequence of elements the* space $C_{[0,1]}$ of *continuous on* [0,1] functions.Let x(t) be its limit i.e.

$$d(x_n, x) \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty.$$

This mean that:

$$\max_t |x_n(t) - x(t)| \longrightarrow \infty \quad \text{as } n \longrightarrow \infty$$

Then for every $\varepsilon > 0$ there exist a natural number $n_0(\varepsilon)$ such that :

$$\max_t |x_n(t) - x(t)| < \varepsilon \quad \forall \quad n \geq n_0(\varepsilon)$$

hence $A\ \varepsilon > 0\ \exists\ n_0$:

$$|x_n(t) - x(t)| < \varepsilon \ \forall\ n \geq n_0(\varepsilon) \quad \text{and} \quad t \in [0,1]$$

This is the definition of the uniformly convergent sequence of functions.

Uniform convergence and continuity

Theorem 4.5:

A uniform limit of a *sequence* of *continuous* functions is also continuous.i.e.

Let $\{f_n(x)\}$ be a sequence of *continuous functions on the same* metric space (X,d) which converges *uniformly* on X to a function f(x). Then the function f(x) is continuous on the domain X.

Proof :

Let $x_0 \in X$ be arbitrary. We will show that the function f(x) is continuous at x_0. i.e. we will show that:

$\forall\ \varepsilon > 0\ \exists\ \ \delta > 0$ such that :

$$d(x,x_0) < \delta \Longrightarrow \rho\{f(x), f(x_0)\} < \varepsilon$$

From the uniform convergence of the sequence of functions we get :

$$\lim_{n \to \infty} c_n = \lim_{n \to \infty} \sup_{x \in D} \rho\{ f_n(x), f(x)\} = 0$$

Therefore,

$$\forall\ \varepsilon > 0\ \exists\ n_0 \in \mathbb{N} \text{ such that :}$$

$$n \geq n_0 \Longrightarrow 0 \leq c_n < \frac{\varepsilon}{3}$$

Hence,

$$n \geq n_0 \Longrightarrow \rho\{f_n(x), f(x)\} < \frac{\varepsilon}{3} \qquad \forall\ x \in X \quad (*)$$

Since the function $f_{n_0}(x)$ is continuous at the point x_0 then, $\forall\ \varepsilon > 0 \quad \exists\ \delta_{n_0} > 0$ such that :

$$d(x,x_0) < \delta_{n_0} \Longrightarrow \rho\{f_{n_0}(x), f_{n_0}(x_0)\} < \frac{\varepsilon}{3}$$

Consequently using (*) we get: $\forall \varepsilon > 0 \; \exists \; \delta > 0$ such that :

$$d(x,x_0) < \delta \Longrightarrow \rho\{f(x), f(x_0)\} \leq$$

$$\leq \rho\{f(x), f_{n_0}(x)\} + \rho\{f_{n_0}(x), f_{n_0}(x_0)\} +$$

$$+ \rho\{f_{n_0}(x_0), f(x_0)\} < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

Remark:

Let $\{f_n(x)\}$ be a sequence of continuous functions which converge pointwisely to a function f(x) on a space X. If the function f(x) is not continuous on the space X then the convergence of the sequence of functions can not be a uniform convergence.

Homeomorphic and isometric metric spaces

Definition : Let X , Y to be two metric spaces and let $\phi : X \longrightarrow Y$ be a bijection (one to one and onto) If ϕ , ϕ^{-1} are continuous then ϕ is called a homeomerphism between X ,Y. The spaces X and Y are called homeomorphic.

Remark :

Let X , Y be to be homeomorphic spaces with a homeomorphism ϕ then :

a) *From* theorem * we see that for a sequence $\{ x_n \}$ of elements of X, it is true that:

$\{ x_n\}$ converges to x iff $\{ \phi(x_n) \}$converges to $\phi(x)$.

i.e.

$$\lim_{n \to \infty} x_n = x \text{ iff } \lim_{n \to \infty} \phi(x_n) = \phi(x) .$$

b) If τ_1 and τ_2 are the topologies induced on the metric spaces X and Y respectively then by theorem (*) we have

$$O \in \tau_1 \Longrightarrow \phi(O) \in \tau_2 \text{ that is } \phi(\tau_1) = \tau_2$$

Definition : Two metric spaces (X ,ρ_1) , (Y ,ρ_2) are called isometric metric space if there exists a bijection

$$\phi : X \longrightarrow Y, \quad \phi^{-1}: Y \longrightarrow X,$$

satisfying:

$$\forall x , x' \in X \qquad \rho_1(x, x') = \rho_2(\phi(x), \phi(x')).$$

In this case ϕ is called an isometry between the spacec X and Y.

Remark:

a) It is easy to see that ismetric metric spaces are homeomorphic metric spaces .

b) Homeomorphism and isometry are equivalence relation on the collection of all metric spaces.Hence the collection of all metric spacse is partitioned to equivalence classes of homeomorphic metric spaces and epuivalence classes of isometric metric spaces.

Finally homeomorphic metric spaces are considered as equivalent metric spaces and isometric metric spaces are considered as identical metric spaces .

Definition:(Equivalent distances)

Let ρ_1, ρ_2 be two metrics defined on the same space X. We say that the two metrics are equivalent if and only if there exists a constant k >0 such that $\forall$ x,y $\in$ X

$$\frac{1}{k}\rho_1(x, y) \leq \rho_2(x,y) \leq k\,\rho_1(x,y) \blacksquare$$

Exercise:

If ρ_1, ρ_2 are two metrics defined on the same space X then prove that metric spaces (X, ρ_1), (X, ρ_2) are homeomorphic.

The Principle of Contraction mapping

Definition 3.12 :

A mapping A :X $\longrightarrow$ X from a metric space (X,d) into itself is

called a contraction mapping if there exists a constant α, $0< \alpha < 1$ such that:

$$d(Ax, Ay) \leq \alpha\, d(x, y) \quad \text{for all } x,y \in X$$

Theorem 3.5 :

Let A be a mapping which maps a complete metric space X into itself such that

$$d(Ax, Ay) \leq \alpha\, d(x,y) \quad \forall\, x,y \in X,$$

where $0 < \alpha < 1$. Then there exists one and only one point $x_0 \in X$ such that

$$Ax_0 = x_0.$$

The point x_0 is called the fixed point of the mapping A .

Proof :

We consider an arbitrary but fixed element x $\in$ X and put

$x_1 = Ax$, $x_2 = A(x_1)$, $x_3 = A(x_2)$, ... $x_n = A(x_{n-1})$.

We show that x_n is a Cauchy sequence, We note that

$$d(x_1, x_2) = d(Ax, Ax_1) \leq \alpha\, d(x, x_1) = \alpha\, d(x, Ax).$$

$$d(x_2, x_3) = d(Ax_1, Ax_2) \leq \alpha\, d(x_1, x_2) = \alpha^2 d(x, Ax).$$

....

....

$$d(x_n, x_{n+1}) \leq \alpha^n d(x, Ax)$$

Then we have :

$$d(x_n, x_{n+p}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots\ldots d(x_{n+p-1}, x_{n+p})$$

$$d(x_n, x_{n+p}) \leq \alpha^n d(x, Ax) + \alpha^{n+1} d(x, Ax) + \ldots + \alpha^{n+p-1} d(x, Ax)$$

$$= [\alpha^n + \alpha^{n+1} + \dots + \alpha^{n+p-1}] \quad d(x, Ax)$$

$$d(x_n, x_{n+p}) \leq \frac{\alpha^n - \alpha^{n+p}}{1 - \alpha} d(x, Ax) < \frac{\alpha^n}{1-\alpha} d(x, Ax)$$

Therefore,

$$d(x_n, x_{n+p}) \longrightarrow 0, \text{ as } n \longrightarrow \infty \quad (\alpha < 1) \qquad \forall p \in N$$

Therefore $\{ x_n \}$ is a Cauchy sequence. Since the metric space X is complete, there exists a point $x_o \in X$ such that

$$\lim_{n \to \infty} x_n = x_o.$$

We shall show that $A x_o = x_o$. We have

$\lim_{n \to \infty} x_n = x_o$, $\lim_{n \to \infty} x_{n-1} = x_o$. And we have

$$d(x_o, Ax_o) \leq d(x_o, x_n) + d(x_n, Ax_o)$$

$$= d(x_o, x_n) + d(Ax_{n-1}, Ax_o)$$

$$\leq d(x_o, x_n) + \alpha\, d(x_{n-1}, x_o)$$

But $\forall \varepsilon > 0 \; \exists n_1 : d(x_n, x_o) < \frac{\varepsilon}{2} \; \forall n \geq n_1$.

$\forall \varepsilon > 0 \; \exists n_2 : d(x_{n-1}, x_o) < \frac{\varepsilon}{2} \; \forall n \geq n_2$.

if $n \geq n_1$, $n \geq n_2$ then

$$d(x_o, x_n) + d(x_{n-1}, x_o) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

i.e. $\forall \varepsilon > 0 \; d(x_o, Ax_o) < \varepsilon$

Since ε is arbitrary, then $d(x_o, Ax_o) = 0$ i.e. $Ax_o = x_o$

Now we prove that x_o is unique. Let $y_o \neq x_o$, $A y_o = y_o$ then we have :

$$d(x_o, y_o) = d(Ax_o, Ay_o) \leq \alpha\, d(x_o, y_o)$$

This contradicts that $d(x_o, y_o) > 0$ and $\alpha < 1$.

The theorem is proved.

Example :

Let R^n be Euclidean space and $A : R^n \longrightarrow R^n$ be a mapping defined for all $x = (x_1, x_2, x_3, \dots, x_n) \in R^n$ by:

$$A(x_1, x_2, x_3, \dots, x_n) = (y_1, y_2, y_3, \dots, y_n)$$

$$y_i = \sum_{j=1}^{n} a_{ij} x_j - b_i$$

Then A is a contraction mapping if $\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2 < 1$.

In other words, for a matrix $[a_{ij}]$ the system of equations

$$y_i = \sum_{j=1}^{n} a_{ij} x_j - b_i, \quad i = 1, 2, \dots, n$$

has a unique solution $x_o = (x_1^o, x_2^o, \dots, x_n^o)$ if

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2 < 1$$

Proof :

We have that

$$\rho(A(x), A(x)) = \rho(y, y') = \sqrt{\sum_{i=1}^{n} (y_i - y_i')^2}$$

$$= \sqrt{\sum_{i=1}^{n} \left\{ \sum_{j=1}^{n} a_{ij} (x_j - x_j') \right\}^2}$$

$$\leq \sqrt{\sum_{i=1}^{n} \left\{ \sum_{j=1}^{n} a_{ij}^2 \sum_{j=1}^{n} (x_j - x_j')^2 \right\}}$$

We have used Holders inequality with p = q = 2

Hence,

$$\left\{ \sum_{j=1}^{n} a_{ij} (x_j - x_j') \right\}^2 \leq \sum_{j=1}^{n} a_{ij}^2 \sum_{j=1}^{n} (x_i - x_j')^2$$

Finally we get that

$$\rho(A(x), A(x')) \le \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}}\ \sqrt{\sum_{j=1}^{n}(x_j - x'_j)^2} = \alpha\, \rho(x,x'),$$

where $\alpha = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^2}$.

Hence the mapping A is contraction if $\alpha < 1$ i.e.if

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^2 < 1.$$

The uniqueness of x_o follows from theorem (*).

Exercises 3.2.

1. Prove that R_d^* is complete metric space.
2. Consider the open interval (0,1), prove *that* $[(0,1),\rho_1]$ is not a complete metric space, and $[(0,1), d_1]$ is a complete metric space, where

$\rho_1(x,y) = |x-y|$

$d_1(x,y) = 1$ if $x \ne y$,

$= 0$ if $x=y$

3. Consider the two metric spaces (R^n, σ), (R^n, τ) where $\sigma(x,y) = \sum_{i=1}^{n} |x_i - y_i|$, $\tau(x,y) = \max_{1 \le i \le n} |x_i - y_i|$

Let $A : \begin{cases} R^n \longrightarrow R^n \\ A(x) = y \end{cases}$ be such that

$$y_i = \sum_{j=1}^{n} a_{ij}\, x_j - b_i, \qquad i = 1, 2, \ldots, n \qquad (1)$$

Give the condition that the system of equations (1) has a unique solution in (R^n, σ), (R^n, τ).

(The conditions are $\sum_{j=1}^{n}(\sum_{i=1}^{n} a_{ij})^2 < 1$, $\sum_{j=1}^{n} |a_{ij}| < 1$ respectively).

4. If $A : [0,1] \longrightarrow [0,1]$ is a differentiable function and if there is a real number α, $0 \le \alpha < 1$ such that:

$$|A'(x)| \le \alpha \quad \forall \quad x \in [0,1]$$

where A' is the derivative of A, prove that A is a contraction mapping on the interval [0,1]

Baire category theorem

Definition 3.24 :

A subset M of a topological space X is called nowhere dense subset if its closure has an empty interior i.e.

$$(\bar{M})^o = \phi$$

Example :-

Any point in the straight line R form a no-where dense subset in R.

In fact any set $\{x\}$ of one point is a closed set in R because it has no limit points. Hence $\overline{\{x\}} = \{x\}$. Clearly $\overline{(\{x\})}^o = \phi$ because any set of one point cannot contain any open sphere.

Remark :-

It is clear that a subset has an empty interior in a metric space iff it cannot contain any sphere.

Example :-

Any real line is a no-where dense subset in the plane R^2 (with the usual topology).

Example :-

Any plane is a no-where dense subset in R^3 (with the usual topology).

Remark :-

Being a no-where dense subset, depends on the space in which the subset lies. For example; R *is nowhere dense in* R^2, while R is everywhere dense in itself.

Definition 3.25 :

A subset Y in a topological space X is called a set of the first category if Y can be written as a countable union of nowhere dense subsets.

Definition 3.26:

A subset in a topological space which is not of the first category is said to be of the second category.

Theorem 3.9 :

A complete metric space X is a set of second category. I.e. any complete metric space cannot be written as a countable union of nowhere dense sets.

Proof :-

Let X be a set of first category, i.e. let X be a countable union,

$$\bigcup_{n \in N} M_n$$

of no-where dense subsets M_n. We will get a contradiction.

Let $N_{r_1}(x_1)$ be any open sphere in X with radius r_1 and center $x_1 \in X$. Clearly $\overline{M_1}$ cannot contain $N_{r_1}(x_1)$. Thus there exists a point $x_2 \in N_{r_1}(x_1)$ and $x_2 \notin \bar{M}_1$. Consequently x_2 is not a limit point of M_1. Then there exists a neighborhood $N_{r_2}(x_2)$: such that $N_{r_2}(x_2) \cap M_1 = \phi$. We can take the radius r_2 small enough (no problem to do that) in order that

$$\bar{N}_{r_2}(x_2) \subseteq N_{r_1}(x_1) \text{ and } r_2 \le r_1/2.$$

In the same manner we see that $\overline{M_2}$ cannot contain the sphere $N_{r_2}(x_2)$. Hence we can choose a point $x_3 \notin \overline{M_2}$ and $x_3 \in N_{r_2}(x_2)$.

Again we choose a sphere $N_{r_3}(x_3)$:

$$\bar{N}_{r_3}(x_3) \subseteq N_{r_2}(x_2) \subseteq \bar{N}_{r_2}(x_2) \subseteq N_{r_1}(x_1)$$

and

$$\bar{N}_{r_3}(x_3) \cap M_2 = \phi$$

we can choose r_3 such that:

$$r_3 \le r_2/2 \le r_1/4$$

In fact we get a sequence of nested closed spheres in a complete metric space. Consequently we get a point x_0 which belong to all the closed spheres $\left\{ \bar{N}_{r_i}(x_i) \right\}_{i \in N}$.

$$x_0 \in \bar{N}_{r_{i+1}}(x_{i+1}) \subset N_{r_i}(x_i) \quad i=1,2,\ldots$$

Since $N_{r_i}(x_i) \cap M_{i-1} = \phi$ for all $i=2,3,\ldots$. Then

$$x_0 \notin M_i \quad i=1,2,3,\ldots$$

Thus

$$x_0 \notin \bigcup_{i\in N} M_i = X.$$

This gives a contradiction. Thus the complete space X cannot be written as a countable union of no where *dense subsets*.

Theorem is completely proved .

Question:

Is the complement of an everywhere dense set, *nowhere-dense*?.

AIN SHAMS UNIVERSITY
FACULTY OF SCIENCE
DEPARTMENT OF MATHEMATICS

THIRD YEAR EXAM
REAL ANALYSIS
MAY 1

ANSWER ONLY FIVE QUESTIONS:

FIRST QUESTION : A) Define the following:

1- A countable set.

2- Interior point of a subset E in a metric space.

3- An open subset E in a metric space.

4- A closd subset F in a metric space.

5- An open cover for a subset.

6- A compact subset K in a metric space.

B) Prove that the set S of all sequences of elements zeros or ones, is not a countable set.

SECOND QUESTION :

A) Define a k- cell in the Eucledian space R^k.

B) Prove that any k-cell in the Eucledian space R^k is compact

THIRD QUSTION : A) Define the following :

1- Upper and lawer Darbaux sums.

2- Upper and lawer Riemann integrals.

3- A Riemann integrable function.

B) Prove that any continuous function on a closed interval [a,b] is Riemann integrable.

C) Let $f : R \longrightarrow R^k$ be a function such that

$$f(x) = (f_1(x), f_2(x), \ldots\ldots\ldots, f_k(x))$$

Prove that the function $f(x)$ is continuous at a point x_0 iff all the functions $f_i(x)$'s are continuous at x_0 .

FOURTH QUESTION

A) Define uniform convergence of a sequence of functions $f_n(x)$ which are defind from a metric space (X,d) into a metric space (Y,ρ).

B) Examine uniform convergence of the sequence of functions

$$f_n(x) = \frac{x}{1 + n x^2} \quad \text{on the closed interval } [0,1] .$$

C) Let $\overline{N_{\varepsilon_1}(x_1)} \supset \overline{N_{\varepsilon_2}(x_2)} \supset \overline{N_{\varepsilon_3}(x_3)} \supset \overline{N_{\varepsilon_4}(x_4)} \ldots,$

be a sequence of closed spheres in a complete metric space (X,d). If the sequence $\{ \varepsilon_n \}$ tends to zero, prove that the intersection

$\bigcap_n \overline{N_{\varepsilon_n}(x_n)} = \{ x_0 \}$ is exactly one point.

FIFTH QUESTION :

A) Let $\{x_n\}$ be a convergent sequence of real numbers .Prove that $\sup_n \inf \{ x_{n+1}, x_{n+2}, \ldots \} = \inf_n \sup \{ x_{n+1}, x_{n+2} \ldots \}$

B)Let $\{x_n\}$be a sequence of elements of a metric space (X,d) which converges to a point x_0. Prove that the set :

$\{ x_1, x_2, x_3, x_4, \ldots \}$ is a compact set.

SIXTH QUESTION :

Prove that a metric space (X,d) is compact iff every family $\{ F_i \}_{i \in I}$ of closed subsets of X such that every finite subfamily of it has a non empty intersection (i.e. any.. inter- section $\bigcap_{j=1}^{j=n} F_{i_j}$ is not empty) must have itself a non-empty intersection $\bigcap_{i \in I} F_i \neq \phi$.

Best Luck · *Nashat Faried*